Introduction

In [FS1] we announced a precise asymptotic formula for the ground–state energy of a large atom. One part of our proof of that formula is a refined analysis of the eigenvalues and eigenfunctions of an ordinary differential operator $\frac{-d^2}{dx^2} + V(x)$. In particular, we need to improve on the standard WKB approximations for rather general potentials V. This paper contains our results on individual eigenvalues and eigenfunctions of ordinary differential operators. Our later papers [FS2,3,4,5] will study sums of eigenvalues and sums of squares of eigenfunctions, and then pass to spherically symmetric three–dimensional problems by separation of variables. In [FS6] we will apply our knowledge of three–dimensional problems to the study of atoms, which is a $3N$–dimensional problem for large N. Finally, in [FS7] we study the Hamiltonian flow in a three–dimensional potential arising in atomic physics, in order to verify a crucial hypothesis in [FS4,5].

We begin our discussion of ordinary differential operators $\frac{-d^2}{dx^2}+V(x)$ by reviewing the standard WKB approximation in one dimension. The approximation applies to large, slowly varying potentials. A basic example is

$$V(x) = \lambda^2 V_1(x) \tag{1}$$

for fixed, smooth V_1 and $\lambda >> 1$. For simplicity, we suppose $V_1'(0) = 0$ and $V_1''(x) > c > 0$ for all x.

According to WKB theory, the eigenvalues E_k of $\frac{-d^2}{dx^2} + V(x)$ are given approximately as the solutions of

$$\phi(E_k) = \pi(k + 1/2) \quad \text{for integers } k, \text{ where} \tag{2}$$

$$\phi(E) = \int_{\{V(x)<E\}} (E - V(x))^{1/2}\, dx. \tag{3}$$

Under our assumptions on V_1, $\{V(x) < E\}$ is an interval $[x_{\text{left}}(E), x_{\text{right}}(E)]$. The endpoints $x_{\text{left}}(E)$, $x_{\text{right}}(E)$ are called turning points.

For E_k not too near the minimum of the potential, WKB theory also gives approximate formulas for an eigenfunction $F_k(x)$ corresponding to E_k. In the simplest case, we take $E_k > (\min V) + c\lambda^2$. Then $F_k(x)$ is given by several different formulas in overlapping regions, and (2) is the consistency condition for the different formulas to agree. The formulas for F_k are as follows.

For $x \in [x_{\text{left}}(E_k), x_{\text{right}}(E_k)]$ not too near the turning points,

$$F_k(x) \approx (E_k - V(x))^{-1/4} \cos(-\frac{\pi}{4} + \int_{x_{\text{left}}(E_k)}^{x} (E_k - V(t))^{1/2}\, dt). \tag{4}$$

For x near $x_{\text{left}}(E_k)$, a smooth change of variable $x \to y = y_k(x)$ transforms

$$\Big(-\frac{d^2}{dx^2} + V(x) - E\Big) F_k(x) = 0 \tag{5}$$

approximately to Airey's equation

$$\Big(\frac{d^2}{dy^2} + \lambda^2 y\Big) \tilde{F}_k(y) = 0. \tag{6}$$

In fact, the function

$$x \mapsto \Big(\frac{3}{2}\lambda^{-1} \int_{x_{\text{left}}(E_k)}^{x} (E_k - V(t))^{1/2}\, dt\Big)^{2/3} \quad \text{for } x \in [x_{\text{left}}(E_k), x_{\text{left}}(E_k) + c] \tag{7}$$

extends to a smooth function $y_k(x)$ on $[x_{\text{left}}(E_k) - c, x_{\text{left}}(E_k) + c]$ that transforms (5) approximately to (6).

Airey's equation (6) has a special solution $\tilde{F}_k(y) = \lambda^{-1/3} A(\lambda^{2/3} y)$, where A is the Airey function, defined by

$$A(y) = (2\sqrt{\pi})^{-1} \lim_{\delta \to 0+} \int_{-\infty}^{\infty} \exp\{-\delta\xi^2 + i\xi y - \frac{i}{3}\xi^3\}\, d\xi. \tag{8}$$

Pulling back $\tilde{F}_k(y)$ by the map $x \mapsto y_k(x)$, we obtain a formula for $F_k(x)$ near $x_{\text{left}}(E)$, namely

$$F_k(x) \approx \lambda^{-1/3} \Big(\frac{dy_k}{dx}\Big)^{-1/2} A(\lambda^{2/3} y_k(x)) \quad \text{for } |x - x_{\text{left}}(E_k)| < c. \tag{9}$$

The Airey function is well–understood, so (9) gives a good understanding of F_k near $x_{\text{left}}(E_k)$.

Similarly,

$$F_k(x) \approx \pm\lambda^{-1/3}\Big(\frac{-d\tilde{y}_k}{dx}\Big)^{-1/2} A(\lambda^{2/3}\tilde{y}_k(x)) \text{ for } |x - x_{\text{right}}(E_k)| < c, \tag{10}$$

where $\tilde{y}_k$ is a smooth change of variable analogous to y_k.

Outside $[x_{\text{left}}(E_k), x_{\text{right}}(E_k)]$, $F_k(x)$ is exponentially small unless x is so near turning point that (9) or (10) applies. Hence we have an approximate description of $F_k(x)$ for all x.

WKB theory also asserts that the L^2–norm of the eigenfunction described by (4), (9), (10) is given by

$$\|F_k\|^2 \approx \frac{1}{2}\int_{\{V(x)<E_k\}} (E_k - V(x))^{-1/2}\, dx. \tag{11}$$

A standard reference containing rigorous results to justify (2), (4), (9), (10), (11) is Erdelyi [E]. For instance, one knows that $\phi(E_k) = \pi(k+1/2) + O(\lambda^{-1})$ for eigenvalues E_k. Equation (4) holds modulo an error $O(\lambda^{-1}(E_k - V(x))^{-1/4})$, provided $x \in [x_{\text{left}}(E_k), x_{\text{right}}(E_k)]$ stays bounded away from the turning points. As x approaches $x_{\text{left}}(E_k)$, the error degrades to $O(\lambda^{-1}(x - x_{\text{left}}(E_k))^{-3/2}(E_k - V(x))^{-1/4})$, and similarly for $x_{\text{right}}(E_k)$.

The first main purpose of this article is to give more accurate approximations than the above for the eigenvalues and eigenfunctions of $\frac{-d^2}{dx^2} + V(x)$. When $V(x)$ is given by (1), our results are essentially as follows.

(A) The eigenvalues E_k satisfy

$$\phi(E_k) + \frac{1}{48}\psi(E_k) = \pi(k+1/2) + O(\lambda^{-2+\varepsilon}), \quad \text{with} \tag{12}$$

$$\psi(E) = \lim_{\delta\to 0+}\Big[\int_{E-V(x)>\delta} V''(x)(E - V(x))^{-3/2}\, dx - q(E)\delta^{-1/2}\Big] \quad \text{and} \tag{13}$$

$$q(E) = \frac{2V''(x_{\text{right}}(E))}{V'(x_{\text{right}}(E))} - \frac{2V''(x_{\text{left}}(E))}{V'(x_{\text{left}}(E))}. \tag{14}$$

If $E_k > (\min V) + c\lambda^2$, then an eigenfunction $F_k(x)$ corresponding to E_k will satisfy the following.

(B) For $x \in [x_{\text{left}}(E_k), x_{\text{right}}(E_k)]$ not too near a turning point, we have

(15)
$$F_k(x) \approx (E_k - V(x))^{-1/4} \cos(-\frac{\pi}{4} + \int_{x_{\text{left}}(E_k)}^{x} (E_k - V(t))^{1/2}\, dt + \omega_k(x)), \quad \text{with}$$

(16)
$$\omega_k(x) = \lim_{\delta \to 0+} \Big[\int_{x_{\text{left}}(E_k)+\delta}^{x} \Big\{ \frac{5}{32} \frac{(V')^2}{(E_k - V)^{5/2}} + \frac{1}{8} \frac{V''}{(E_k - V)^{3/2}} \Big\}\, dt - \sum_{\ell=1}^{3} q_\ell(E_k) \delta^{-\ell/2} \Big]$$

and $q_\ell(E_k)$ uniquely specified by demanding the finiteness of the limit (16). "Not too near a turning point" means that $|x - x_{\text{left}}(E_k)|, |x - x_{\text{right}}(E_k)| > \lambda^{\varepsilon - 2/3}$. Equation (15) holds modulo an error $O(\lambda^{-2}(E_k - V(x))^{-1/4})$, provided x stays bounded away from the turning points. When x is close to $x_{\text{left}}(E_k)$, the error degrades to $O(\lambda^{-2}(x - x_{\text{left}}(E_k))^{-3}(E_k - V(x))^{-1/4})$, and similarly for $x_{\text{right}}(E_k)$.

(C) Near the turning points, we can describe F_k with great accuracy by formulas (9) and (10) for suitable smooth coordinate changes $y_k(x)$, $\tilde{y}_k(x)$. In fact,

(17)
$$F_k(x) = \lambda^{-1/3} \Big(\frac{dy_k}{dx} \Big)^{-1/2} A(\lambda^{2/3} y_k(x)) + O(\lambda^{-N}) \text{ for } |x - x_{\text{left}}(E_k)| < \lambda^{-\varepsilon}$$

and

(18)
$$F_k(x) = \pm b_k \lambda^{-1/3} \Big(-\frac{d\tilde{y}_k}{dx} \Big)^{-1/2} A(\lambda^{2/3} \tilde{y}_k(x)) + O(\lambda^{-N}) \text{ for } |x - x_{\text{right}}(E_k)| < \lambda^{-\varepsilon},$$

with $\varepsilon > 0$ arbitrarily small and $N > 0$ arbitrarily large, and with b_k a real constant satisfying $|b_k - 1| < C\lambda^{\varepsilon - 2}$. However, the formula (7) for $y_k(x)$ and its analogue for $\tilde{y}_k(x)$ have to be corrected to achieve (17), (18). This was known already to Cherry [C].

Away from the intervals where (B) or (C) holds, we know already from standard WKB theory that $F_k(x)$ is exponentially small.

(D) The L^2–norm of F_k is given by

$$\|F_k\|^2 = \frac{1}{2}\int_{\{V(x)<E_k\}} (E_k - V(x))^{-1/2}\, dx \cdot (1 + O(\lambda^{\varepsilon-2})). \tag{19}$$

Thus we have improved on the accuracy of standard WKB theory in describing the eigenvalues and eigenfunctions for potentials of the form (1).

The second main point of this article is to deal with potentials more general than (1). We need the added generality in order to understand atoms. Already from the Hydrogen atom, we get

$$V(x) = \frac{\ell(\ell+1)}{x^2} - \frac{1}{x}, \tag{20}$$

which is certainly not of the form (1). Hughes [H] extended the standard WKB theory to potentials relevant to atomic physics. For the potential (20), our results yield analogues of (A)...(D) above, with ℓ playing the rôle of the large parameter λ.

To capture the behavior of a general potential V, we introduce positive weight functions $S(x)$, $B(x)$, and assume the estimates

$$\left|\left(\frac{d}{dx}\right)^{\alpha} V(x)\right| \le C_\alpha S(x)(B(x))^{-\alpha}, \quad \text{on a suitable interval } I. \tag{21}$$

For instance, if $V(x)$ is given by (1), then (21) holds with $S(x) = \lambda^2$, $B(x) = 1$, $I = [-1, 1]$, and our results apply to eigenvalues E for which $[x_{\text{left}}(E), x_{\text{right}}(E)] \subset\subset I$.

For the potential (20) we can take $S(x) = \frac{1}{x}$, $B(x) = x$, and (21) holds on $I = (\frac{\ell(\ell+1)}{10}, \infty)$. In (21), $S(x)$ controls the size of V, while $B(x)$ controls the lengths of intervals over which V is essentially constant. Thus, (21) gives a precise meaning to the notion of a large, slowly varying potential. A closely related idea underlies the Beals–Fefferman pseudodifferential operator calculus [BF].

Our general WKB theorems apply for instance to potentials that satisfy (21) and have no critical points on I except for a single non–degenerate minimum. More precisely, for an $x_0 \in I$ we assume:

$$V'(x_0) = 0 \tag{22}$$

$$V''(x) > cS(x)B^{-2}(x) \quad \text{for} \quad |x - x_0| \leq c_1 B(x_0) \tag{23}$$

$$|V'(x)| > cS(x)B^{-1}(x) \quad \text{for } x \in I \quad \text{with} \quad |x - x_0| \geq c_1 B(x_0). \tag{24}$$

In addition, we make various technical assumptions on $V(x)$, $S(x)$, $B(x)$. Under these conditions, we can prove analogues of (A)...(D) above.

We focus on the eigenvalues E_k that are not too close to the minimum of the potential V. Our results are essentially as follows.

(A′) If $E > V(x_0) + cS(x_0)$ is an eigenvalue of $-\frac{d^2}{dx^2} + V(x)$, then

$$\phi(E) + \frac{1}{48}\psi(E) = \pi(k + 1/2) + O(\Lambda^{-2}) \quad \text{for an integer } k, \tag{25}$$

with ϕ and ψ defined by (3) and (13), and with

$$\Lambda^{-1} = \int_{x_{\text{left}}(E)}^{x_{\text{right}}(E)} \frac{dx}{S^{1/2}(x)B^2(x)}. \tag{26}$$

Moreover, an eigenfunction $F(x)$ corresponding to E has the following properties.

(B′) For $x \in [x_{\text{left}}(E), x_{\text{right}}(E)]$ not too near the turning points, we have

$$F(x) \approx (E - V(x))^{-1/4} \cos(-\frac{\pi}{4} + \int_{x_{\text{left}}(E)}^{x} (E - V(t))^{1/2}\, dt + w(x)) \tag{27}$$

with $w(x)$ as in (16). "Not too near the turning points" means $|x - x_{\text{left}}(E)| > \lambda_{\text{left}}^{\varepsilon - 2/3} B(x_{\text{left}}(E))$ and $|x - x_{\text{right}}(E)| > \lambda_{\text{right}}^{\varepsilon-2/3} B(x_{\text{right}}(E))$, with

$$\lambda_{\text{left}} = S^{1/2}(x_{\text{left}}(E))B(x_{\text{left}}(E)) \quad \text{and} \quad \lambda_{\text{right}} = S^{1/2}(x_{\text{left}}(E))B(x_{\text{left}}(E)). \tag{28}$$

Equation (27) holds modulo an error $O(\Lambda^{\varepsilon-2}(E - V(x))^{-1/4})$, provided $|x - x_{\text{left}}(E)| > cB(x_{\text{left}}(E))$ and $|x - x_{\text{right}}(E)| > cB(x_{\text{right}}(E))$. When $|x - x_{\text{left}}(E)| \leq cB(x_{\text{left}}(E))$, the error is degraded to $O(\Lambda^{\varepsilon-2} \cdot \left(\frac{x - x_{\text{left}}(E)}{B(x_{\text{left}}(E))}\right)^{-3} \cdot (E - V(x))^{-1/4})$, and similarly for $x_{\text{right}}(E)$.

(C′) For $|x - x_{\text{left}}(E)| < \lambda_{\text{left}}^{-\varepsilon} B(x_{\text{left}}(E))$ we have

$$F(x) = \lambda_{\text{left}}^{-1/3} \left(\frac{dY}{dx}\right)^{-1/2} A(\lambda_{\text{left}}^{2/3} Y) + O(\Lambda^{-N} B^{1/2}(x_{\text{left}}(E))), \tag{29}$$

with Y a smooth function of $\frac{x - x_{\text{left}}(E)}{B(x_{\text{left}}(E))}$.

Similarly, for $|x - x_{\text{right}}(E)| < \lambda_{\text{right}}^{-\varepsilon} B(x_{\text{right}}(E))$ we have

$$F(x) = \pm b \lambda_{\text{right}}^{-1/3} \left(\frac{-d\tilde{Y}}{dx}\right)^{-1/2} A(\lambda_{\text{right}}^{2/3} \tilde{Y}) + O(\Lambda^{-N} B^{1/2}(x_{\text{right}}(E))) \tag{30}$$

with $\tilde{Y}$ a smooth function of $\frac{x - x_{\text{right}}(E)}{B(x_{\text{right}}(E))}$, and with a real constant b satisfying $|b - 1| < C\Lambda^{\varepsilon - 2}$.

Outside the regions where (B′), (C′) hold, F is exponentially small. Finally, the L^2–norm of F is given by

(D′) $\|F\|^2 = \frac{1}{2} \int_{V(x)<E} (E - V(x))^{-1/2}\, dx \cdot (1 + O(\Lambda^{\varepsilon - 2}))$.

These are our basic results on the eigenvalues and eigenfunctions of ordinary differential operators. Their accuracy is determined by the number Λ in (26). For $V(x)$ as in (1) one checks that $\Lambda \sim \lambda$, while for $V(x)$ as in (20) one gets $\Lambda \sim \ell$. In view of (A′), (B′), the uncertainty in the phase at an eigenvalue and the percentage error in $F(x)$ at a typical point of $[x_{\text{left}}(E), x_{\text{right}}(E)]$ are $O(\Lambda^{\varepsilon-2})$. These errors are $O(\lambda^{-1})$ for the standard WKB approximation applied to potentials (1). Thus (A′) ... (D′) sharpen the standard approximations. We will need both the precision and generality of (A′)... (D′) in order to deal with atoms in our later papers.

We will investigate what happens to (A′)... (D′) when the eigenvalue E gets close to the minimum of the potential. As in standard WKB theory for potentials (1), we find that the accuracy of (B′)... (D′) is degraded for E near (min V), but (A′) retains its full accuracy. Also, we will work out a very crude WKB theory that applies, for instance, to the Hydrogen atom (20) for small ℓ. The reader should consult the main body of this paper for a careful statement of each of our WKB theorems.

Note that (8) differs slightly from the usual Airey function. In our normalization, $A(y)$ satisfies

$$\left(\frac{d^2}{dy^2}+y\right)A(y)=0$$

$$A(y)\sim \mathrm{Re}\Big[e^{-\frac{\pi}{4}i}\,\frac{e^{\frac{2}{3}iy^{3/2}}}{y^{1/4}}(1+\sum_{s=1}^{\infty}c_s y^{-\frac{3}{2}s})\Big] \quad \text{as } y\to+\infty$$

$$A(y)=O(|y|^{-M}), \text{any } M, \quad \text{as } y\to-\infty.$$

See [AS].

We are grateful to Maureen Kirkham for expertly texing our paper.

Let us now begin the work of solving ordinary differential equations.

Approximate Local Solutions of Ordinary Differential Equations

Let $p\in C^\infty[-1,1]$ with $p(0)=0$, $p'(0)>0$. Our goal in this section is to write down in closed form a function $F(x,\lambda)$ defined for $|x|\le c$, $\lambda>C$ that solves the ordinary differential equation

$$\text{(1)}\qquad \frac{\partial^2}{\partial x^2}F(x,\lambda)+\lambda^2 p(x)F(x,\lambda)=0$$

modulo a high power of $1/\lambda$.

Our approximate solution F is patched together from three different approximations $F_-(x,\lambda)$, $F_0(x,\lambda)$, $F_+(x,\lambda)$ defined on the three regions $\mathcal{J}_-=\{x<-\lambda^{\varepsilon-2/3}\}$, $\mathcal{J}_0=\{|x|<\lambda^{-\varepsilon}\}$, $\mathcal{J}_+=\{x>+\lambda^{\varepsilon-2/3}\}$. Here $\varepsilon>0$ is a small constant to be picked (much) later.

In $\mathcal{J}_-$ we simply set $F_-(x,\lambda)=0$.

In $\mathcal{J}_0$ we attempt to transform (1) into Airey's equation

$$\text{(2)}\qquad \frac{\partial^2}{\partial y^2}A(y,\lambda)+\lambda^2 yA(y,\lambda)=0$$

by a change of variable $y=y(x,\lambda)$. The change of variable makes (1) equivalent to (2) via the transformation law $F(x,\lambda)=\left(\frac{\partial y(x,\lambda)}{\partial x}\right)^{-1/2}A(y(x,\lambda),\lambda)$, provided y

satisfies the equation

$$(3) \qquad p(x) = \Big(\frac{\partial y}{\partial x}\Big)^2 y + \lambda^{-2}\{y,x\} \quad \text{with} \quad \{y,x\} = \tfrac{1}{2}\frac{\partial_x^3 y}{\partial_x y} - \tfrac{3}{4}\Big(\frac{\partial_x^2 y}{\partial_x y}\Big)^2.$$

We construct a formal power series y in x and λ^{-2} that solves (3) to infinite order, and then truncate y at degree N to obtain a function $y_N(x,\lambda)$ that solves (3) modulo errors $O(\lambda^{-\varepsilon N})$ in $\mathcal{J}_0$. As our solution of (2) we take $A(y,\lambda) = \lambda^{-1/3}A(\lambda^{2/3}y)$, where $A(t)$ is the Airey function defined in the introduction. Thus our approximate solution of (1) in $\mathcal{J}_0$ is

$$(4) \qquad F_0(x,\lambda) = \lambda^{-1/3}\Big(\frac{\partial y_N}{\partial x}(x,\lambda)\Big)^{-1/2} A(\lambda^{2/3}y_N(x,\lambda)), \quad |x| < \lambda^{-\varepsilon}.$$

We call this the <u>N^{th} Airey approximation</u>. Note that F_- and F_0 differ by $O(\lambda^{-M})$ on $\mathcal{J}_- \cap \mathcal{J}_0$ for any M, since $A(t)$ decreases exponentially as $t \to -\infty$.

In $\mathcal{J}_+ = \{x > \lambda^{\varepsilon - 2/3}\}$, our approximate solution of (1) is

$$(5) \qquad F_+(x,\lambda) = \mathrm{Re}\Bigg[\frac{e^{\pm i\pi/4} e^{i\int_0^x (\lambda^2 p(t))^{1/2}dt}}{(\lambda^2 p(x))^{1/4}}\Big(\sum_{k=0}^{N'} u_k(x)\lambda^{-k}\Big)\Bigg],$$

where $u_0(x), u_1(x), \ldots, u_{N'}(x)$ are defined for $x \in (0,c]$ and satisfy the <u>transport equations</u>:

$$(6) \qquad u_0 \equiv 1$$

(6)

$$2iu'_{k+1} + (\tfrac{5}{16}(p')^2p^{-5/2} - \tfrac{1}{4}p''p^{-3/2})u_k - \tfrac{1}{2}p'p^{-3/2}u'_k + p^{-1/2}u''_k = 0, \quad 0 \le k < N'.$$

If we put (5) into (1) and expand formally in decreasing powers of λ, we get (6), which is the classical motivation for the transport equations. We expect $u_k(x)$ to have a singularity like $x^{-\frac{3}{2}k}$ at $x = 0$. Hence the successive terms $u_k(x)\lambda^{-k}$ decrease rapidly, provided we take $x > \lambda^{\varepsilon - 2/3}$. This is the classical motivation for using (5) only on $\mathcal{J}_+$.

The transport equations (6) let us solve successively for each u_{k+1} in terms of u_k. Each time we do this, we get an arbitrary constant of integration. There is a unique way to pick all these constants of integration in order to make (4) and (5) agree on the overlap $\mathcal{J}_0 \cap \mathcal{J}_+$ modulo a high power of $1/\lambda$. With the integration constants picked in this way, we get the canonical solution $(u_0(x), u_1(x), \ldots, u_{N'}(x))$ to the transport equations.

Thus we produce three excellent approximate solutions of (1) on the regions $\mathcal{J}_-$, $\mathcal{J}_0$, $\mathcal{J}_+$, which agree closely on the overlaps. A partition of unity lets us patch these solutions into a single approximate solution $F(x, \lambda)$ defined in $|x| \leq c$, $\lambda > C$.

It will be important to understand how $F_0(x, \lambda)$ and $F_+(x, \lambda)$ depend on p. To keep track of this, we introduce the following notation. $\mathcal{P}$ denotes the set of polynomials in the derivatives $(p^{(m)}(0))_{m \geq 1}$ and in $p'(0)^{-1/6}$. By c_*, C_*, C'_*, etc. with a subscript $*$, we denote positive constants that depend only on an upper bound for $\|p\|_{C^m[-1,1]}$ for some m, and on a positive lower bound for $p'(0)$. Similarly, $C_*^{\alpha\beta}$ denotes a constant depending only on α, β, upper bounds for some $\|p\|_{C^m[-1,1]}$, and a postive lower bound for $p'(0)$.

We now indicate in more detail how to carry out the program outlined above.

LEMMA 1. *There is one and only one formal power series* $y(x, \lambda) = \sum\limits_{k,\ell \geq 0} f_{k\ell} x^\ell \lambda^{-2k}$ *with* $f_{00} = 0$, $f_{01} > 0$, *and satisfying (3) as formal power series. Moreover, each* $f_{k\ell}$ *belongs to* $\mathcal{P}$. *We have* $f_{01} = (p'(0))^{1/3}$.

Sketch of proof: Set $y_k(x) = \sum\limits_{\ell \geq 0} f_{k\ell} x^\ell$ so that $y(x, \lambda) = \sum\limits_{k \geq 0} y_k(x) \cdot \lambda^{-2k}$, and define $z_k(x)$ by setting $(\partial_x y)^2 y + \lambda^{-2}\{y, x\} - p(x) = \sum\limits_{k \geq 0} z_k(x) \lambda^{-2k}$. By induction on k we show that y_k can be picked uniquely in terms of $p(x)$ and earlier $y_{k'}$ to make $z_k = 0$. If $k = 0$, this amounts to justifying the obvious solution $y_0 = (\frac{3}{2} \int_0^x p^{1/2}(t) dt)^{2/3}$ in terms of formal power series and checking that $f_{0\ell} \in \mathcal{P}$.

If $k > 0$ then the contribution of $\lambda^{-2}\{y, x\}$ to z_k is determined by $y_0 \ldots y_{k-1}$, and the contribution of $(\partial_x y)^2 y$ to z_k has the form $2y_0 (\frac{\partial y_0}{\partial x}) \frac{\partial y_k}{\partial x} + (\frac{\partial}{\partial x} y_0)^2 y_k +$ [stuff determined by $y_0 \ldots y_{k-1}$]. Hence to make $z_k = 0$ we must solve the regular singular equation $2y_0(\frac{\partial y_0}{\partial x})\frac{\partial y_k}{\partial x} + (\frac{\partial y_0}{\partial x})^2 y_k =$ given formal power series.

To solve this we take $y_k = \sum_{\ell \geq 0} f_{k\ell} x^\ell$ and solve successively for $f_{k0}, f_{k1}, \ldots$. It is again easy to check that each $f_{k\ell} \in \mathcal{P}$. ■

Once we have found $y(x, \lambda)$ by Lemma 1, the truncation $y_N(x, \lambda)$ is defined as

(7) $$y_N(x, \lambda) = \sum_{0 \leq k, \ell \leq N} f_{k\ell} x^\ell \lambda^{-2k}, \text{ with } N >> \tfrac{1}{\varepsilon} \text{ to be picked later.}$$

Thus

(8) $$p(x) - \left(\frac{\partial}{\partial x} y_N(x, \lambda)\right)^2 y_N(x, \lambda) - \lambda^{-2}\{y_N(x, \lambda), x\} \text{ vanishes to order } N - 3 \text{ at } x = 0, \lambda^{-2} = 0.$$

Lemma 2. *For* $|x|,\ \lambda^{-1} \le c_*^N$ *we have* $\frac{\partial y_N}{\partial x} > c_*$, *so the Airey approximation (4) is well-defined. Moreover,* $|\frac{\partial^2}{\partial x^2} F_0(x,\lambda) + \lambda^2 p(x) F_0(x,\lambda)| \le C_*^N\, \lambda^{-\frac{\varepsilon N}{2}}$ *for* $|x| < \lambda^{-\varepsilon}$ *and* $\lambda \ge C_*^{\varepsilon N}$. *For* $-\lambda^{-\varepsilon} < x < -\lambda^{\varepsilon - 2/3}$ *and* $\lambda \ge C_*^{\varepsilon N}$ *we have* $|(\frac{\partial}{\partial x})^\alpha F_0(x,\lambda)| \le c_*^{\varepsilon N} \lambda^{-\frac{\varepsilon N}{2}}$ *for* $\alpha = 0, 1, 2$.

Sketch of proof: Airey's equation (2) holds exactly for $A(y,\lambda) = \lambda^{-1/3} A(\lambda^{2/3} y)$, hence $F_0(x,\lambda)$ satisfies $\frac{\partial^2 F_0}{\partial x^2} + \lambda^2 \tilde{p}(x,\lambda) F_0 = 0$ with

$$\tilde{p}(x,\lambda) = \Big(\frac{\partial y_N}{\partial x}\Big)^2 y_N + \lambda^{-2}\{y_N, x\}.$$

Thus $\frac{\partial^2 F_0}{\partial x^2} + \lambda^2 p(x) F_0 = \lambda^2 (p(x) - \tilde{p}(x,\lambda)) F_0$.

Regard (x, λ^{-2}) as the independent variables in y_N, F_0. Then $p(x) - \tilde{p}(x,\lambda)$ vanishes to order $N-3$ at the origin, and $|\partial_x^\alpha \partial_{\lambda^{-2}}^\beta [p(x) - \tilde{p}(x,\lambda)]| \le C_*^{\alpha\beta N}$ for $|x|$, $|\lambda^{-2}| \le c_*^N$. Taylor's theorem with remainder yields $|p(x) - \tilde{p}(x,\lambda)| \le C_*^N (|x| + \lambda^{-2})^{N-3}$ for $|x|,\ \lambda^{-2} \le c_*^N$, hence

(8bis) $$|p(x) - \tilde{p}(x,\lambda)| \le C_*^N \lambda^{-\varepsilon(N-3)} \text{ if } \quad |x| < \lambda^{-\varepsilon} \quad \text{and} \quad \lambda \ge C_*^N.$$

Since the Airey function $A(t)$ is bounded, a look at (4) gives $|F_0| \le C_*^N \lambda^{-1/3}$, so $|\lambda^2 (p(x) - \tilde{p}(x,\lambda)) F_0| \le C_*^N\, \lambda^{2 - \frac{1}{3} - \varepsilon(N-3)}$ for $|x| < \lambda^{-\varepsilon}$, $\lambda \ge C_*^N$. ■

Thus, F_0 is an excellent approximate solution of the basic ODE (1) in $\mathcal{J}_0$, and F_0 agrees very well with $F_- \equiv 0$ on $\mathcal{J}_0 \cap \mathcal{J}_-$.

We prepare to construct a solution (5) that agrees very well with F_0 in $\mathcal{J}_0 \cap \mathcal{J}_+$. Clearly the first task is to understand how F_0 behaves in $\mathcal{J}_0 \cap \mathcal{J}_+$.

Lemma 3. *For* $\lambda^{\varepsilon - 2/3} < x < \lambda^{-\varepsilon}$ *and* $\lambda > C_*^{\varepsilon N}$ *we have*

$$F_0(x,\lambda) = Re\Big[\frac{e^{\pm i\pi/4} e^{i\int_0^x (\lambda^2 p(t))^{1/2} dt}}{(\lambda^2 p(x))^{1/4}} \{1 + \sum_{k=1}^N \lambda^{-k} w_k(x)\}\Big] + \text{ Error}(x,\lambda)$$

where $|(\frac{\partial}{\partial x})^\alpha\ \text{Error}(x,\lambda)| \le C_*^{\varepsilon N} \lambda^{-\frac{\varepsilon N}{10}}$ $(\alpha = 0,1,2)$ *and* $w_k(x) = \sum_{-3k \le \ell \le M_k} w_{k\ell} x^{\ell/2}$, $w_{k\ell} \in \mathcal{P}$. *We have* $w_{10} = 0$.

Sketch of proof: We use the asymptotic form of the Airey function:

$$A(t) = \mathrm{Re}\Big[\frac{e^{\pm i\pi/4}\, e^{\frac{2}{3}it^{3/2}}}{t^{1/4}}(1+\sum_{s=1}^{M} c_s t^{-\frac{3}{2}s})\Big] + \mathrm{Error}(t)$$

with $|\partial_t^\alpha \mathrm{Error}(t)| \leq Ct^{-M}$ for $t > 1$ and $\alpha = 0, 1, 2$. Put $t = \lambda^{2/3} y_N(x,\lambda)$ and substitute the result in the definition (4) of F_0 to obtain the following.

(9)

$$F_0(x,\lambda) = \lambda^{-1/3}\Big(\frac{\partial}{\partial x} y_N(x,\lambda)\Big)^{-1/2}\Big(\lambda^{2/3} y_N(x,\lambda)\Big)^{-1/4} \mathrm{Re}\Big[e^{\pm i\pi/4} e^{i\lambda\cdot\frac{2}{3}(y_N(x,\lambda))^{3/2}} \cdot \Big(1+\sum_{s=1}^{M} c_s(\lambda^{2/3} y_N(x,\lambda))^{-\frac{3}{2}s}\Big)\Big] + \mathrm{junk}_{10}(x,\lambda).$$

Throughout the proof of Lemma 3, $\mathrm{junk}_s(x,\lambda)$ denotes a function $j(x,\lambda)$ which satisfies

$$|\partial_x^\alpha j(x,\lambda)| \leq C_*^{\varepsilon N}\, \lambda^{-\frac{\varepsilon N}{s}} \quad \text{for} \quad \alpha = 0,1,2 \quad \text{and} \quad \lambda^{\varepsilon - 2/3} < x < \lambda^{-\varepsilon}, \lambda \geq C_*^{\varepsilon N}.$$

(In particular $\mathrm{Error}(\lambda^{2/3} y_N(x,\lambda))$ contributes the term $\mathrm{junk}_{10}(x,\lambda)$ in (9), provided we take M large enough.)

To understand how F_0 behaves by using (9), we must understand the three expressions

$$\exp(\tfrac{2}{3} i\lambda(y_N(x,\lambda))^{3/2}), (\tfrac{\partial}{\partial x} y_N(x,\lambda))^{-1/2}(y_N(x,\lambda))^{-1/4}, \text{ and } (y_N(x,\lambda))^{-\frac{3}{2}s}.$$

We begin with the exponential.

Recall that $y_N(x,\lambda) = \sum_{k,\ell=0}^{N} f_{k\ell} x^\ell \lambda^{-2k}$ with $f_{k\ell} \in \mathcal{P}$, and that $y_0(x) = \sum_{\ell=0}^{N} f_{0\ell} x^\ell$ satisfies $(y_0')^2 y_0 = p(x)$ to order $\geq N-1$. It follows that

$$\frac{2}{3} y_0^{3/2} = \int_0^x (p(t))^{1/2} dt + \mathrm{junk}_2(x,\lambda). \tag{10}$$

(An error of order $\geq N-1$ is small since we are looking at the region $|x| \leq \lambda^{-\varepsilon}$.)

Also,

$$(11)\quad (y_N(x,\lambda))^{3/2} = (y_0(x))^{3/2}\cdot\Big(1+\Big\{(\frac{\lambda^{-2}}{x})(\frac{x}{y_0(x)})(\sum_{k=1}^{N}\sum_{\ell=0}^{N} f_{k\ell}x^\ell\lambda^{-2(k-1)})\Big\}\Big)^{3/2}$$

Regarding the factors in curly brackets, we note the following:

$$\frac{x}{y_0(x)} = (p'(0))^{-1/3}\cdot\Big(1+\Big[\sum_{\ell=1}^{N}(p'(0))^{-1/3}f_{0\ell}x^\ell\Big]\Big)^{-1}.$$

Expanding the factor $(1+[\dots])^{-1}$ in a high-order Taylor series with remainder, we find that

$$(12)\qquad \frac{x}{y_0(x)} = \sum_{\ell=0}^{N} f_\ell^{\#}x^\ell + \mathrm{junk}_2(x,\lambda)\quad\text{with}\quad f_\ell^{\#}\in\mathcal{P}.$$

A high enough power of $\frac{\lambda^{-2}}{x}$ has the form $\mathrm{junk}_2(x,\lambda)$, since we are working in the region $\lambda^{\varepsilon-2/3} < x < \lambda^{-\varepsilon}$. Hence substituting (12) in (11) and then expanding $(1+\{\dots\})^{3/2}$ in a high-order Taylor series with remainder, we get

$$(y_N(x,\lambda))^{3/2} = (y_0(x))^{3/2}\cdot\Big(1+\sum_{k=1}^{N}\lambda^{-2k}\Big\{\sum_{\ell=-k}^{M} f_{k\ell}^{\#\#}x^\ell\Big\}+\mathrm{junk}_2(x,\lambda)\Big)$$
$$\text{with } f_{k\ell}^{\#\#}\in\mathcal{P}.$$

Putting in (10), we conclude that

$$(13)\quad \tfrac{2}{3}\lambda(y_N(x,\lambda))^{3/2} = \int_0^x(\lambda^2p(t))^{1/2}dt + \tfrac{2}{3}\lambda(y_0(x))^{3/2}\sum_{k=1}^{N}\sum_{\ell=-k}^{N} f_{k\ell}^{\#\#}x^\ell\lambda^{-2k}$$
$$+\,\mathrm{junk}_3(x,\lambda).$$

The proof of (12) shows also† that $(\frac{y_0(x)}{x})^{3/2} = \sum_{\ell=0}^{N}\tilde f_\ell^{\#}x^\ell + \mathrm{junk}_2(x,\lambda)$ with $\tilde f_\ell^{\#}\in\mathcal{P}$. Writing the factor $(y_0(x))^{3/2}$ in (13) as $(\frac{y_0(x)}{x})^{3/2}\cdot x^{3/2}$ and putting in our result for $(\frac{y_0(x)}{x})^{3/2}$, we obtain from (13) the following:

$$(14)\quad \tfrac{2}{3}i\lambda(y_N(x,\lambda))^{3/2} = i\int_0^x(\lambda^2p(t))^{1/2}dt + \sum_{k=1}^{N}\sum_{\ell=-k}^{N} h_{k\ell}x^{(\ell+\frac{3}{2})}\lambda^{-(2k-1)} +$$
$$+\ \mathrm{junk}_4(x,\lambda),\ h_{k\ell}\in\mathcal{P}.$$

†Note that the factor $(p'(0))^{-1/3}$ in the proof of (13) becomes now $(p'(0))^{1/2}$. This still belongs to $\mathcal{P}$, since we carefully included $(p'(0))^{-1/6}$ in the definition of $\mathcal{P}$.

We prepare to exponentiate both sides of (14). First of all, $\exp(\text{junk}_4(x,\lambda)) = 1+\text{junk}_4(x,\lambda)$. Secondly, set $X = \sum_{k=1}^{N}\sum_{\ell=-k}^{N} h_{k\ell}x^{(\ell+3/2)}\lambda^{-(2k-1)}$. A high enough power of X will be of the form $\text{junk}_4(x,\lambda)$, so we can evaluate $\exp(X)$ using a high-order Taylor series with remainder. If $s \geq 2$ then X^s contains no terms of the form $x^a\lambda^{-1}$; while for $s=1$, $X^s = X$ contains terms of the form $x^a\lambda^{-1}$ only for non-integer a. Consequently, $\sum_{s=1}^{M}\frac{X^s}{s!}$ contains no terms of the form $x^a\lambda^{-1}$ with a an integer. Moreover, a monomial $x^{(\ell+3/2)}\lambda^{-(2k-1)}$ with $1\leq k\leq N$, $-\ell \leq k \leq N$ (i.e. the sort that appears in X) has the form

$$x^{\ell'/2}\lambda^{-k'} \text{ with } \quad 1\leq k' \leq 2N,\ 2-k'\leq \ell' \leq 2N+3.$$

Hence $\sum_{s=1}^{M}\frac{X^s}{s!} = \sum_{k'=1}^{\overline{M}}\sum_{\ell'=2-k'}^{\overline{M}} h^{\#}_{k'\ell'}x^{\ell'/2}\lambda^{-k'}, h^{\#}_{k'\ell'}\in\mathcal{P}$, for some large $\overline{M}$.

Putting together the conclusions of the preceding paragraph, we obtain

$$\exp\Big(\sum_{k=1}^{N}\sum_{\ell=-k}^{N} h_{k\ell}x^{(\ell+\frac{3}{2})}\lambda^{-(2k-1)}\Big) = \Big(1+\sum_{k=1}^{2N}\sum_{\ell=2-k}^{3N} h^{\#}_{k\ell}x^{\ell/2}\lambda^{-k} + \text{junk}_4(x,\lambda)\Big)$$

with $h^{\#}_{k\ell}\in\mathcal{P}$, and with $h^{\#}_{1\ell}=0$ for ℓ even.

Substituting this into (14) gives our basic result on the exponential term, namely

$$\exp(\tfrac{2}{3}i\lambda(y_N(x,\lambda))^{3/2}) = \exp(i\int_0^x(\lambda^2p(t))^{1/2}dt)\cdot\Big[1+\sum_{k=1}^{2N}\sum_{\ell=2-k}^{3N}h^{\#}_{k\ell}x^{\ell/2}\lambda^{-k} + \text{junk}_4(x,\lambda)\Big] \tag{15}$$

with $h^{\#}_{k\ell}\in\mathcal{P}$ and $h^{\#}_{k\ell}=0$ for $k=1$, ℓ even.

Next we study the expression $(\frac{\partial y_N}{\partial x}(x,\lambda))^{-\frac{1}{2}}(y_N(x,\lambda))^{-\frac{1}{4}}$.

Since $(y_0')^2y_0 = p(x)$ to order $\geq N-1$ and both sides vanish to first order at $x=0$, we have $(y_0')^2y_0 = p(x)\cdot(1+(x^{N-2}g(x))$ with $|\partial_x^\alpha g(x)|\leq C_*^N$, $\alpha=0,1,2$ for $|x|\leq c_*^N$. We know also that $(\frac{\partial}{\partial x}y_N)^2y_N = (y_0')^2y_0 + \sum_{k=1}^{3N}\sum_{\ell=0}^{3N} q_{k\ell}x^\ell\lambda^{-2k}$ with

$q_{k\ell} \in \mathcal{P}$, since $y_N(x,\lambda) = \sum\limits_{k,\ell=0}^{N} f_{k\ell} x^\ell \lambda^{-2k}$. Thus

$$\left(\tfrac{\partial}{\partial x} y_N\right)^2 y_N = (1 + x^{N-2} g(x)) p(x) + \lambda^{-2} \sum_{k=1}^{3N} \sum_{\ell=0}^{3N} q_{k\ell} x^\ell \lambda^{-2(k-1)}$$

(16) $$= p(x) \cdot \Big(1 + \Big[x^{N-2} g(x) + (\frac{\lambda^{-2}}{x})(\frac{x}{p(x)})(\sum_{k=1}^{3N}\sum_{\ell=0}^{3N} q_{k\ell} x^\ell \lambda^{-2(k-1)})\Big]\Big).$$

As a function of (x,λ), we have $x^{N-2} g(x) = \text{junk}_2(x,\lambda)$. (Recall we're working in the region $|x| \le \lambda^{-\varepsilon}$). Also $\frac{x}{p(x)} = (p'(0))^{-1} \cdot \Big(1 + \Big[\sum\limits_{\ell=2}^{M} \frac{p^{(\ell)}(0)}{\ell! p'(0)} x^{\ell-1} + \text{junk}_1(x,\lambda)\Big]\Big)^{-1}$ for large M, so that by expanding $(1+X)^{-1}$ in a high-order Taylor series with remainder, we get

(16bis) $$\frac{x}{p(x)} = \sum_{\ell=0}^{N} \tilde{p}_\ell x^\ell + \text{junk}_2(x,\lambda) \quad \text{with} \quad \tilde{p}_\ell \in \mathcal{P}, \quad \tilde{p}_0 = (p'(0))^{-1}.$$

Hence (16) yields

(17)
$$(\frac{\partial}{\partial x} y_N)^2 y_N = p(x) \cdot (1 + \Big[\frac{\lambda^{-2}}{x} \sum_{k=0}^{4N} \sum_{\ell=0}^{4N} \tilde{q}_{k\ell} x^\ell \lambda^{-2k} + \text{junk}_3(x,\lambda)\Big]\Big) \quad \text{with} \quad \tilde{q}_{k\ell} \in \mathcal{P}.$$

We prepare to raise both sides to the power $-\frac{1}{4}$. Since a high enough power of $\frac{\lambda^{-2}}{x}$ has the form $\text{junk}_1(x,\lambda)$ (recall we're working in the region $\lambda^{\varepsilon - 2/3} < x < \lambda^{-\varepsilon}$), we may expand $(1+X)^{-1/4}$ in a high-order Taylor series with remainder, taking $X =$ the expression in square brackets in (17). The remainder will have the form $\text{junk}_3(x,\lambda)$. Therefore, from (17) we get

(18)
$$\Big(\frac{\partial y_N(x,\lambda)}{\partial x}\Big)^{-\frac{1}{2}} \big(y_N(x,\lambda)\big)^{-1/4} = (p(x))^{-1/4} \cdot \Big(1 + \sum_{k=1}^{N} \sum_{\ell=-k}^{N} q^{\#}_{k\ell} x^\ell \lambda^{-2k} + \text{junk}_4(x,\lambda)\Big)$$
$$\text{with} \quad q^{\#}_{k\ell} \in \mathcal{P}.$$

This is our basic result on $(\frac{\partial}{\partial x} y_N)^{-1/4} y_N^{-1/2}$.

We turn now to $(y_N(x,\lambda))^{-\frac{3}{2}s}$.

¿From (17) we conclude that

(19)
$$\Big(\frac{\partial y_N(x,\lambda)}{\partial x}\Big)^{-3s}\Big(y_N(x,\lambda)\Big)^{-\frac{3}{2}s}=\Big(p(x)\Big)^{-\frac{3}{2}s}\cdot\Big(1+\sum_{k=1}^{N}\sum_{\ell=-k}^{N}q^s_{k\ell}x^\ell\lambda^{-2k}+\text{junk}_4(x,\lambda)\Big)$$
$$\text{with } q^s_{k\ell}\in\mathcal{P}.$$

This follows the same way as (18), except that we raise (17) to the power $-\frac{3}{2}s$ instead of $-\frac{1}{4}$.

Also

$$\lambda^{-s}\Big(p(x)\Big)^{-\frac{3}{2}s}=\lambda^{-s}x^{-\frac{3}{2}s}\Big(\frac{x}{p(x)}\Big)^{\frac{3}{2}s}.$$

Equation (16bis) shows that

$$\Big(\frac{x}{p(x)}\Big)^{\frac{3}{2}s}=\sum_{\ell=0}^{N}p^s_\ell x^\ell+\text{junk}_3(x,\lambda)\quad\text{with}\quad p^s_\ell\in\mathcal{P}.$$

(To derive this we use a high-order Taylor expansion with remainder for $(1+X)^{\frac{3}{2}s}$.) Hence

(20)
$$\lambda^{-s}\Big(p(x)\Big)^{-\frac{3}{2}s}=\lambda^{-s}x^{-\frac{3}{2}s}\Big(\sum_{k=0}^{N}\sum_{\ell=-k}^{N}\hat{q}^s_{k\ell}x^\ell\lambda^{-2k}+\text{junk}_4(x,\lambda)\Big)\quad\text{with}\quad\hat{q}^s_{k\ell}\in\mathcal{P}.$$

Since $y_N(x,\lambda)=\sum_{k,\ell=0}^{N}f_{k\ell}x^\ell\lambda^{-2k}$ we have also

$$(21)\qquad\Big(\frac{\partial y_N}{\partial x}\Big)^{+3s}=\sum_{k,\ell=0}^{3N}\hat{f}^s_{k\ell}x^\ell\lambda^{-2k}\quad\text{exactly,}\quad\hat{f}^s_{k\ell}\in\mathcal{P}.$$

Rewriting (19) as

$$\lambda^{-s}\big(y_N(x,\lambda)\big)^{-\frac{3}{2}s}=\Big(\frac{\partial y_N}{\partial x}\Big)^{+3s}\cdot\lambda^{-s}\big(p(x)\big)^{-\frac{3}{2}s}\cdot\Big(1+\sum_{k=1}^{N}\sum_{\ell=-k}^{N}q^s_{k\ell}x^\ell\lambda^{-2k}$$
$$+\text{junk}_4(x,\lambda)\Big)$$

and substituting (20), (21) into the right-hand side, we obtain

$$\lambda^{-s}(y_N(x,\lambda))^{-\frac{3}{2}s}=\lambda^{-s}x^{-\frac{3}{2}s}\cdot\Big(\sum_{k=0}^{N}\sum_{\ell=-k}^{N}h^s_{k\ell}x^\ell\lambda^{-2k}+\text{junk}_5(x,\lambda)\Big)\quad\text{with}\quad h^s_{k\ell}\in\mathcal{P}.$$

This is our basic result on $(y_N)^{-\frac{3}{2}s}$. It gives at once

$$(22)\quad \Big(1+\sum_{s=1}^{M} c_s(\lambda^{2/3}y_N(x,\lambda))^{-\frac{3}{2}s}\Big) = \Big(1+\sum_{k=1}^{N}\sum_{\ell=-3k}^{N} \hat{h}_{k\ell}x^{\ell/2}\lambda^{-k} + \mathrm{junk}_6(x,\lambda)\Big)$$

with $\hat{h}_{k\ell}\in\mathcal{P}$ and $\hat{h}_{1\ell}=0$ for ℓ even.

¿From (15), (18), (22) we get

$$(23)\quad \frac{e^{\pm i\pi/4}\, e^{\frac{2}{3}i\lambda(y_N(x,\lambda))^{\frac{3}{2}}}}{\lambda^{1/2}(\frac{\partial y_N}{\partial x})^{1/2}y_N^{1/4}}\Big(1+\sum_{s=1}^{M} c_s(\lambda^{2/3}y_N(x,\lambda))^{-\frac{3}{2}s}\Big) =$$

$$\frac{e^{\pm i\pi/4}\, e^{i\int_0^x(\lambda^2p(t))^{1/2}dt}}{(\lambda^2p(x))^{1/4}}\cdot\Bigg[\Big(1+\sum_{k=1}^{2N}\sum_{\ell=2-k}^{3N} h^{\#}_{k\ell}x^{\ell/2}\lambda^{-k}+\mathrm{junk}_4(x,\lambda)\Big)\cdot \Big(1+\sum_{k=1}^{N}\sum_{\ell=-k}^{N} q^{\#}_{k\ell}x^{\ell}\lambda^{-2k}+\mathrm{junk}_4(x,\lambda)\Big)\cdot \Big(1+\sum_{k=1}^{N}\sum_{\ell=-3k}^{N} \hat{h}_{k\ell}x^{\ell/2}\lambda^{-k}+\mathrm{junk}_6(x,\lambda)\Big)\Bigg].$$

The product of the three series has the form $\big(1+\sum_{k=1}^{10N}\sum_{\ell=-3k}^{10N} h^{+}_{k\ell}x^{\ell/2}\lambda^{-k}+\mathrm{junk}_7(x,\lambda)\big)$, with $h^+_{k\ell}\in\mathcal{P}$. Since $h^{\#}_{1\ell}=0$ and $\hat{h}_{1\ell}=0$ for even ℓ, it follows at once that $h^+_{1\ell}=0$ for even ℓ. In particular $h^+_{10}=0$. Now the conclusion of Lemma 3 is immediate from (9) and (23). ■

Lemma 4. *The functions $w_k(x)$ in the statement of the previous lemma satisfy the approximate transport equations*

$$|2iw'_{k+1}(x)+\mathcal{L}w_k(x)|\le C_*^{\varepsilon N}x^{N/32}\quad \text{for}\quad 0<x<c_*^{\varepsilon N}\quad \text{and}\quad 0\le k\le \tfrac{\varepsilon N}{32},$$

where we set $w_0(x)\equiv 1$ and

$$\mathcal{L}w\equiv(\tfrac{5}{16}(p')^2p^{-5/2}-\tfrac{1}{4}p''p^{-3/2})w-\tfrac{1}{2}p'p^{-3/2}w'+p^{-1/2}w''.$$

Sketch of Proof. We start with a seeming digression. The Airey function $A(t)$ may be written as the real part of a complex solution $A_c(t)$ of Airey's equation

$\frac{d^2A_c(t)}{dt^2} + tA_c(t) = 0$, having the asymptotic expansion

$$A_c(t) = \frac{e^{\pm i\pi/4}\, e^{\frac{2}{3}it^{3/2}}}{t^{1/4}}\Big(1 + \sum_{s=1}^{M} c_s t^{-\frac{3}{2}s}\Big) + \text{ Error}_M(t)$$

with

$$\Big|(\frac{d}{dt})^\alpha \text{Error}_M(t)\Big| \le C_M\, t^{-M} \quad \text{for} \quad \alpha = 0,1,2 \quad \text{and} \quad t > 10.$$

Whereas $A(t)$ is bounded on the whole real line $A_c(t)$ is bounded on $[-1,\infty)$ but grows rapidly as $t \to -\infty$. Using $A_c(t)$ in place of $A(t)$, we define a complex analogue of $F_0(x,\lambda)$, namely

$$F_c(x,\lambda) = \lambda^{-1/3}\Big(\frac{\partial y_N(x,\lambda)}{\partial x}\Big)^{-1/2} A_c(\lambda^{2/3} y_N(x,\lambda)).$$

Thus $F_0(x,\lambda) = \text{Re}[F_c(x,\lambda)]$. Note that $A_c(\lambda^{2/3}y_N(x,\lambda))$ remains bounded by $C_*^{\varepsilon N}$ when $0 < x < c_*^{\varepsilon N}$ and $\lambda > C_*^{\varepsilon N}$. To see this, we write $y_N(x,\lambda) = y_0(x) + O(\lambda^{-2})$ with $y_0(x) > 0$ for $0 < x < c_*^{\varepsilon N}$. Hence $\lambda^{2/3}y_N(x,\lambda) \ge -1$ and $A_c(\lambda^{2/3}y_N(x,\lambda))$ remains bounded as claimed.

Now we can repeat the proofs of Lemmas 2 and 3 above to derive their complex analogues:

$$\Big|\frac{\partial^2}{\partial x^2} F_c(x,\lambda) + \lambda^2 p(x) F_c(x,\lambda)\Big| \le C_*^{\varepsilon N}\lambda^{-\frac{\varepsilon N}{2}} \quad \text{for} \quad 0 < x < \lambda^{-\varepsilon}, \lambda \ge C_*^{\varepsilon N}. \tag{24}$$

(25)

$$F_c(x,\lambda) = \frac{e^{\pm i\pi/4}\, e^{i\int_0^x (\lambda^2 p(t))^{1/2} dt}}{(\lambda^2 p(x))^{1/4}}\Big(1 + \sum_{k=1}^{N} \lambda^{-k} w_k(x)\Big) + \text{Error}(x,\lambda) \quad \text{with}$$

$$\big|\partial_x^\alpha \text{Error}(x,\lambda)\big| \le C_*^{\varepsilon N}\lambda^{-\frac{\varepsilon N}{10}} \quad \text{for } \lambda^{\varepsilon - 2/3} < x < \lambda^{-\varepsilon},\ \lambda \ge C_*^{\varepsilon N},\ \alpha = 0,1,2.$$

The proof of Lemma 2 used the boundedness of $A(\lambda^{2/3}y_N(x,\lambda))$, which we have just verified for A_c when $x > 0$. Otherwise, we can repeat the proofs of Lemmas 2 and 3 virtually word for word.

Now we return to the proof of Lemma 4. Setting $w_0(x) \equiv 1$ and $w_{N+1} \equiv 0$ we obtain from (25) by elementary calculation the formula:

$$\left(\tfrac{\partial^2}{\partial x^2} + \lambda^2 p(x)\right) F_c(x,\lambda) = \frac{e^{\pm i\pi/4}\, e^{i\int_0^x (\lambda^2 p(t))^{1/2}dt}}{(\lambda^2 p(x))^{1/4}} \sum_{k=0}^{N} \lambda^{-k}\{2iw'_{k+1}(x) + \mathcal{L}w_k(x)\} + \left(\tfrac{\partial^2}{\partial x^2} + \lambda^2 p(x)\right)\mathrm{Error}(x,\lambda),$$

for $\lambda^{\varepsilon-2/3} < x < \lambda^{-\varepsilon}, \lambda \geq C_*^{\varepsilon N}$.

The last expression on the right-hand side is dominated by $C_*^{\varepsilon N}\lambda^{-\frac{\varepsilon N}{2}}$, by virtue of (24). Therefore,

$$\left|\sum_{k=0}^{N} \lambda^{-k}\{2i\, w'_{k+1}(x) + \mathcal{L}w_k(x)\}\right| \leq C_*^{\varepsilon N}\lambda^{-\frac{\varepsilon N}{15}}(\lambda^2 p(x))^{1/4} \leq C_*^{\varepsilon N}\lambda^{-\frac{\varepsilon N}{16}} \tag{26}$$

for $\lambda^{\varepsilon-2/3} < x < \lambda^{-\varepsilon}$, $\lambda \geq C_*^{\varepsilon N}$.

Had we not taken the trouble to remove the real part from the statement of Lemma 3 by passing to $F_c(x,\lambda)$, the derivation of (26) would have been more difficult. For $0 < x < c_*^{\varepsilon N}$ and $\tilde{\lambda} \in [\frac{1}{2}, 1]$, set $\lambda = \tilde{\lambda}x^{-\frac{1}{\varepsilon}}$. Thus $\lambda^{\varepsilon-2/3} < x < \lambda^{-\varepsilon}$ and $\lambda \geq C_*^{\varepsilon N}$, so (26) holds. Since $\lambda = \tilde{\lambda}x^{-\frac{1}{\varepsilon}}$ with $\tilde{\lambda} \in [\frac{1}{2}, 1]$, (26) becomes

$$\left|\sum_{k=0}^{N} \tilde{\lambda}^{-k}\left[x^{k/\varepsilon}\{2iw'_{k+1}(x) + \mathcal{L}w_k(x)\}\right]\right| \leq C_*^{\varepsilon N}x^{\frac{N}{16}}. \tag{27}$$

Fix $x \in (0, c_*^{\varepsilon N}]$. Since (27) holds for all $\tilde{\lambda} \in [\frac{1}{2}, 1]$, it follows that $|x^{k/\varepsilon}\{2iw'_{k+1}(x) + \mathcal{L}w_k(x)\}| \leq C_*^{\varepsilon N}x^{\frac{N}{16}}$ for $0 \leq k \leq N$. The conclusion of Lemma 4 is now obvious. ■

Clearly Lemmas 3 and 4 put us in position to pick an exact solution $(u_0(x), u_1(x), \ldots, u'_N(x))$ of the transport equations to match $F_0(x,\lambda)$ with $F_+(x,\lambda)$ defined by (5). Put $N' = [N/100]$.

Lemma 5. *There is a solution $(u_0(x), u_1(x), \ldots, u_{N'}(x))$ of the transport equations, defined for $x \in (0, c_*^{\varepsilon N})$, with*

$$\left|\left(\tfrac{d}{dx}\right)^{\alpha}\{u_k(x) - w_k(x)\}\right| \leq C_*^{\varepsilon N}x^{N/100} \quad \textit{for} \quad 0 \leq \alpha \leq N/100,\ x \in (0, c_*^{\varepsilon N}].$$

With $F_+(x,\lambda)$ defined by (5) in terms of this solution, we have

$$\left|\left(\tfrac{d}{dx}\right)^\alpha \{F_0(x,\lambda) - F_+(x,\lambda)\}\right| \le C_*^{\varepsilon N} \lambda^{-\varepsilon N/200} \quad \textit{for} \quad \lambda^{\varepsilon - 2/3} < x < \lambda^{-\varepsilon},$$
$$\lambda \ge C_*^{\varepsilon N}, \; \alpha = 0, 1, 2.$$

Proof. With $\mathcal{L}$ as in Lemma 4, we start by studying $v_k = 2i\, w'_{k+1} + \mathcal{L} w_k$.

For M as large as we please, we have

$$p(x) = p'(0)x \cdot \left(1 + \sum_{\ell=2}^{M} \frac{p^{(\ell)}(0)}{\ell! p'(0)} x^{\ell-1} + x^{M+1} g_M(x)\right)$$

with $|\partial_x^\alpha g_M(x)| \le C_*^{M\alpha}$ for $|x| \le 1$. We can raise this equation to the powers $-5/2$, $-3/2$, $-1/2$ using a high-order Taylor series with remainder for $(1+X)^{-m/2}$. As a result, the coefficients of $\mathcal{L}$ all have the form $x^{-5/2}(\sum_{\ell=0}^{M} e_\ell x^\ell + x^{M+1} e_M^{\#}(x))$ with $e_\ell \in \mathcal{P}$ and $|\partial_x^\alpha e_M^{\#}(x)| \le C_*^{M\alpha}$ for $|x| \le c_*^M$, all α. The functions $w_k(x)$, to which we apply $\mathcal{L}$, have the form $\sum_{\ell=-3k}^{N} w_{k\ell} x^{\ell/2}$ with $w_{k\ell} \in \mathcal{P}$. Therefore,

$$v_k(x) = \sum_{\ell=-3k-100}^{N+10M} v_{k\ell} x^{\ell/2} + x^{M-3k-100}\left[g_{k1}^M(x) x^{1/2} + g_{k0}^M(x)\right] \tag{28}$$

with $v_{k\ell} \in \mathcal{P}$ and $|\partial_x^\alpha g_{k0}^M|$, $|\partial_x^\alpha g_{k1}^M| \le C_*^{M\alpha k}$ for $|x| \le c_*^M$.

Lemma 4 shows that $v_{k\ell} = 0$ for $k \le \varepsilon N/32$, $\ell < N/32$ (provided we take $M = 1000N$, which we now do.) Hence (28) shows that $v_k(x)$ may be written as

(28bis)

$$v_k(x) = x^{\overline{N}} f_k(x^{1/2}) \text{ with } \overline{N} = [\tfrac{N}{33}] \text{ and } |\partial_y^\alpha f_k(y)| \le C_*^{\alpha N} \text{ for } |y| \le c_*^N, \text{ all } \alpha,$$

for $0 \le k \le \varepsilon N/32$.

Now we define $u_k(x) = w_k(x) + U_k(x)$ with U_k picked to make u_k satisfy the transport equations. That is, $U_0 \equiv 0$, and

$$2iU'_{k+1} = -\mathcal{L} U_k + v_k. \tag{29}$$

By induction on k we check that $U_0, U_1, \dots, U_k$ can be picked to satisfy (29), with U_k of the form

$$U_k(x) = x^{\overline{N}-5k} V_k(x^{1/2}) \quad \text{with} \quad |\partial_y^\alpha V_k(y)| \le C_*^{\alpha k N} \quad \text{for} \quad |y| \le c_*^N, \ \text{all } \alpha.$$

In fact, for $k = 0$ it's clear. To pass from k to $k+1$, we use inductive hypothesis and (28bis) to write (29) as

$$2i \tfrac{d}{dx} U_{k+1}(x) = -\mathcal{L}(x^{\overline{N}-5k} V_k(x^{1/2})) + x^{\overline{N}} f_k(x^{1/2}). \tag{30}$$

The right-hand side may be written as $x^{\overline{N}-5(k+1)} h_k(x^{1/2})$ with $|\partial_y^\alpha h_k(y)| \le C_*^{\alpha k N}$ for $|y| \le c_*^N$, all α. Putting $U_{k+1} = x^{\overline{N}-5(k+1)} V_{k+1}(x^{1/2})$ for unknown V_{k+1}, we may therefore rewrite (30) in terms of $y = x^{1/2}$ as

$$y^{-1} \tfrac{d}{dy} \Big(y^{2\overline{N}-10(k+1)} V_{k+1}(y) \Big) = (\text{const.}) y^{2\overline{N}-10(k+1)} h_k(y)$$

This has a solution V_{k+1} with $|\partial_y^\alpha V_{k+1}(y)| \le C_*^{\alpha k N}$, completing the inductive transition.

Our solution $(u_0(x), u_1(x), \dots u_{N'}(x))$ of the transport equations has $u_k(x) - w_k(x) = x^{\overline{N}-5k} V_k(x^{1/2})$ with $|\partial_y^\alpha V_k(y)| \le C_*^{\alpha k N}$ for $|y| \le c_*^N$, all α. Therefore by induction on $\alpha = 0, 1, \dots, \overline{N} - 5k$ we have

$$\partial_x^\alpha \{u_k(x) - w_k(x)\} = x^{\overline{N}-5k-\alpha} V_{k\alpha}(x^{1/2}) \quad \text{with} \quad |\partial_y^\beta V_{k\alpha}(y)| \le C_*^{\alpha \beta k N}$$
$$\text{for} \quad |y| \le c_*^N, \quad \text{all} \quad \beta.$$

In particular $|\partial_x^\alpha \{u_k(x) - w_k(x)\}| \le C_*^N x^{N/100}$ for $\alpha \le N/100$, $k \le N/100$, which is one of the conclusions of the Lemma.

By the defining equation (5) for $F_+(x, \lambda)$, and by Lemma 3 for $F_0(x, \lambda)$ we have

$$F_0(x,\lambda) - F_+(x,\lambda) = \text{Re}\Big[\frac{e^{\pm i\pi/4} \, e^{i \int_0^x (\lambda^2 p(t))^{1/2} dt}}{(\lambda^2 p(x))^{1/4}} \Big(\sum_{k=1}^{N'} \lambda^{-k} (w_k(x) - u_k(x))$$
$$+ \sum_{k=N'+1}^{N} \lambda^{-k} w_k(x) \Big) \Big] + \text{Error}(x, \lambda)$$

for $\lambda^{\varepsilon-2/3} < x < \lambda^{-\varepsilon}$, $\lambda \geq C_*^{\varepsilon N}$.

Hence for $\alpha = 0, 1, 2$ we get

$$(31)\quad |\partial_x^\alpha \{F(x,\lambda) - F_+(x,\lambda)\}| \leq \frac{C_*^{\varepsilon N}\lambda^{10}}{(p(x))^{10}}\Big\{\sum_{k=1}^{N'}\sum_{\beta=0}^{2} |\partial_x^\beta(w_k(x) - u_k(x))|\lambda^{-k} + \sum_{k=N'+1}^{N} \lambda^{-k}|\partial_x^\beta w_k(x)|\Big\} + |\partial_x^\alpha \operatorname{Error}(x,\lambda)|.$$

The last term on the right is at most $C_*^{-\varepsilon N}\lambda^{-\varepsilon N/100}$ by Lemma 3, and the double sum in braces is dominated by

$$C_*^N x^{N/100} \leq C_*^{\varepsilon N}\lambda^{-\varepsilon N/100} \qquad \text{for} \quad \lambda^{\varepsilon-2/3} < x < \lambda^{-\varepsilon}.$$

The single sum in braces is dominated by

$$\sum_{k=N'+1}^{N} \lambda^{-k}(C_*^{kN}x^{-\frac{3}{2}k})x^{-\beta} \qquad \text{since} \qquad w_k(x) = \sum_{\ell=-3k}^{N} w_{k\ell}x^{\ell/2} \quad \text{with} \quad w_{k\ell} \in \mathcal{P}.$$

Since $\lambda^{-2/3+\varepsilon} < x < \lambda^{-\varepsilon}$, this sum is in turn dominated by $C_*^{N\varepsilon}(\lambda^{-\varepsilon})^{N'}$. Hence (31) yields

$$|\partial_x^\alpha\{F_0(x,\lambda) - F_+(x,\lambda)\}| \leq \frac{C_*^{N\varepsilon}\lambda^{10}}{(p(x))^{10}} \cdot \lambda^{-\varepsilon N/100} \quad \text{for} \quad \lambda^{\varepsilon-2/3} < x < \lambda^{-\varepsilon}, \quad \lambda \geq C_*^{\varepsilon N}, \alpha = 0, 1, 2.$$

Since also $(p(x))^{-10} \leq C_*^{\varepsilon N}\lambda^{\frac{20}{3}}$ for $\lambda^{\varepsilon-2/3} < x < \lambda^{-\varepsilon}$, the final conclusion of Lemma 5 is obvious. ■

Corollary. *The solution of the transport equations constructed in Lemma 5 may be written as* $u_k(x) = x^{-\frac{3}{2}k}f_k(x^{1/2})$ *with* $f_k(y)$ *smooth in a neighborhood of the origin.*

Proof. We defined $u_k(x) = w_k(x) + x^{\overline{N}-5k}V_k(x^{1/2})$ with $V_k(y)$ smooth in a neighborhood of the origin. Since $w_k(x) = \sum_{\ell=-3k}^{N} w_{k\ell}x^{\ell/2}$, the corollary is obvious. ■

Next we understand how a general solution of the transport equations depends on arbitrary constants of integration.

Lemma 6. *Let* $(u_0(x), u_1(x), \dots, u_{N'}(x))$ *be a solution of the transport equations. Then the general solution* $(\tilde{u}_0(x), \tilde{u}_1(x), \dots, \tilde{u}'_N(x))$ *of the transport equations is given by* $\tilde{u}_k(x) = \sum_{\ell=0}^{k} c_\ell u_{k-\ell}(x)$, *where* $c_1, \dots c_{N'}$ *are arbitrary constants and* $c_0 = 1$. *The constants* (c_k) *are uniquely determined by the* (u_k) *and* $(\tilde{u}_k)$.

Proof. Trivial induction on N'. ■

Corollary. *Every solution of the transport equations may be written as* $u_k(x) = x^{-\frac{3}{2}k} f_k(x^{1/2})$ *with* $f_k(y)$ *smooth in a neighborhood of the origin.*

Proof. Immediate from Lemma 6 and the Corollary to Lemma 5. ■

This corollary is not immediately clear by inspection of the transport equations, since successive integrations of $x^{-\text{power}}$ singularities might give rise to logarithmic terms.

Lemma 5 constructed a solution of the transport equations to match $F_0(x, \lambda)$, but we haven't yet checked the uniqueness of $(u_0, u_1, \dots, u_N)$. This is taken care of in the next result.

Lemma 7. *Let* $(\tilde{u}_0(x), \tilde{u}_1(x), \dots, \tilde{u}_{N'}(x))$ *be a solution of the transport equations, for which* $F_0(x, \lambda)$ *defined by (4) matches* $F_+(x, \lambda)$ *defined by (5) in the sense that* $|F_0(x, \lambda) - F_+(x, \lambda)| \leq \tilde{C}\lambda^{-\varepsilon N/200}$ *for* $\lambda^{\varepsilon - 2/3} < x < \lambda^{-\varepsilon}$, $\lambda \geq \tilde{C}$. *(Here* $\tilde{C}$ *denotes a positive number independent of* x *and* λ*.) Then for* $0 \leq k \leq \varepsilon N/300$, *the function* $\tilde{u}_k$ *is exactly the same as the function* u_k *constructed in Lemma 5.*

Proof. Let $\tilde{C}$ denote any positive number independent of x and λ. By Lemma 6 we have $\tilde{u}_k = \sum_{\ell=0}^{k} c_\ell u_{k-\ell}$, where $c_0 \dots c_{N'}$ are constants with $c_0 = 1$, and (u_k) are the functions constructed in Lemma 5. We want to show that $c_k = 0$ for

$1 \le k \le \varepsilon N/300$, for then $\tilde{u}_k = u_k$ in that range. We assume $c_{k_0} \neq 0$ for some k_0 between 1 and $\varepsilon N/300$ and deduce a contradiction. Take k_0 to be as small as possible. Thus $\tilde{u}_k - u_k = \sum_{\ell=k_0}^{k} c_\ell u_{k-\ell}$. By the Corollary to Lemma 5, we have therefore

$$\tilde{u}_k - u_k = 0 \quad \text{for} \quad k < k_0,\ \tilde{u}_{k_0} - u_{k_0} = c_{k_0},\ |\tilde{u}_k(x) - u_k(x)| \le \tilde{C} x^{-\frac{3}{2}(k-k_0)} \qquad \text{for} \quad k_0 < k \le N'.$$

Hence

$$\Big|\sum_{k=0}^{N'} \lambda^{-k}(\tilde{u}_k(x) - u_k(x)) - \lambda^{-k_0} c_{k_0}\Big| \le \sum_{k=k_0+1}^{N'} \lambda^{-k} \cdot \tilde{C}\, x^{-\frac{3}{2}(k-k_0)} \le \tilde{C}\, \lambda^{-k_0 - \frac{3}{2}\varepsilon}$$

for $\lambda^{\varepsilon - 2/3} < x < \lambda^{-\varepsilon}$, and therefore

(32)

$$\left|\mathrm{Re}\Big[\frac{e^{\pm i\pi/4}\, e^{i\int_0^x (\lambda^2 p(t))^{1/2} dt}}{(\lambda^2 p(x))^{1/4}} \Big\{\sum_{k=0}^{N'} \lambda^{-k}(\tilde{u}_k(x) - u_k(x)) - \lambda^{-k_0} c_{k_0}\Big\}\Big]\right| \le \frac{\tilde{C}\, \lambda^{-k_0 - \frac{3}{2}\varepsilon}}{(\lambda^2 p(x))^{1/4}} \qquad \text{for } \lambda^{\varepsilon - 2/3} < x < \lambda^{-\varepsilon}.$$

On the other hand, since $F_+(x, \lambda)$ matches $F_0(x, \lambda)$ as in the hypothesis of the Lemma, both for $(u_k(x))$ and for $(\tilde{u}_k(x))$, we have

$$\left|\mathrm{Re}\Big[\frac{e^{\pm i\pi/4}\, e^{i\int_0^x (\lambda^2 p(t))^{3/2}] dt}}{\lambda^2 p(x))^{1/4}} \Big(\sum_{k=0}^{N'} \lambda^{-k}(\tilde{u}_k(x) - u_k(x))\Big)\Big]\right| \le \tilde{C}\lambda^{-\frac{\varepsilon N}{200}} \qquad \text{for} \quad \lambda^{\varepsilon - 2/3} < x < \lambda^{-\varepsilon}, \lambda \ge \tilde{C}.$$

Combining this with (32), we obtain the following

$$\text{(33)} \quad \left|\mathrm{Re}\Big[\frac{e^{\pm i\pi/4}\, e^{i\int_0^x (\lambda^2 p(t))^{1/2} dt}}{(\lambda^2 p(x))^{1/4}} \cdot \lambda^{-k_0} c_{k_0}\Big]\right| \le \tilde{C}\lambda^{-\frac{\varepsilon N}{200}} + \frac{\tilde{C}\lambda^{-k_0 - \frac{3}{2}\varepsilon}}{(\lambda^2 p(x))^{1/4}} \qquad \text{for} \quad \lambda^{\varepsilon - 2/3} < x < \lambda^{-\varepsilon}, \lambda \ge \tilde{C}.$$

For $\lambda^{\varepsilon-2/3} < x < \lambda^{-\varepsilon}$ we have $(\lambda^2 p(x))^{-\frac{1}{4}} \ge \tilde{c}(\lambda^2 \cdot \lambda^{-\varepsilon})^{-\frac{1}{4}} \ge \tilde{c}\lambda^{-10}$. Since $k_0 \le \frac{\varepsilon N}{300}$, it follows that the second term on the right-hand side of (33) dominates the first term on the right. Thus,

$$(34)\quad \left|\mathrm{Re}\Big[\frac{e^{\pm i\pi/4}\, e^{\lambda_i \int_0^x (p(t))^{1/2}dt}}{(\lambda^2 p(x))^{1/4}}\cdot \lambda^{-k_0} c_{k_0}\Big]\right| \le \frac{\tilde{C}\lambda^{-k_0-\frac{3}{2}\varepsilon}}{(\lambda^2 p(x))^{1/4}}$$

$$\text{for}\quad \lambda^{\varepsilon-2/3} < x < \lambda^{-\varepsilon}, \lambda \ge \tilde{C}.$$

For small positive x, let $I_x = \{\lambda \mid \frac{1}{2}x^{-\frac{1}{\varepsilon}} < \lambda < x^{-\frac{1}{\varepsilon}}\}$. Note that $\lambda^{\varepsilon-2/3} < x < \lambda^{-\varepsilon}$, $\lambda \ge \tilde{C}$ if x is small enough and $\lambda \in I_x$. Thus (34) holds if x is small and $\lambda \in I_x$. Also, $\int_0^x (p(t))^{1/2}dt \ge \tilde{c}x^{3/2}$, while the length of I_x is large compared to $x^{-3/2}$ when x is small. Therefore, as λ varies over I_x, $\lambda \int_0^x (p(t))^{1/2}dt$ varies over an interval whose length is large, say more than 100π. Hence we can pick some $\lambda_x \in I_x$ for which

$$\pm\tfrac{\pi}{4} + \lambda_x \int_0^x (p(t))^{1/2}dt + \arg(c_{k_0}) \equiv 0 \mod 2\pi.$$

Putting this into (34) yields $\frac{\lambda_x^{-k_0}|c_{k_0}|}{(\lambda_x^2 p(x))^{1/4}} \le \frac{\tilde{C}\lambda_x^{-k_0-\frac{3}{2}\varepsilon}}{(\lambda_x^2 p(x))^{1/4}}$. Since $|c_{k_0}| \ne 0$, and since $\lambda_x \to \infty$ as $x \to 0$, this is clearly false for small enough positive x. Thus we have derived a contradiction, and the Lemma is proven. ■

Set $N'' = [\varepsilon N/300]$.

Definition. A solution $(u_0(x), \dots, u_{N''}(x))$ of the transport equations is called the canonical solution (for potential $p(x)$ at the turning point $x = 0$) if we can define $u_{N''+1}(x) \dots u_{N'}(x)$ for which $(u_0(x), \dots, u_{N'}(x))$ still solves the transport equations, and for which

$$|F_0(x,\lambda) - F_+(x,\lambda)| \le \tilde{C}\lambda^{-\varepsilon N/200} \quad \text{for} \quad \lambda^{\varepsilon-2/3} < x < \lambda^{-\varepsilon}, \lambda \ge \tilde{C}.$$

with F_0 defined by (4), F_+ defined by (5), and $\tilde{C}$ independent of (x, λ).

Lemmas 5 and 7 show that there is one and only one canonical solution of the transport equations. We want to write down the canonical solution rather explicitly. Our method is to write down an "elementary solution" of the transport equations, whose properties are easily understood, and then to relate the canonical solution to the elementary one. To write down the elementary solution we proceed we follows.

If $u = (u_0(x), \dots, u_{N''}(x))$ solves the transport equations, then define the residue

$$\mathrm{Res}_k(u) = \lim_{x \to 0+} \Big(u_k(x) - \sum_{\ell=1}^{3k} g_{k\ell} x^{-\ell/2} \Big),$$

where the $g_{k\ell}$ are uniquely specified by demanding the finiteness of the limit. The Corollary to Lemma 6 shows that $\mathrm{Res}_k(u)$ is well-defined. If $u = (u_0(x), u_1(x), \dots, u_{N''}(x))$ and $\tilde{u} = (\tilde{u}_0(x), \dots, \tilde{u}_{N''}(x))$ are two solutions of the transport equations related by $\tilde{u}_k = \sum_{\ell=0}^{k} c_\ell u_{k-\ell}$ as in Lemma 6, then we have

$$\mathrm{Res}_k(\tilde{u}) = \sum_{\ell=0}^{k} c_\ell \mathrm{Res}_{k-\ell}(u).$$

Note that $\mathrm{Res}_0(u) \equiv 1$.

Definition. A solution $\tilde{u} = (\tilde{u}_0(x), \tilde{u}_1(x), \dots, \tilde{u}_{N''}(x))$ of the transport equations is called the elementary solution (for potential $p(x)$ and turning point $x = 0$) if $\mathrm{Res}_k(\tilde{u}) = 0$ for $k = 1, 2, \dots, N''$.

Lemma 8. *There is one and only one elementary solution of the transport equations.*

Proof. Let $u = (u_0, u_1, \dots, u_{N''})$ be any solution of the transport equations. We show that there is one and only one sequence $c_0, c_1 \dots c_{N''}$ with $c_0 = 1$, for which $\tilde{u}_k = \sum_{\ell=0}^{k} c_\ell u_{k-\ell}$ has residue zero $k = 1, 2, \dots, N''$).

Since the equations to be solved for the (c_k) may be written as $c_k + + \sum_{\ell=0}^{k-1} c_\ell \mathrm{Res}_{k-\ell}(u) = 0$, the c_k are uniquely determined by an obvious induction. ■

The elementary solution is given rather explicitly by the following result.

Lemma 9. *The elementary solution* $(u_0(x), u_1(x), \dots, u_{N''}(x))$ *is obtained from the inductive procedure* $u_0(x) \equiv 1$,

$$-2i\, u_{k+1}(x) = \lim_{\delta \to 0+} \Big[\int_\delta^x \mathcal{L} u_k(t) dt + \sum_{\ell=1}^{3k+3} g_{k\ell} \delta^{-\ell/2} \Big],$$

with $\mathcal{L}$ as in Lemma 4, and $g_{k\ell}$ uniquely specified by demanding the finiteness of the limit. The $g_{k\ell}$ all belong to $\mathcal{P}$. We can write u_k in the form $u_k(x) = x^{-\frac{3}{2}k} f_k(x^{1/2})$ with f_k smooth and $(\frac{d}{dy})^\alpha f_k(0) \in \mathcal{P}$ for all α.

Proof. By the corollary to Lemma 6 we can write

$$-2i\, u_{k+1}(t) = \sum_{\ell=1}^{3k+3} q_{k\ell} t^{-\ell/2} + \tilde{q}_k(t^{1/2}) \tag{35}$$

with $\tilde{q}_k(y)$ smooth.

In particular, $\lim_{t\to 0_+}[2iu_{k+1}(t) + \sum_{\ell=1}^{3k+3} q_{k\ell} t^{-\ell/2}] = -\tilde{q}_k(0)$, so $2i\,\mathrm{Res}_{k+1}(u) = -\tilde{q}_k(0)$. For the elementary solution the residues are zero, so $\tilde{q}_k(0) = 0$. Now we return to (35) and differentiate, obtaining

$$-2iu'_{k+1}(t) = \sum_{\ell=1}^{3k+3} (-\tfrac{\ell}{2}) q_{k\ell} t^{-\frac{\ell}{2}-1} + \tfrac{1}{2} t^{-1/2} \tilde{q}'_k(t^{1/2}). \tag{36}$$

Integrating from $t = \delta$ to $t = x$, we get

$$\int_\delta^x (-2i\, u'_{k+1}(t))dt = \sum_{\ell=1}^{3k+3} q_{k\ell}(x^{-\ell/2} - \delta^{-\ell/2}) + \tilde{q}_k(x^{1/2}) - \tilde{q}_k(\delta^{1/2}).$$

Since $\tilde{q}_k$ is smooth and vanishes at the origin, it follows that

$$\begin{aligned}\lim_{\delta\to 0+}\Big[\int_\delta^x (-2i\, u'_{k+1}(t))dt + \sum_{\ell=1}^{3k+3} q_{k\ell}\delta^{-\ell/2}\Big] &= \sum_{\ell=1}^{3k+3} q_{k\ell}\, x^{-\ell/2} + \tilde{q}_k(x^{1/2}) \\ &= -2i\, u_{k+1}(x).\end{aligned}$$

The inductive formula for $-2i\, u_{k+1}(x)$ in the statement of the Lemma now follows at once, since $-2i\, u'_{k+1}(t) = \mathcal{L}u_k(t)$ by the transport equation. The $g_{k\ell}$ are equal to the $q_{k\ell}$. Again using the Corollary to Lemma 6 to write $u_k(x) = x^{-\frac{3}{2}k} f_k(x^{1/2})$ with $f_k(y)$ smooth, we see from (35) that the $q_{k\ell}$ are proportional to the derivatives of $f_{k+1}(y)$ at $y = 0$. Hence the assertion $g_{k\ell} \in \mathcal{P}$ will follow, once we prove that $(\frac{d}{dy})^\alpha f_k(0) \in \mathcal{P}$. So it remains only to verify the latter assertion. We proceed by induction on k. For $k=0$ we have $f_0(y)\equiv 1$ and there is nothing to prove. Suppose we know that $(\frac{d}{dy})^\alpha f_k(0)\in\mathcal{P}$ for all α. Then

$$-2i\,u'_{k+1}(t)=\mathcal{L}u_k(t)=\{(\tfrac{5}{16}(p')^2p^{-5/2}-\tfrac{1}{4}p''p^{-3/2})-\tfrac{1}{2}p'p^{-3/2}\tfrac{d}{dt}+p^{-1/2}\tfrac{d^2}{dt^2}\}\cdot \{t^{-\frac{3}{2}k}f_k(t^{1/2})\}\equiv X.$$

With $y=t^{1/2}$ we have $\frac{d}{dt}=\frac{1}{2}y^{-1}\frac{d}{dy}$, $p(t)=g_1(y)$, $p'(t)=g_2(y)$, $p''(t)=g_3(y)$, $y^5/p^{5/2}(t)=g_4(y)$ with $g_i(y)$ smooth functions and $(\frac{d}{dy})^\alpha g_i(0)\in\mathcal{P}$. Hence

$$X=\frac{(g_5(y)+g_6(y)\frac{d}{dy}+g_7(y)\frac{d^2}{dy})f_k(y)}{y^{3k+5}}$$

where again $g_i(y)$ are smooth and $(\frac{d}{dy})^\alpha g_i(0)\in\mathcal{P}$. By inductive hypothesis, the numerator is smooth, and its derivatives at $y=0$ all belong to $\mathcal{P}$. Taylor's theorem with remainder gives therefore

$$X=\sum_{\ell=1}^{3k+5}\hat{q}_{k\ell}\,y^{-\ell}+\hat{q}_k(y)\quad\text{with}\quad \hat{q}_{k\ell}\in\mathcal{P},$$
$$\hat{q}_k(y)\quad\text{smooth},\quad (\tfrac{d}{dy})^\alpha\hat{q}_k(0)\in\mathcal{P}.$$

That is,

$$\text{(37)}\qquad -2i\,u'_{k+1}(t)=\sum_{\ell=1}^{3k+5}\hat{q}_{k\ell}\,t^{-\ell/2}+\hat{q}_k(t^{1/2}),\quad\text{by definition of } t \text{ and } X.$$

Comparing (36) and (37) gives

$$\text{(38)}\qquad \sum_{\ell=1}^{3k+5}\hat{q}_{k\ell}\,t^{-\ell/2}-\sum_{\ell=1}^{3k+3}(-\tfrac{\ell}{2})q_{k\ell}\,t^{-\frac{\ell}{2}-1}=\tfrac{1}{2}t^{-1/2}\tilde{q}'_k(t^{1/2})-\hat{q}_k(t^{1/2}),$$

where the $'$ on the right is taken with respect to y. The right-hand side here is $O(t^{-1/2})$ at the origin, while the left-hand side is a sum of negative half-integer powers of t. We conclude by matching like powers of t that

$$\text{(39)}\qquad q_{k\ell}=-\tfrac{2}{\ell}\hat{q}_{k\ell+2}\ (\ell=1,\dots,3k+3)$$

and that $\hat{q}_{k2} = 0$. Hence (38) simplifies to $\hat{q}_{k1}\, t^{-1/2} = \frac{1}{2} t^{-1/2}\, \tilde{q}_k'(t^{1/2}) - \hat{q}_k(t^{1/2})$, i.e., $\frac{d}{dy}\tilde{q}_k(y) = 2\hat{q}_{k1} + 2y\,\hat{q}_k(y)$. Since $\tilde{q}_k(0) = 0$, we conclude that

$$\tilde{q}_k(y) = 2\hat{q}_{k1}\, y + \int_0^y 2s\,\hat{q}_k(s)\,ds. \tag{40}$$

Since $\hat{q}_{k\ell} \in \mathcal{P}$ and $(\frac{d}{dy})^\alpha \hat{q}_k(0) \in \mathcal{P}$ for all α, equations (39), (40) show that $q_{k\ell} \in \mathcal{P}$ and $(\frac{d}{dy})^\alpha \tilde{q}_k(0) \in \mathcal{P}$ for all α. Therefore, (35) allows us to write

$$u_{k+1}(x) = x^{-\frac{3}{2}(k+1)} f_{k+1}(x^{1/2}) \quad \text{with}$$

$$f_{k+1}(y) = \frac{1}{-2i}\Big(\sum_{\ell=1}^{3k+3} q_{k\ell}\, y^{3k+3-\ell} + y^{3k+3}\tilde{q}_k(y)\Big)$$

and thus $(\frac{d}{dy})^\alpha f_{k+1}(0) \in \mathcal{P}$ for all α. The induction step is complete, and the Lemma is proven. ■

Now it is trivial to relate the canonical to the elementary solution.

Lemma 10. *Let $u = (u_0(x), u_1(x), \dots, u_{N''}(x))$ be the canonical solution of the transport equations, and let $\tilde{u} = (\tilde{u}_0(x), \dots, \tilde{u}_{N''}(x))$ be the elementary solution. Then we have $u_k(x) = \sum_{\ell=0}^{k} c_{k-\ell}^+ \tilde{u}_\ell(x)$ with $c_0^+ = 1$ and $c_k^+ \in \mathcal{P}$. In particular, $c_1^+ = 0$. The coefficients (c_k^+) are called* <u>*matching coefficients*</u>.

Proof. In view of Lemma 6, the only points to be checked are that $c_k^+ \in \mathcal{P}$ and that $c_1^+ = 0$. Now $\mathrm{Res}_\ell(\tilde{u}) = \begin{cases} 1 & \text{if } \ell = 0 \\ 0 & \text{if } \ell > 0 \end{cases}$ by definition of $\tilde{u}$, so $\mathrm{Res}_k(u) = \sum_{\ell=0}^{k} c_{k-\ell}^+ \mathrm{Res}_\ell(\tilde{u}) = c_k^+$. We must therefore show that $\mathrm{Res}_k(u) \in \mathcal{P}$.

Lemmas 3 and 5 give

$$u_k(x) = (u_k(x) - w_k(x)) + \sum_{-3k \le \ell \le M_k} w_{k\ell}\, x^{\ell/2} \quad \text{with} \quad w_{k\ell} \in \mathcal{P},$$

$$w_{10} \equiv 0, \quad \text{and} \quad \lim_{x \to 0^+} (u_k(x) - w_k(x)) = 0.$$

Therefore $\lim_{x\to 0^+} [u_k(x) - \sum_{-3k \le \ell \le -1} w_{k\ell}\, x^{\ell/2}] = w_{k0}$, which shows that $c_k^+ = \mathrm{Res}_k(u) = w_{k0} \in \mathcal{P}$. Also, $c_1^+ = w_{10} = 0$. The Lemma is proven. ■

Remark. It would be interesting to exhibit the first few matching coefficients beyond

c_1^+. In principle, this is routine. In practice we don't even know whether the higher c_k^+ are all zero.

Next, we want a-priori bounds on the C^∞ seminorms of the functions f_k in Lemma 9. For later application, we take the potential $p(x)$ to depend smoothly on an additional parameter τ, and investigate how $f_k(x,\tau)$ depends on (x,τ) together.

Lemma 11. *Let $p(x,\tau)$ be C^∞ on $\{|x|,|\tau| \leq 1\}$ with $p(0,\tau) = 0$ and $\frac{\partial}{\partial x}p(0,\tau) \geq c_1 > 0$. For each fixed τ, let $(u_0(x,\tau), u_1(x,\tau), \ldots u_{N''}(x,\tau))$ be the elementary solution to the transport equations for potential $x \mapsto p(x,\tau)$. Then define $f_k(y,\tau)$ by $u_k(x,\tau) = x^{-\frac{3}{2}k} f_k(x^{1/2},\tau)$ as in Lemma 9. The C^∞ seminorms of f_k can be bounded a-priori in terms of the C^∞ seminorms of $p(x,\tau)$ and the constant c_1.*

Proof. We use induction on k. For $k = 0$ we have $f_k(y,\tau) \equiv 1$, so there is nothing to prove. For the inductive step, we use C_{**}, C_{**}^α, etc. to denote constants determined a-priori in terms of the C^∞ seminorms of $p(x,\tau)$ and the lower bound c_1 for $\frac{\partial p}{\partial x}(0,\tau)$. Thus we must prove that $|\partial_{y,\tau}^\alpha f_{k+1}| \leq C_{**}^\alpha$ for all α, given that $|\partial_{y,\tau}^\alpha f_k| \leq C_{**}^\alpha$ for all α.

Let Q be a small square about the origin with $\frac{p(x,\tau)}{x} > \frac{1}{2}c_1$ for $x > 0$ and $(x^{1/2},\tau) \in Q$. We will make our estimates on Q. Note that the size of Q is bounded below by c_{**}.

We have $\mathcal{L}u_k(x,\tau) = \mathcal{L}(x^{-\frac{3}{2}k} f_k(x^{1/2},\tau)) = x^{-\frac{3}{2}k-\frac{5}{2}}(\hat{\mathcal{L}}_k f_k)(x^{1/2},\tau)$ with

$$\hat{\mathcal{L}}_k w = y^{3k+5}\Big[\big(\tfrac{5}{16}(p')^2 p^{-5/2} - \tfrac{1}{4}p''p^{-3/2}\big) - \tfrac{1}{2}p'p^{-3/2} \cdot \tfrac{1}{2y}\tfrac{\partial}{\partial y} + p^{-1/2}\big(\tfrac{1}{2y}\tfrac{\partial}{\partial y}\big)^2\Big] y^{-3k} w,$$

where p, p', p'' are evaluated at (y^2,τ).

The defining formula for $\hat{\mathcal{L}}_k$ may be written in the form

$$\hat{\mathcal{L}}_k = a_k^{(2)}(y,\tau)\frac{\partial^2}{\partial y^2} + a_k^{(1)}(y,\tau)\frac{\partial}{\partial y} + a_k^{(0)}(y,\tau)$$

with $a_k^{(i)}$ smooth on Q and $|\partial_{y,\tau}^\alpha a_k^{(i)}(y,\tau)| \le C_{**}^{\alpha k}$, $(y,\tau) \in Q$, all α. Since also $|\partial_{y,\tau}^\alpha f_k(y,\tau)| \le C_{**}^{\alpha k}$ by inductive hypothesis, we have for $g_k = \hat{\mathcal{L}}_k f_k$ the estimates $|\partial_{y,\tau}^\alpha g_k(y,\tau)| \le C_{**}^{\alpha k}$ on Q, all α.

We have also $\mathcal{L}u_k(x,\tau) = x^{-\frac{3}{2}k-\frac{5}{2}} g_k(x^{1/2},\tau)$. Applying Taylor's theorem with remainder to g_k, we get smooth functions $\tilde{g}_{k\ell}(\tau)$ and a smooth function $\tilde{g}_k(y,\tau)$ that satisfy $\mathcal{L}u_k(x,\tau) = \sum_{\ell=1}^{3k+5} \tilde{g}_{k\ell}(\tau)x^{-\ell/2} + \tilde{g}_k(x^{1/2},\tau)$, $|\partial_\tau^\alpha \tilde{g}_{k\ell}(\tau)| \le C_{**}^{\alpha k\ell}$ and $|\partial_{y,\tau}^\alpha \tilde{g}_k(y,\tau)| \le C_{**}^{\alpha k}$ on Q. Therefore, Lemma 9 says that

$$(41) \qquad -2i\, u_{k+1}(x,\tau) = \lim_{\delta\to 0+} \Big[\int_\delta^x \Big\{ \sum_{\ell=1}^{3k+5} \tilde{g}_{k\ell}(\tau) t^{-\ell/2} + \tilde{g}_k(t^{1/2},\tau) \Big\}\, dt + \sum_{\ell=1}^{3k+3} g_{k\ell}(\tau)\delta^{-\ell/2} \Big]$$

for suitable functions $g_{k\ell}(\tau)$. In particular, the limit exists, which shows that $\tilde{g}_{k2} \equiv 0$. (Otherwise the integral would produce a logarithmic singularity in δ, which could not be remedied by any choice of the $g_{k\ell}(\tau)$.) Hence (41) means that

$$(42) \qquad -2i\, u_{k+1}(x,\tau) = \sum_{\ell=3}^{3k+5} \left(\tfrac{-2}{\ell-2}\right) x^{-(\ell-2)/2} \tilde{g}_{k\ell}(\tau) + g_k^{\#}(x^{1/2},\tau)$$

$$\text{with } g_k^{\#}(y,\tau) = 2\tilde{g}_{k1}(\tau)y + \int_0^y \tilde{g}_k(s,\tau)\cdot 2s\,ds.$$

We have $|\partial_{y,\tau}^\alpha\, g_k^{\#}(y,\tau)| \le C_{**}^{\alpha k}$ on Q, by virtue of our estimates for $\tilde{g}_{k\ell}(\tau)$ and $\tilde{g}_k(y,\tau)$.

By (42) we have $u_{k+1}(x,\tau) = x^{-\frac{3}{2}(k+1)} f_{k+1}(x^{1/2},\tau)$ with

$$f_{k+1}(y,\tau) = \sum_{\ell=3}^{3k+5} \tfrac{1}{i(\ell-2)} \tilde{g}_{k\ell}(\tau) y^{(3k+3)-(\ell-2)} - \tfrac{1}{2i} y^{3k+3} g_k^{\#}(y,\tau).$$

Our estimates for $\tilde{g}_{k\ell}(\tau)$ and $g_k^{\#}(y,\tau)$ show that $|\partial_{y,\tau}^\alpha f_{k+1}(y,\tau)| \le C_{**}^{\alpha k}$ on Q for all α. The induction is complete, and the Lemma is proven. ■

Corollary. *The conclusion of Lemma 11 holds also for the canonical solution of the transport equation.*

Proof. Express the canonical solution in terms of the elementary solution using the matching coefficients c_k^+. Since $c_k^+ \in \mathcal{P}$, the matching coefficients depend smoothly on τ, and the Corollary follows at once from Lemma 11. ∎

It remains to verify that $F_+(x,\lambda)$ approximately satisfies the basic differential equation (1). Since we only established the uniqueness of the canonical solution $(u_0(x), u_1(x), \dots, u_k(x), \dots)$ for $k \le N''$, we prefer to truncate the sum (5) defining $F_+(x,\lambda)$ below $k = N''$. We will check that the truncated solution still agrees well with $F_0(x,\lambda)$ as in Lemma 5. (This problem arises only from the technicalities of our exposition. Clearly the canonical $u_k(x)$ are defined for all k, by taking N large enough depending on k. We didn't want to bother with the N dependence of u_k.)

Lemma 12. *Set* $N''' = [\varepsilon N/500]$ *and define* $F_{++}(x,\lambda) = Re\Big[\frac{e^{\pm i\pi/4} e^{i\int_0^x (\lambda^2 p(t))^{1/2}\,dt}}{(\lambda^2 p(x))^{1/4}} \cdot \cdot \sum_{k=0}^{N'''} \lambda^{-k} u_k(x)\Big]$ *with* $(u_0(x), \dots u_{N''}(x))$ *the canonical solution of the transport equations. Then for* $\lambda^{\varepsilon-2/3} < x < \lambda^{-\varepsilon}$ *and* $\lambda \ge C_*^{\varepsilon N}$ *we have* $|\partial_x^\alpha\{F_{++}(x,\lambda) - F_0(x,\lambda)\}| \le C_*^{\varepsilon N}(\lambda x^{3/2})^{-N'''}\lambda^{10}$ $(\alpha = 0,1,2)$*. Also* $|(\frac{\partial^2}{\partial x^2} + \lambda^2 p(x))F_{++}(x,\lambda)| \le C_*^{\varepsilon N}(\lambda x^{3/2})^{-N'''}\lambda^{10}$ *for* $\lambda^{\varepsilon-\frac{2}{3}} < x < c_*^{\varepsilon N}$, $\lambda \ge C_*^{\varepsilon N}$.

Proof. For x, λ as in the Lemma, we have from Lemma 5

$$|\partial_x^\alpha\{F_+(x,\lambda) - F_0(x,\lambda)\}| \le C_*^{\varepsilon N}\lambda^{-\varepsilon N/200} \le C_*^{\varepsilon N}(\lambda x^{3/2})^{-N'''}\lambda^{10} \qquad (\alpha = 0,1,2).$$

So the first conclusion of Lemma 12 will follow if we can show that

$$|\partial_x^\alpha\{F_{++}(x,\lambda) - F_+(x,\lambda)\}| \le C_*^{\varepsilon N}(\lambda x^{3/2})^{-N'''}\lambda^{10} \quad (\alpha = 0,1,2). \tag{43}$$

Now $F_{++} - F_+$ is a sum of real parts of expressions

$$X = \frac{e^{\pm i\pi/4}}{(\lambda^2 p(x))^{1/4}} e^{i\int_0^x (\lambda^2 p(t))^{1/2}dt}\lambda^{-k}u_k(x) \quad \text{with} \quad N''' < k \le N'.$$

Also, Lemma 5 gives $|\partial_x^\alpha\{u_k(x) - w_k(x)\}| \le C_*^{\varepsilon N}$ $(\alpha = 0,1,2)$, while $|\partial_x^\alpha w_k(x)| \le C_*^{\varepsilon N} x^{-\frac{3}{2}k-\alpha}$ $(\alpha = 0,1,2)$ since $w_k(x) = \sum_{\ell=-3k}^{M_k} w_{k\ell}\, x^{\ell/2}$, $w_{k\ell} \in \mathcal{P}$. Hence $|\partial_x^\alpha u_k(x)|$

$\leq C_*^{\varepsilon N} x^{-\frac{3}{2}k-\alpha}$ $(\alpha = 0,1,2)$. Putting this into the definition of X, we find that $|\partial_x^\alpha X| \leq C_*^{\varepsilon N} \lambda^{-k} x^{-\frac{3}{2}k} \cdot x^{-3}\lambda^3$ for $\alpha = 0,1,2$. In the region $\lambda^{\varepsilon-2/3} < x < \lambda^{-\varepsilon}$ this implies $|\partial_x^\alpha X| \leq C_*^{\varepsilon N} (\lambda x^{3/2})^{-k} \lambda^{10}$ $(\alpha = 0,1,2)$, which proves (43).

To establish the second conclusion of the Lemma, note that

(44)
$$\Big(\frac{\partial^2}{\partial x^2} + \lambda^2 p(x)\Big) F_{++}(x,\lambda) = \mathrm{Re}\Big[\frac{e^{\pm i\pi/4}\, e^{i\int_0^x (\lambda^2 p(t))^{1/2}dt}}{(\lambda^2 p(x))^{1/4}}\, p(x)^{1/2} \mathcal{L} u_{N'''}(x) \lambda^{-N'''}\Big],$$

because $(u_k(x))$ satisfy the transport equations.

In proving Lemma 11, we showed that $\mathcal{L}\tilde{u}_k(x) = x^{-\frac{3}{2}k-\frac{5}{2}} g_k(x^{1/2})$ with $|\partial_y^\alpha g_k(y)| \leq C_*^{\varepsilon N \alpha}$, $0 \leq k \leq N''$, all α and all $|y| \leq c_*^{\varepsilon N}$. Here $(\tilde{u}_k(x))$ denotes the elementary solution of the transport equations.

Since the canonical and elementary solutions are related by the matching coefficients $c_k^+ \in \mathcal{P}$, we have the same result for the $\mathcal{L}u_k(x)$ arising from the canonical solution. In particular, $|\mathcal{L}u_k(x)| \leq C_*^{\varepsilon N} x^{-\frac{3}{2}k-\frac{5}{2}}$ for $0 \leq k \leq N''$ and all $x \in (0, c_*^{\varepsilon N}]$. Taking $k = N'''$ and substituting in (44), we obtain the second conclusion of the Lemma, since we are in the region $\lambda^{\varepsilon-2/3} < x,\ \lambda \geq C_*^{\varepsilon N}$. ■

We pause to write down in detail the function $u_1(x)$ for the canonical solution of the transport equations. Since the first two matching coefficients are $c_0^+ = 1$ and $c_1^+ = 0$ by Lemma 10, the elementary and canonical solutions have the same $u_1(x)$. Therefore, recalling that $u_0(x) \equiv 1$, we may apply Lemma 9 and the definition of $\mathcal{L}$ to derive:

(45)
$$-2iu_1(x) = \lim_{\delta\to 0+}\Big[\int_\delta^x \big(\tfrac{5}{16}(p')^2 p^{-5/2} - \tfrac{1}{4}p'' p^{-3/2}\big)dt - g_1\delta^{-1/2} - g_2\delta^{-1} - g_3\delta^{-3/2}\Big]$$

with g_1, g_2, g_3 uniquely specified by demanding the finiteness of the limit. We needn't bother to compute g_1, g_2, g_3.

Next we rescale our approximate ODE solutions.

Let $F_0(x,\lambda)$, $F_+(x,\lambda)$ be the matching approximate solutions for

(46)
$$\Big(\frac{\partial^2}{\partial x^2} + \lambda^2 p(x)\Big) F = 0 \quad \text{given by (4), (5).}$$

For $t > 0$ define $p_{\#}(x) = t^2 p(x)$. Then let $F_0^{\#}(x, \lambda_{\#})$, $F_+^{\#}(x, \lambda_{\#})$ be the matching approximate solutions for

$$\left(\frac{\partial^2}{\partial x^2} + \lambda_{\#}^2 p_{\#}(x)\right) F^{\#} = 0 \quad \text{given similarly.} \tag{47}$$

If we set $\lambda_{\#} = t^{-1}\lambda$, then (46) and (47) are the same equation. One checks that $y^{\#}(x, \lambda_{\#}) = t^{2/3} y(x, t\lambda_{\#})$ solves the analogue of (3) for $p_{\#}(x)$. Therefore $F_0^{\#}(x, \lambda_{\#}) = F_0(x, \lambda)$, i.e. our Airey approximation really depends only on the potential, not how it is factored. Since $F_+^{\#}(x, \lambda_{\#})$ and $F_+(x, \lambda)$ match $F_0^{\#}(x, \lambda_{\#})$ and $F_0(x, \lambda)$, it follows that $F_+^{\#}(x, \lambda_{\#}) = F_+(x, \lambda)$ as well, and the canonical solutions $(u_0(x), u_1(x), \ldots, u_{N''}(x))$, $(u_0^{\#}(x), u_1^{\#}, \ldots, u_{N''}^{\#}(x))$ are related by $u_k^{\#}(x) = t^{-k} u_k(x)$. Thus, $\lambda^{-k} u_k(x)$ depends only on the potential and not how it is factored.

There is a second kind of rescaling for our approximate solutions. With $t > 0$ we take $p_{\#}(x) = t^2 p(tx)$. The solution of equation (3) for $p_{\#}(x)$ is $y_{\#}(x, \lambda) = y(tx, \lambda)$. Thus $\left(\frac{\partial y_{\#}(x,\lambda)}{\partial x}\right)^{-1/2} = t^{-1/2}\left(\frac{\partial y}{\partial x}\,|_{tx,\lambda)}\right)^{-1/2}$, so the Airey approximation for $p_{\#}(x)$ is $F_0^{\#}(x, \lambda) = t^{-1/2} F_0(tx, \lambda)$.

Again, since $F_+^{\#}$ matches $F_0^{\#}$ and F_+ matches F_0, we can deduce the scaling laws for F_+ and for the canonical solution of the transport equations we get $F_+^{\#}(x, \lambda) = t^{-1/2} F_+(tx, \lambda)$,

$$u_k^{\#}(x) = u_k(tx).$$

For both notions of rescaling, the elementary solutions of the transport equations scale in the same way as the canonical solutions. Details of the rescaling are easily filled in by the reader.

The main application of our local results is to the one-parameter family of potentials $\tilde{p}(x, \tau) = p(x + x(\tau)) - \tau$, where $x(\tau)$ is the solution of $p(x(\tau)) = +\tau$. If $p(x)$ is C^∞ with $p'(0) > 0$, then $\tilde{p}(x, \tau)$ is C^∞ with $\tilde{p}(0, \tau) = 0$, $\frac{\partial \tilde{p}}{\partial x}(0, \tau) > c_1 > 0$ for $|p(0) - \tau|$ small. For each fixed τ we apply the local ODE results of this section to $\tilde{p}(x, \tau)$.

Quantities belonging to $\mathcal{P}$, such as the matching coefficients c_k^+ or the $f_{k\ell}$ in Lemma 1, evidently depend smoothly on τ. The τ-dependence of the canonical solutions to the transport equation is given by the Corollary to Lemma 11. Therefore, after a small change of notation, we obtain the following result.

Local WKB Lemma. *Let ε, N be given, and put $N' = [\varepsilon N/500]$. Let $p(x)$ be a smooth function on $[-1,1]$ with $p'(0) > 0$. For $|E - p(0)| < c$ let $x(E)$ denote the solution of $p(x) = E$, nearest to $x = 0$. Then there exist functions $y(x,E,s)$ and $u_k(x,E)$ $(0 \le k \le N')$ with the following properties.*

(A) The function $y(x,E,s)$ is smooth and satisfies $\frac{\partial y}{\partial x} > c > 0$ on $\{|x - x(E)| < c, |E - p(0)| < c, |s| < c\}$.

(B) The function $F_0(x,E,\lambda) = \lambda^{-1/3}(\partial_x y(x,E,\lambda^{-2}))^{-1/2} A(\lambda^{2/3} y(x,E,\lambda^{-2}))$ satisfies $|(\frac{\partial^2}{\partial x^2} + \lambda^2(p(x) - E))F_0(x,E,\lambda)| \le C\lambda^{-N'}$ for $|x - x(E)| < \lambda^{-\varepsilon}$, $|E - p(0)| < c$, $\lambda > C$.

(C) We have $|\partial_x^\alpha F_0(x,E,\lambda)| \le C\lambda^{-N'}$ for $-\lambda^{-\varepsilon} < x - x(E) < -\lambda^{\varepsilon - 2/3}$, $|E - p(0)| < c$, $\lambda > C$, $\alpha \le 2$.

(D) The functions $u_k(x,E)$ are defined for $0 < x - x(E) < c$, $|E - p(0)| < c$ and have there the form $u_k(x,E) = (x - x(E))^{-\frac{3}{2}k} f_k((x - x(E))^{1/2}, E)$ with $f_k(y,E)$ smooth.

(E) The function $F_+(x,E,\lambda) = Re\Big[\frac{\exp(\pm i\frac{\pi}{4} + i\lambda\int_{x(E)}^x (p(t)-E)^{1/2}dt)}{\lambda^{1/2}(p(x)-E)^{1/4}}\Big\{1 + \sum_{k=1}^{N'} \lambda^{-k} u_k(x,E)\Big\}\Big]$ satisfies $|(\frac{\partial^2}{\partial x^2} + \lambda^2(p(x) - E))F_+(x,E,\lambda)| \le C\lambda^{10} \cdot (x - x(E))^{-\frac{3}{2}N'}\lambda^{-N'}$ for $\lambda^{\varepsilon-2/3} < x - x(E) < c$, $|E - p(0)| < c$, $\lambda > C$.

(F) We have $|\partial_x^\alpha\{F_+(x,E,\lambda) - F_0(x,E,\lambda)\}| \le C\lambda^{10} \cdot (x - x(E))^{-\frac{3}{2}N'}\lambda^{-N'}$ for $\alpha \le 2$, $\lambda^{\varepsilon-2/3} < x - x(E) < \lambda^{-\varepsilon}$, $|E - p(0)| < c$, $\lambda > C$.

More precise information on the functions $y(x,E,s)$ and $u_k(x,E)$ is given by the following:

(G) We have $(\partial_x y(x,E,s))^2 y(x,E,s) = p(x) - E + (x - x(E))^{N'} g_0(x,E) + s g_1(x,E,s)$ *with* $g_0(x,E)$ *smooth on* $\{|x - x(E)| \le c,\ |E - p(0)| \le c\}$ *and* $g_1(x,E,s)$ *smooth on* $\{|x - x(E)|, |E - p(0)|, |s| \le c\}$. *For* $|E - p(0)| < c$, $|x - x(E)| < \lambda^{-\varepsilon}$, $s = \lambda^{-2}$, $\lambda > C$ *we have* $|(\partial_x y)^2 + s\{y,x\} - (p(x) - E)| \le C\lambda^{-N'-2}$, *with* $\{y,x\}$ *given by (3).*

(H) For each fixed E, $(u_k(x,E))$ *is the canonical solution of the transport equations for the potential* $p(x) - E$ *and the turning point* $x(E)$. *In particular* $u_0(x,E) \equiv 1$, *and* $u_1(x,E) = \frac{i}{2}\lim_{\delta\to 0+}\left[\int_{x(E)+\delta}^{x}\left(\frac{5}{16}(p')^2(p-E)^{-5/2} - \frac{1}{4}p''(p-E)^{-3/2}\right)dt - \sum_{\ell=1}^{3} q_\ell(E)\delta^{-\ell/2}\right]$, *with* $q_\ell(E)$ *uniquely specified by demanding the finiteness of the limit.*

We can give a positive lower bound for c *and positive upper bounds for* C *and for the* C^∞ *seminorms of the functions* $y(x,E,s)$, $f_k(y,E)$, $g_0(x,E)$, $g_1(x,E,s)$ *depending only on upper bounds for the* C^∞ *seminorms of* $p(x)$ *and on a positive lower bound for* $p'(0)$.

Approximate Global Solutions of Ordinary Differential Equations

In this section, we attempt to write down explicit, highly accurate approximate solutions of the ordinary differential equation

$$\left[\frac{d^2}{dx^2} + (E - V(x))\right]F = 0 \tag{1}$$

in an interval I containing two turning points $x_{\text{left}} < x_{\text{rt}}$. First we cover I by two subintervals I_{left}, I_{rt}, each containing only one of the turning points. Next we apply the local theory of the previous section to construct an accurate approximate solution $F_{\text{left}}(x,E)$ of (1) in a small neighborhood $J_{\text{left}} \subset I_{\text{left}}$ of the turning point. We then extend $F_{\text{left}}(x,E)$ from the small neighborhood J_{left} to the whole of I_{left}. It is quite simple to define the extension: Near the left endpoint of J_{left}, $F_{\text{left}}(x,E)$ is very tiny. Hence we may simply set $F_{\text{left}} = 0$ to be the left of J_{left}. Near the right endpoint of J_{left}, F_{left} is given in terms of the canonical solution (u_k) of the transport equations. A glance at the transport equations shows that the solution (u_k) can be continued until we get to the second turning point. Thus (u_k) may be defined in all of $I_{\text{left}} \cap (x_{\text{left}}, \infty)$, and then $F_{\text{left}}(x, E)$ may be defined in terms of (u_k) to the right of J_{left} by the usual formula. Thus we have extended our approximate solution to all of I_{left}. Of course we still have to see how F_{left} behaves on its enlarged domain, and check that it approximately satisfies (1). Also, there is an approximate solution $F_{\text{rt}}(x, E)$ on I_{rt}, completely analogous to F_{left}.

Finally, we attempt to patch together F_{left} and F_{rt} into a single approximate solution of (1) defined on the whole of I. Unless E lies near an eigenvalue of $\frac{-d^2}{dx^2} + V(x)$, we expect F_{left} and F_{rt} to disagree, so that the patching cannot be done. We will define complex approximate solutions $F_c^{\text{left}}(x, E)$, $F_c^{\text{rt}}(x, E)$ on $I_{\text{left}} \cap I_{\text{rt}}$ and a complex number $R(E)e^{i\Phi(E)}$ with the following properties: $F_{\text{left}} = \text{Re}(F_c^{\text{left}})$, $F_{\text{rt}} = \text{Re}(F_c^{\text{rt}})$, and F_c^{left} is very well approximated by $R(E)e^{i\Phi(E)} F_c^{\text{rt}}(x, E)$. Thus, when $\Phi(E) \equiv 0 \mod \pi$ the two solutions F_{left}, F_{rt} are nearly proportional, and can therefore be patched together into an approximate eigenfunction. It is natural to conjecture that the true eigenvalues of $\frac{-d^2}{dx^2} + V(x)$ are well-approximated by the solutions of $\Phi(E) \equiv 0 \mod \pi$, and that the true eigenfunctions are nearly proportional to F_{left} on I_{left}, and to F_{rt} on I_{rt}. However, we postpone discussion of the true eigenvalues and eigenfunctions to a later section. Our task here is to construct the approximate solutions $F_{\text{left}}(x, E)$, $F_{\text{rt}}(x, E)$ and the complex factor $R(E)\exp(i\Phi(E))$.

We start by formulating our hypotheses on the potential $V(x)$. They involve auxiliary functions $S(x)$, $B(x)$ used to capture the size and smoothness of V. Our hypotheses are as follows.

(H0) $V(x)$, $S(x)$, $B(x)$ are functions defined on an interval I, and E_0 is a real number.

(H1) $S(x), B(x) > 0$ on I. If $x, y \in I$ and $|x - y| < cB(x)$, then $cB(x) < B(y) < CB(x)$ and $cS(x) < S(y) < CS(x)$.

(H2) For $x \in I$ and $\alpha \geq 0$ we have $|(\frac{d}{dx})^\alpha V(x)| \leq C_\alpha S(x)(B(x))^{-\alpha}$.

(H3) The equation $V(x) = E_0$ has two solutions $x_{\text{left}} < x_{\text{rt}}$ in I.

(H4) We have distance $(x_{\text{left}}, \partial I) > cB(x_{\text{left}})$ and distance $(x_{\text{rt}}, \partial I) > cB(x_{\text{rt}})$.

(H5) For $x \in [x_{\text{left}}, x_{\text{left}} + c_1 B(x_{\text{left}})]$ we have $-V'(x) > cS(x_{\text{left}})/B(x_{\text{left}})$.

(H6) For $x \in [x_{\text{rt}} - c_1 B(x_{\text{rt}}), x_{\text{rt}}]$ we have $+V'(x) > cS(x_{\text{rt}})/B(x_{\text{rt}})$.

(H7) For $x \in [x_{\text{left}} + c_1 B(x_{\text{left}}), x_{\text{rt}} - c_1 B(x_{\text{rt}})]$ we have $E_0 - V(x) > cS(x)$.

We will denote by $C_\#$, $c_\#$, $C_\#^{\alpha\beta}$ etc. a positive constant depending only on the constants c, C, C_1, C_α appearing in (H1)…(H7), but not on $V(x)$, $S(x)$, $B(x)$.

Example. Take $E_0 = 0$, $B(x) \equiv 1$, $S(x) \equiv \lambda^2$, $I = [-1, 1]$, $V(x) = \lambda^2 p(x)$. Then (H1)…(H7) hold if p is smooth and $\{p < 0\} = (x_{\text{lt}}, x_{\text{rt}})$ with $p' \neq 0$ at x_{lt}, x_{rt}. Constants $C_\#$ depend on the C^∞ seminorms of $p(x)$ and on lower bounds for $|p'(x_{\text{lt}})|$, $|p'(x_{\text{rt}})|$, but are independent of λ.

The following quantities play the role of the large parameter λ in the local WKB theory of the previous section. Set $\lambda(x) = S^{1/2}(x)B(x)$ and define Λ by the equation

$$\frac{1}{\Lambda} = \int_{x_{\text{left}}}^{x_{\text{right}}} \frac{1}{\lambda(x)} \frac{dx}{B(x)} = \int_{x_{\text{left}}}^{x_{\text{right}}} \frac{dx}{S^{1/2}(x) B^2(x)}.$$

In our elementary example above, $\lambda(x) \equiv \lambda$, $\Lambda = (\text{const.})\lambda$. Our approximate solutions will satisfy (1) modulo a large negative power of Λ.

Another basic quantity is $\sigma(\beta) = \int_{x_{\text{left}}}^{x_{\text{right}}} \frac{\Lambda\, dy}{\lambda(y) S^\beta(y) B(y)}$. We note that

(Hö A) $$\sigma(\beta_1) \leq (\sigma(\beta))^{\beta_1/\beta} \quad \text{for } 0 \leq \beta_1 \leq \beta,$$

as follows from Hölder's inequality and the definition of Λ. This implies the useful inequality

(Hö B) $$\sigma(\beta_1)\sigma(\beta_2) \leq \sigma(\beta_1 + \beta_2)$$

We define also $\sigma = \min\limits_{x \in I} S(x)$.

We begin the work of constructing the approximate solutions F_{left}, F_{rt}. Let ε, N be given, and let $N' = [\varepsilon N/500]$ as in the local WKB lemma. We change notation slightly and allow the constants $c_\#$, $C_\#$ etc. to depend on ε, N as well as on c, c_1, C, C_α in (H0)...(H7). We construct an approximate solution of

$$\Big(\frac{\partial^2}{\partial z^2} + \tilde{E} - V(z)\Big)\hat{F}(z, \tilde{E}) = 0 \tag{2}$$

in a small neighborhood of the turning point x_{left}. To do so, we simply rescale the problem to reduce matters to the setting of the local WKB lemma. Specifically, set $z(\tilde{E}) =$ (solution of $V(z) = \tilde{E}$ near $z = x_{\text{left}}$) for $|\tilde{E} - E_0| < c_\# S(x_{\text{left}})$. Then define $\lambda_{\text{left}} = S^{1/2}(x_{\text{left}}) B(x_{\text{left}})$, $p(x) = -V(z)/S(x_{\text{left}})$, $E = -\tilde{E}/S(x_{\text{left}})$, where $z = x_{\text{left}} + c_\# B(x_{\text{left}}) x$.

The point is that $p(x)$ satisfies the hypotheses of the local WKB lemma, with constants depending only on c, c_1, C, C_α in (H0)...(H7). We assume $\lambda_{\text{left}} \geq C_\#$. Then the local WKB lemma produces approximate solutions $F_0(x, E, \lambda)$, $F_+(x, E, \lambda)$ to $(\frac{\partial^2}{\partial x^2} + \lambda^2(p(x) - E))F = 0$.

Recall that F_0 and F_+ are defined in terms of auxiliary functions $y(x, E, s)$, $u_k(x, E)$. Conclusion (G) of the local WKB lemma shows that y can be written as $y(x, E, s) = y_0(x, E) + s y_1(x, E, s)$ with y_0, y_1 smooth and $y_0(\frac{\partial y_0}{\partial x})^2 = p(x) - E$ to order $\geq N'$ at $x(E) =$ (solution to $p(x) = E$). We pull back F_0, F_+, y, y_0, y_1, u_k as follows. With $z = x_{\text{left}} + c_\# B(x_{\text{left}}) \cdot x$ and $E = -\tilde{E}/S(x_{\text{left}})$ as before, define

$$\hat{F}_0(z, \tilde{E}) = B^{1/2}(x_{\text{left}}) \cdot F_0(x, E, \lambda_{\text{left}})$$

$$\hat{F}_+(z, \tilde{E}) = B^{1/2}(x_{\text{left}}) \cdot F_+(x, E, \lambda_{\text{left}})$$

$$Y(z, \tilde{E}) = y(x, E, \lambda_{\text{left}}^{-2}), Y_0(z, \tilde{E}) = y_0(x, E),$$

$$Y_\#(z, \tilde{E}) = y_1(x, E, \lambda_{\text{left}}^{-2})$$

$$U_k(z, \tilde{E}) = \lambda_{\text{left}}^{-k} u_k(x, E).$$

Note that $B^{1/2}(x_{\text{left}})(\frac{\partial y}{\partial x}(x,E,\lambda_{\text{left}}^{-2}))^{-1/2} = (\frac{\partial Y(z,\tilde{E})}{\partial z})^{-1/2}$, so that $\hat{F}_0(z,\tilde{E}) = \lambda_{\text{left}}^{-1/3}(\frac{\partial Y(z,\tilde{E})}{\partial z})^{-1/2} A(\lambda_{\text{left}}^{2/3} Y(z,\tilde{E}))$. Also, $Y = Y_0 + \lambda_{\text{left}}^{-2} Y_{\#}$ and $\lambda_{\text{left}}^2 Y_0 (\frac{\partial Y_0}{\partial z})^2 = S(x_{\text{left}}) y_0 (\frac{\partial y_0}{\partial x})^2 = S(x_{\text{left}})(p(x)-E) = \tilde{E} - V(z)$ to order at least N' at $z = z(\tilde{E})$.

Regarding $\hat{F}_+$, we note that $\frac{B^{1/2}(x_{\text{left}})}{\lambda_{\text{left}}^{1/2}(p(x)-E)^{1/4}} = \frac{1}{(\tilde{E}-V(z))^{1/4}}$ and $\lambda_{\text{left}} \int_{x(E)}^{x} (p(t) - E)^{1/2}\, dt = \int_{z(\tilde{E})}^{z} (\tilde{E} - V(t))^{1/2}\, dt$. Thus,

$$\hat{F}_+(z,\tilde{E}) = \mathrm{Re}\Big[\frac{e^{\pm i\frac{\pi}{4}}\, e^{i\int_{z(\tilde{E})}^{z}(\tilde{E}-V(t))^{1/2}\,dt}}{(\tilde{E}-V(z))^{1/4}} \sum_{k=0}^{N'} U_k(z,\tilde{E})\Big].$$

Our discussion of rescaling in the previous section shows that for fixed $\tilde{E}$, $(U_k(z,\tilde{E}))$ is the canonical solution of the transport equations for the potential $\tilde{E} - V(z)$ and the turning point $z = z(E)$. Hence after an obvious change of notation, we arrive at the following rescaled version of the local WKB lemma.

Lemma 1. *Suppose $V(x)$, $S(x)$, $B(x)$ satisfy (H0)…(H7). For $|E - E_0| < c_{\#} S(x_{\text{left}})$, let $x_{\text{left}}(E)$ be the solution of $V(x) = E$ near x_{left}. Set $\lambda_{\text{left}} = S^{1/2}(x_{\text{left}}) B(x_{\text{left}})$, and assume $\lambda_{\text{left}} > C_{\#}$. Let $(u_k^{\text{left}}(x,E))$ be the canonical solution to the transport equations for the potential $E - V(x)$ and the turning point $x_{\text{left}}(E)$. Then there exist auxiliary functions $Y(x,E)$, $Y_0(x,E)$, $Y_{\#}(x,E)$ with the following properties.*

(I) *Set $F_0^{\text{left}}(x,E) = \lambda_{\text{left}}^{-1/3}(\frac{\partial Y(x,E)}{\partial x})^{-1/2} A(\lambda_{\text{left}}^{2/3} Y(x,E))$. Then for $|x - x_{\text{left}}(E)| < \lambda_{\text{left}}^{-\varepsilon} B(x_{\text{left}})$ we have $|(\frac{\partial^2}{\partial x^2} + E - V(x)) F_0^{\text{left}}| \leq C_{\#} \lambda_{\text{left}}^{-N'} B^{-3/2}(x_{\text{left}})$. Also, for $0 \leq \alpha \leq 2$ and for $-\lambda_{\text{left}}^{-\varepsilon} B(x_{\text{left}}) < x - x_{\text{left}}(E) < -\lambda_{\text{left}}^{\varepsilon - 2/3} B(x_{\text{left}})$ we have $|\partial_x^\alpha F_0^{\text{left}}| \leq C_{\#} \lambda_{\text{left}}^{-N'} B^{\frac{1}{2}-\alpha}(x_{\text{left}})$.*

(II) *Set $F_+^{\text{left}}(x,E) = Re\Big[\frac{e^{\pm i\frac{\pi}{4}}\, e^{i\int_{x_{\text{left}}(E)}^{x}(E-V(t))^{1/2}\,dt}}{(E-V(x))^{1/4}} \sum_{k=0}^{N'} u_k^{\text{left}}(x,E)\Big]$. Then for $\lambda_{\text{left}}^{\varepsilon-2/3} B(x_{\text{left}}) < x - x_{\text{left}}(E) < c_{\#} B(x_{\text{left}})$ we have $|(\frac{\partial^2}{\partial x^2} + E - V(x)) F_+^{\text{left}}| \leq C_{\#} \lambda_{\text{left}}^{10} (\lambda_{\text{left}}^{2/3}(x - x_{\text{left}}(E))/B(x_{\text{left}}))^{-\frac{3}{2}N'} B^{-\frac{3}{2}}(x_{\text{left}})$. Also, for $0 \leq \alpha \leq 2$ and $\lambda_{\text{left}}^{\varepsilon-2/3} B(x_{\text{left}}) < x - x_{\text{left}}(E) < \lambda_{\text{left}}^{-\varepsilon} B(x_{\text{left}})$ we have $|\partial_x^\alpha \{F_0^{\text{left}} - F_+^{\text{left}}\}| \leq C_{\#} \lambda_{\text{left}}^{10} (\lambda_{\text{left}}^{2/3}(x - x_{\text{left}}(E))/B(x_{\text{left}}))^{-\frac{3}{2}N'} B^{\frac{1}{2}-\alpha}(x_{\text{left}})$.*

(III) *We have* $Y(x,E) = Y_0(x,E) + \lambda_{\text{left}}^{-2} Y_\#(x,E)$ *with* $|\partial_x^\alpha \partial_E^\beta (Y_0, Y_\#)| \leq C_\#^{\alpha\beta} B^{-\alpha}(x_{\text{left}}) S^{-\beta}(x_{\text{left}})$, *(all* α, β*) for* $|x - x_{\text{left}}(E)| < c_\# B(x_{\text{left}})$ *and* $\lambda_{\text{left}}^2 Y_0 (\frac{\partial Y_0}{\partial x})^2 = E - V(x)$ *to order at least* N' *at* $x = x_{\text{left}}(E)$*. For* $|x - x_{\text{left}}(E)| < \lambda_{\text{left}}^{-\varepsilon} B(x_{\text{left}})$ *we have* $|\lambda_{\text{left}}^2 Y (\frac{\partial Y}{\partial x})^2 + \{Y, x\} - (E - V(x))| \leq C_\# \lambda^{-N'} S(x_{\text{left}})$.

(IV) *The functions* $u_k^{\text{left}}(x,E)$ *may be written as* $u_k^{\text{left}}(x,E) = \lambda_{\text{left}}^{-k} (\frac{x - x_{\text{left}}(E)}{B(x_{\text{left}})})^{-\frac{3}{2}k} \cdot f_k((\frac{x - x_{\text{left}}(E)}{B(x_{\text{left}})})^{1/2}, \frac{E - E_0}{S(x_{\text{left}})})$ *with* $|\partial_y^\alpha \partial_s^\beta f_k(y,s)| \leq C_\#^{\alpha\beta}$ *(all* α, β*) for* $|y|$, $|s| < c_\#$.

(V) *We have* $u_0(x,E) \equiv 1$, *and*

$$u_1(x,E) = \lim_{\delta \to 0+} \Big[\int_{x_{\text{left}}(E)+\delta}^{x} \Big\{ \frac{5i}{32} \frac{(V')^2}{(E-V)^{5/2}} + \frac{i}{8} \frac{V''}{(E-V)^{3/2}} \Big\} dt - \sum_{\ell=1}^{3} q_\ell(E) \delta^{-\ell/2} \Big]$$

with $q_\ell(E)$ *uniquely determined by demanding the finiteness of the limit.*

Remark. The subscript in $Y_\#$ has nothing to do with our convention for constants $C_\#$. Note the signs in (V), which are correct for the potential $E - V(x)$.

Thus, we have succeeded in finding a good approximate solution of our ODE (1) in $U = \{|E - E_0| < c_\# S(x_{\text{left}}),\ -\lambda_{\text{left}}^{-\varepsilon} B(x_{\text{left}}) < x - x_{\text{left}}(E) < c_\# B(x_{\text{left}})\}$. We simply patch together the two solutions F_0^{left}, F_+^{left}, with a partition of unity. The intersection of U with a line $E = \text{const.}$ is the interval which we called J_{left} at the beginning of this section.

Our next task is to continue the approximate solution of (1) into a larger domain by extending the canonical solution of the transport equations. Thus define $U_{\text{center}} = \{|E - E_0| < c_\#^{(0)} \sigma,\ x_{\text{left}} + c_\#^{(1)} B(x_{\text{left}}) < x < x_{\text{rt}} - c_\#^{(1)} B(x_{\text{rt}})\}$. The intersection of U_{center} with a line $E = \text{const.}$ is an interval which we call I_{center}. If we first take $c_\#^{(1)}$ small enough and then take $c_\#^{(0)}$ small depending on $c_\#^{(1)}$, then the following properties hold: $I_{\text{center}} \cap J_{\text{left}} \neq \emptyset$ if $|E - E_0| < c_\#^{(0)} \sigma$. Also $x - x_{\text{left}}(E) > c_\# B(x_{\text{left}})$

for $|E-E_0| < c_{\#}^{(0)}\sigma$ and $x \in J_{\text{left}} \cap I_{\text{center}}$. Hence for such (x,E) we have from Lemma 1 (IV) the estimates:

$$(3)\quad |\partial_x^\alpha \partial_E^\beta u_k^{\text{left}}(x,E)| \le C_{\#}^{\alpha,\beta} \lambda_{\text{left}}^{-k} B^{-\alpha}(x_{\text{left}}) S^{-\beta}(x_{\text{left}})$$

$$(\text{all } \alpha,\beta)\ (x,E) \in U \cap U_{\text{center}}.$$

We continue $(u_k^{\text{left}}(x,E))$ to a solution of the transport equations in U_{center}. This can be done, as follows by inspection of the transport equations and the fact that $E - V(x) \ge E_0 - V(x) - c_{\#}^{(0)}\sigma \ge c_{\#} S(x)$ in U_{center}. We must estimate $u_k^{\text{left}}(x,E)$ and its derivatives in U_{center}. This is accomplished by the following result.

Lemma 2. *In U_{center} we have $|\partial_x^\alpha \partial_E^\beta u_k^{\text{left}}(x,E)| \le C_{\#}^{\alpha\beta} \Lambda^{-k} B^{-\alpha}(x)\sigma(\beta)$.*

Proof. Cover I_{center} by intervals $I_\nu = \{x \in I \mid |x - x_\nu| < b_\nu\}$, $1 \le \nu \le \nu_{\max}$, with $b_\nu \sim B(x_\nu)$. We may arrange the b_ν, x_ν so that $x_{\nu+1} - x_\nu \sim B(x_\nu)$, $I_\nu \cap I_{\nu+1} \neq \emptyset$, but $I_\mu \cap I_\nu = \emptyset$ for $|\mu - \nu| > 1$. In each I_ν, we take $(u_k^\nu(x,E))_{0 \le k \le N'}$ to be the solution of the transport equations

$$(4)\qquad u_0^\nu(x,E) \equiv 1$$

$$(4)\quad 2i\frac{\partial u_{k+1}^\nu(x,E)}{\partial x} + \Big(\frac{5}{16}\frac{(V'(x))^2}{(E-V(x))^{5/2}} + \frac{1}{4}\frac{V''(x)}{(E-V(x))^{3/2}}\Big) u_k^\nu(x,E) + \\ + \frac{1}{2}\frac{V'(x)}{(E-V(x))^{3/2}}\frac{\partial}{\partial x} u_k^\nu(x,E) + \frac{1}{(E-V(x))^{1/2}}\frac{\partial^2}{\partial x^2} u_k^\nu(x,E) = 0$$

for $x \in I_\nu$, $|E - E_0| < c_{\#}^{(0)}\sigma$, with boundary conditions $u_k^\nu(x_\nu, E) = 0$ for $k \ge 1$. In this region we have $E - V(x) \ge E_0 - V(x) - c_{\#}^{(0)}\sigma \ge c_{\#} S(x_\nu)$. Therefore the transport equations take the form

$$u_0^\nu \equiv 1, \frac{\partial u_{k+1}^\nu}{\partial x} = \sum_{\ell=0}^{2} P_\ell(x,E)(\frac{\partial}{\partial x})^\ell u_k^\nu,\ u_{k+1}^\nu = 0 \quad \text{at} \quad x = x_\nu,$$

with $|\partial_x^\alpha \partial_E^\beta P_\ell| \le C_{\#}^{\alpha\beta} S^{-\frac{1}{2}-\beta}(x_\nu) B^{-2+\ell-\alpha}(x_\nu)$ all α,β for $x \in I_\nu$, $|E-E_0| < c_{\#}^{(0)}\sigma$. Hence an easy induction on k shows that

$$(5)\qquad |\partial_x^\alpha \partial_E^\beta u_k^\nu(x,E)| \le C_{\#}^{\alpha\beta} \lambda^{-k}(x_\nu) B^{-\alpha}(x_\nu) S^{-\beta}(x_\nu) \quad (\text{all } \alpha,\beta)$$

for $x \in I_\nu$, $|E - E_0| < c_{\#}^{(0)}\sigma$.

Next we relate $(u_k^\nu(x,E))$ to $(u_k^{\nu+1}(x,E))$ on $I_\nu \cap I_{\nu+1}$. Lemma 6 in the previous section shows that

$$u_k^\nu(x,E) = \sum_{\ell=0}^{k} h_\ell^\nu(E) u_{k-\ell}^{\nu+1}(x,E) \tag{6}$$

for uniquely determined $(h_\ell^\nu(E))$ with $h_0^\nu(E) \equiv 1$. In particular, $h_k^\nu(E) = u_k^\nu(x,E) - \sum_{\ell=0}^{k-1} h_\ell^\nu(E) u_{k-\ell}^{\nu+1}(x,E)$ on $I_\nu \cap I_{\nu+1}$. Hence an easy induction on k using (5) shows that

$$|\partial_E^\beta h_k^\nu(E)| \le C_{\#}^\beta \lambda^{-k}(x_\nu) S^{-\beta}(x_\nu) \quad \text{for } |E - E_0| < c_{\#}^{(0)}\sigma, \quad \text{all } \beta. \tag{7}$$

Using (6), we can patch together the $(u_k^\nu(x,E))$ for $1 \le \nu \le \nu_{\max}$ into a solution $(U_k(x,E))$ of the transport equations on U_{center}. We define

$$U_k(x,E) = \sum_{\ell=0}^{k} H_\ell^{1\nu}(E) u_{k-\ell}^\nu(x,E) \quad \text{for } x \in I_\nu, |E - E_0| < c_{\#}^{(0)}\sigma, \tag{8}$$

with

$$H_\ell^{\mu\nu}(E) = \sum_{\ell_\mu + \ell_{\mu+1} + \ldots + \ell_{\nu-1} = \ell} \prod_{j=\mu}^{\nu-1} h_{\ell_j}^j(E) \quad \text{for } \mu < \nu$$

$$H_\ell^{\mu\nu}(E) = \begin{cases} 1 & \text{for } \ell = 0 \\ 0 & \text{for } \ell \ne 0 \end{cases} \quad \text{for } \mu = \nu.$$

On each region $\Omega_\nu = \{x \in I_\nu, |E - E_0| < c_{\#}^{(0)}\sigma\}$, equation (8) defines a solution of the transport equations. Moreover, our descriptions of $U_k(x,E)$ on Ω_ν and on $\Omega_{\nu+1}$ agree on $\Omega_\nu \cap \Omega_{\nu+1}$ by virtue of (6) and the definition of $H_\ell^{\mu\nu}(E)$.

Next we estimate the global solution $U_k(x,E)$. To do so, we use the inequality

$$\sum_{\nu=1}^{\nu_{\max}} \frac{\Lambda}{\lambda(x_\nu) S^\beta(x_\nu)} \le C_{\#}\sigma(\beta), \tag{9}$$

which follows from the definition of $\sigma(\beta)$ and the geometry of the x_ν. We begin with the size of the $H_\ell^{\mu\nu}$. We have for $\mu < \nu$:

$$\sum_{\ell=0}^{N'} \Lambda^\ell |H_\ell^{\mu\nu}(E)| \le \sum_{\ell_\mu + \ldots + \ell_{\nu-1} \le N'} \prod_{j=\mu}^{\nu-1} \Lambda^{\ell_j} |h_{\ell_j}^j(E)|$$

$$\le \prod_{j=\mu}^{\nu-1} \Big(\sum_{\ell=0}^{N'} \Lambda^\ell |h_\ell^j(E)| \Big). \tag{10}$$

For each j we have $\sum_{\ell=0}^{N'} \Lambda^\ell |h^j_\ell(E)| \le 1 + \sum_{\ell=1}^{N'} C_\# \Lambda^\ell \lambda^{-\ell}(x_j)$ by (7), and the right–hand side is less than or equal to $\exp(C_\# \Lambda/\lambda(x_j))$. Putting this into (10), we find that

$$\sum_{\ell=0}^{N'} \Lambda^\ell |H^{\mu\nu}_\ell(E)| \le \exp\Big(\sum_{j=\mu}^{\nu-1} \frac{C_\# \Lambda}{\lambda(x_j)}\Big).$$

The right–hand side is at most $C_\#$ by (9) with $\beta = 0$ and therefore

$$(11) \qquad |H^{\mu\nu}_\ell(E)| \le C_\# \Lambda^{-\ell} \quad (0 \le \ell \le N') \quad \text{for } |E - E_0| < c^{(0)}_\# \sigma, \mu < \nu.$$

Obviously, (11) holds also for $\mu = \nu$.

Next we control the derivatives of $H^{\mu\nu}_\ell$. We will show that

$$(12) \qquad |\partial^\beta_E H^{\mu\nu}_k(E)| \le C^\beta_\# \Lambda^{-k} \sigma(\beta) \quad (\text{all } \beta)$$

$$\text{for } 0 \le k \le N', |E - E_0| < c^{(0)}_\# \sigma, \mu \le \nu.$$

We use induction on β. The case $\beta = 0$ is already known. We assume (12) for $0 \le \beta \le \overline{\beta}$ and then prove (12) for $\beta = \overline{\beta} + 1$. The case $\mu = \nu$ is trivial, so suppose $\mu < \nu$. Differentiating the sum of products defining $H^{\mu\nu}_\ell$, we get

$$(13) \qquad (\frac{\partial}{\partial E}) H^{\mu\nu}_k(E) = \sum_{j=\mu}^{\nu-1} \sum_{k_1+k_2+k_3=k} H^{\mu j}_{k_1}(E) \Big[\frac{\partial h^j_{k_2}(E)}{\partial E}\Big] H^{j+1\nu}_{k_3}(E), \quad \text{and}$$

therefore

$$(14) \quad (\partial^{\overline{\beta}+1}_E H^{\mu\nu}_k) = \sum_{j=\mu}^{\nu-1} \sum_{\substack{k_1+k_2+k_3=k \\ \beta_1+\beta_2+\beta_3=\overline{\beta}}} \text{coeff } (\beta_1\beta_2\beta_3)(\partial^{\beta_1}_E H^{\mu j}_{k_1})(\partial^{\beta_2+1}_E h^j_{k_2})(\partial^{\beta_3}_E H^{j+1\nu}_{k_3}).$$

We use inductive hypothesis (12) together with (7) to estimate the terms on the right of (14). Note that we may restrict the sum in (14) to $k_2 > 0$, since for $k_2 = 0$ we have $h^j_{k_2}(E) \equiv 1$ so that $\partial^{\beta_2+1}_E h^j_{k_2} \equiv 0$. Hence from (7), (14) and inductive hypothesis we get

$$|\partial^{\overline{\beta}+1}_E H^{\mu\nu}_k| \le C^{\overline{\beta}}_\# \sum_{j=\mu}^{\nu-1} \sum_{\substack{k_1+k_2+k_3=k \\ \beta_1+\beta_2+\beta_3=\overline{\beta} \\ k_2>0}} \Lambda^{-k_1} \sigma(\beta_1) \lambda^{-k_2}(x_j) S^{-\beta_2-1}(x_j) \Lambda^{-k_3} \sigma(\beta_3).$$

Since $\lambda(x_\nu) \geq \Lambda$, this implies by virtue of (9) and (Hö B) that

$$|\partial_E^{\overline{\beta}+1} H_k^{\mu\nu}| \leq \sum_{\beta_1+\beta_2+\beta_3=\overline{\beta}} C_\#^{\overline{\beta}} \Lambda^{-k} \sigma(\beta_1)\sigma(\beta_3) \sum_{j=\mu}^{\nu-1} \Big(\sum_{k_2=1}^{k} (\frac{\Lambda}{\lambda(x_j)})^{k_2} \Big) S^{-(\beta_2+1)}(x_j)$$

$$\leq C_\#^{\overline{\beta}} \Lambda^{-k} \sum_{\beta_1+\beta_2+\beta_3=\overline{\beta}} \sigma(\beta_1)\sigma(\beta_3) \sum_{j=\mu}^{\nu-1} \frac{\Lambda}{\lambda(x_j) S^{\beta_2+1}(x_j)}$$

$$\leq \frac{C_\#^{\overline{\beta}}}{\Lambda^k} \sum_{\beta_1+\beta_2+\beta_3=\overline{\beta}} \sigma(\beta_1)\sigma(\beta_2+1)\sigma(\beta_3) \leq \frac{C_\#}{\Lambda^k} \sigma(\overline{\beta}+1)$$

Thus we have proven (12) for $\beta = \overline{\beta}+1$, completing the induction step. So (12) is proven for all β.

Now from (5), (8), (12), (Hö B) we obtain easily the estimate

$$|\partial_x^\alpha \partial_E^\beta U_k(x,E)| \leq C_\#^{\alpha\beta} \Lambda^{-k} B^{-\alpha}(x)\sigma(\beta) \quad \text{for } (x,E) \in U_{\text{center}}. \tag{15}$$

To complete the proof of Lemma 2, we relate $u_k^{\text{left}}(x,E)$ to $U_k(x,E)$. By Lemma 6 in the previous section, we have

$$u_k^{\text{left}}(x,E) = \sum_{\ell=0}^{k} h_\ell(E) U_{k-\ell}(x,E) \quad \text{in } U_{\text{center}} \tag{16}$$

for a unique set of $(h_\ell(E))$ with $h_0(E) \equiv 1$.

To estimate $h_\ell(E)$, we take for each E some x_0 so that $(x_0,E) \in U \cap U_{\text{center}}$. Estimate (3) applies at (x_0,E), from which we deduce the weaker estimate

$$|\partial_E^\beta u_k^{\text{left}}(x_0,E)| \leq C_\#^\beta \Lambda^{-k} \sigma(\beta). \tag{17}$$

Since $h_k(E) = u_k^{\text{left}}(x_0,E) - \sum_{\ell=0}^{k-1} h_\ell(E) U_{k-\ell}(x_0,E)$, an obvious induction on k, using (15) and (17) and (Hö B), implies the estimate

$$|\partial_E^\beta h_k(E)| \leq C_\#^\beta \Lambda^{-k} \sigma(\beta) \quad \text{for } |E-E_0| < c_\#^{(0)}\sigma, \text{ all } \beta. \tag{18}$$

From (15), (16), (18) and (Hö B) we get

$$|\partial_x^\alpha \partial_E^\beta u_k^{\text{left}}(x,E)| \leq C_\#^{\alpha\beta} \Lambda^{-k} B^{-\alpha}(x)\sigma(\beta) \quad \text{for } (x,E) \in U_{\text{center}},\ 0 \leq k \leq N', \quad \text{all } \alpha,\beta.$$

This is the conclusion of Lemma 2. ■

It is now trivial to show that

$$F_+^{\text{left}}(x,E) = \text{Re}\Big[\frac{e^{\pm i\frac{\pi}{4}}\, e^{i\int_{x_{\text{left}}(E)}^{x}(E-V(t))^{1/2}\,dt}}{(E-V(x))^{1/4}}\sum_{k=0}^{N'} u_k^{\text{left}}(x,E)\Big]$$

approximately solves the ODE (1) in U_{center}. The result is as follows.

Corollary.

$$|(\frac{\partial^2}{\partial x^2}+E-V(x))F_+^{\text{left}}(x,E)| \le C_\# \Lambda^{-N'} S^{-1/4}(x) B^{-2}(x) \quad in \quad U_{\text{center}}.$$

Proof. A formal calculation using the transport equations gives

$$(19) \qquad (\frac{\partial^2}{\partial x^2}+E-V(x))F_+^{\text{left}} = \text{Re}\Big[e^{\pm i\pi/4} e^{i\int_{x_{\text{left}}(E)}^{x}(E-V(t))^{1/2}\,dt} \cdot (E-V(x))^{1/4}\mathcal{L}u_{N'}(x,E)\Big]$$

with $\mathcal{L} = \left(\frac{5}{16}\frac{(V')^2}{(E-V)^{5/2}} + \frac{1}{4}\frac{V''}{(E-V)^{3/2}}\right) + \frac{1}{2}\frac{V'}{(E-V)^{3/2}}\frac{\partial}{\partial x} + \frac{1}{(E-V)^{1/2}}\frac{\partial^2}{\partial x^2}$. Since only negative powers of $(E-V)$ appear in $(E-V)^{1/4}\mathcal{L}$, and since $E-V(x) > c_\# S(x)$ (as we saw in the proof of Lemma 2), we see that $(E-V)^{1/4}\mathcal{L} = \sum_{\ell=0}^{2} \hat{P}_\ell(x,E)(\frac{\partial}{\partial x})^\ell$ with $|\hat{P}_\ell| \le C_\# S^{-1/4}(x)B^{-2+\ell}(x)$ in U_{center}. Therefore $|(E-V)^{1/4}\mathcal{L}u_{N'}| \le \sum_{\ell=0}^{2} C_\# S^{-1/4}(x)B^{-2+\ell}(x)|\partial_x^\ell u_{N'}(x,E)| \le \sum_{\ell=0}^{2} C_\# S^{-1/4}(x)B^{-2+\ell}(x)\cdot C_\#\Lambda^{-N'}B^{-\ell}(x)$ by Lemma 2. That is, $|(E-V)^{1/4}\mathcal{L}u_{N'}(x,E)| \le C_\#\Lambda^{-N'}S^{-1/4}(x)B^{-2}(x)$ on U_{center}. The Corollary now follows at once from (19). ■

We have carried out the task of extending $F_+^{\text{left}}(x,E)$ to an approximate solution of (1) in U_{center}. Completely analogous to our solution of (1) is another approximate solution defined near $x = x_{\text{rt}}(E)$. This solution has the form $F_0^{\text{rt}}(x,E) = \lambda_{\text{rt}}^{-1/3}\left(\frac{-\partial Y^{\text{rt}}(x,E)}{\partial x}\right)^{-1/2} A(\lambda_{\text{rt}}^{2/3}Y^{\text{rt}}(x,E))$ for $|x - x_{\text{rt}}(E)| < \lambda_{\text{rt}}^{-\varepsilon}B(x_{\text{rt}})$, $|E-E_0| < c_\# S(x_{\text{rt}})$, with $\lambda_{\text{rt}} = S^{1/2}(x_{\text{rt}})B(x_{\text{rt}})$ and a suitable function $Y^{\text{rt}}(x,E)$. For $-c_\# B(x_{\text{rt}}) < x - x_{\text{rt}}(E) < -\lambda_{\text{rt}}^{\varepsilon-2/3}B(x_{\text{rt}})$, $|E-E_0| < c_\# S(x_{\text{rt}})$ we have

$$F_+^{\text{rt}}(x,E) = \text{Re}\Big[\frac{e^{\mp i\frac{\pi}{4}}\, e^{-i\int_{x}^{x_{\text{rt}}(E)}(E-V(t))^{1/2}\,dt}}{(E-V(x))^{1/4}}\sum_{k=0}^{N'} u_k^{\text{rt}}(x,E)\Big]$$

where, in an obvious sense $(u_k^{\mathrm{rt}}(x,E))$ is the canonical solution of the transport equations for the potential $E - V(x)$ and the turning point $x_{\mathrm{rt}}(E)$. As in the preceding lemma, $(u_k^{\mathrm{rt}}(x,E))$ continues to a solution of the transport equations in U_{center}, and the function $F_+^{\mathrm{rt}}(x,E)$ approximately solves the ODE (1) there. In particular $u_0^{\mathrm{rt}}(x,E) \equiv 1$ and

$$(20) \quad u_1^{\mathrm{rt}}(x,E) = -\lim_{\delta \to 0+} \Big[\int_x^{x_{\mathrm{rt}}(E)-\delta} \Big\{ \frac{5i}{32} \frac{(V')^2}{(E-V)^{5/2}} + \frac{i}{8} \frac{V''}{(E-V)^{3/2}} \Big\} dt - \sum_{\ell=1}^{3} q_\ell^{\mathrm{rt}}(E) \delta^{-\ell/2} \Big]$$

with $q_\ell^{\mathrm{rt}}(E)$ uniquely specified by demanding the finiteness of the limit.

Our next task is to compare $(u_k^{\mathrm{left}}(x,E))$ with $(u_k^{\mathrm{rt}}(x,E))$ in U_{center}. Both solve the same transport equations, so by Lemma 6 in the previous section, they are related by the equation

$$(21) \qquad u_k^{\mathrm{left}}(x,E) = \sum_{\ell=0}^{k} G_\ell(E) u_{k-\ell}^{\mathrm{rt}}(x,E) \quad (x,E) \in U_{\mathrm{center}}$$

for uniquely determined coefficients $(G_\ell(E))_{0 \le \ell \le N'}$ with $G_0(E) \equiv 1$. We can estimate $G_\ell(E)$ by using Lemma 2 and its analogue for $u_k^{\mathrm{rt}}(x,E)$. In particular, we have

$$(22) \qquad |\partial_E^\beta u_k^{\mathrm{left}}|, |\partial_E^\beta u_k^{\mathrm{rt}}| \le C_\#^\beta \Lambda^{-k} \sigma(\beta) \quad \text{in } U_{\mathrm{center}}, \quad \text{all } \beta.$$

Since $G_k(E) = u_k^{\mathrm{left}}(x,E) - \sum_{\ell=0}^{k-1} G_\ell(E) u_{k-\ell}^{\mathrm{rt}}(x,E)$, an obvious induction on k using (22), (Hö B) shows that

$$(23) \qquad |\partial_E^\beta G_k(E)| \le C_\#^\beta \Lambda^{-k} \sigma(\beta) \quad \text{for } |E - E_0| < c_\#^{(0)} \sigma, \quad \text{all } \beta.$$

As a simple but important consequence of (23), we have

$$(24) \qquad \Big|\sum_{k=1}^{N'} G_k(E)\Big| < \frac{1}{10} \quad \text{for } |E - E_0| < c_\#^{(0)} \sigma,$$

provided we have $\Lambda \geq C_{\#}$.

We need an explicit formula for $G_1(E)$. Since $u_0^{\text{left}} \equiv u_0^{\text{rt}} \equiv 1$ and $G_0(E) \equiv 1$, equation (21) for $k = 1$ becomes

$$G_1(E) = u_1^{\text{left}}(x, E) - u_1^{\text{rt}}(x, E), \quad (x, E) \in U_{\text{center}}.$$

Using the known formulas (20) and Lemma 1 (IV) for u_1^{left}, u_1^{rt}, we obtain

$$(25) \quad G_1(E) = \lim_{\delta \to 0+} \Big[\int_{x_{\text{left}}(E)+\delta}^{x_{\text{rt}}(E)-\delta} \Big\{ \frac{5i}{32} \frac{(V')^2}{(E-V)^{5/2}} + \frac{i}{8} \frac{V''}{(E-V)^{3/2}} \Big\} dt - \sum_{\ell=1}^{3} q_\ell^{\text{total}}(E) \delta^{-\ell/2} \Big]$$

with $q_\ell^{\text{total}}(E)$ uniquely specified by demanding the finiteness of the limit. Formula (25) can be simplified, using the elementary identity $\left(\frac{5i}{48} \frac{V'}{(E-V)^{3/2}}\right)' = \frac{5i}{32} \frac{(V')^2}{(E-V)^{5/2}} + \frac{5i}{48} \frac{V''}{(E-V)^{3/2}}$, which integrates to

$$(26) \qquad \int_{x_{\text{left}}(E)+\delta}^{x_{\text{rt}}(E)-\delta} \Big\{ \frac{5i}{32} \frac{(V')^2}{(E-V)^{5/2}} + \frac{5i}{48} \frac{V''}{(E-V)^{3/2}} \Big\} dt = \frac{5i}{48} \frac{V'(x)}{((E-V)(x))^{3/2}} \Big|_{x=x_{\text{left}}(E)+\delta}^{x=x_{\text{rt}}(E)-\delta}.$$

For $x = x_{\text{left}}(E) + \delta$ we have $\frac{V'(x)}{(E-V(x))^{3/2}} = \delta^{-3/2} \Big[\big(\frac{E-V(x)}{\delta} \big)^{-3/2} V'(x) \Big]$, and the quantity in square brackets is a smooth function of δ. Hence $\frac{5i}{48} \cdot \frac{V'(x)}{(E-V(x))^{3/2}} \Big|_{x=x_{\text{left}}(E)+\delta} = a(E)\delta^{-3/2} + b(E)\delta^{-1/2} + O(\delta^{+1/2})$ for small $\delta > 0$. A similar formula holds for $x = x_{\text{rt}}(E) - \delta$, and therefore (26) implies

$$\lim_{\delta \to 0+} \Big[\int_{x_{\text{left}}(E)+\delta}^{x_{\text{rt}}(E)-\delta} \Big\{ \frac{5i}{32} \frac{(V')^2}{(E-V)^{5/2}} + \frac{5i}{48} \frac{V''}{(E-V)^{3/2}} \Big\} dt - \sum_{\ell=1}^{3} q_\ell^{\text{extra}}(E) \delta^{-\ell/2} \Big] = 0,$$

with $(q_\ell^{\text{extra}}(E))$ uniquely specified by demanding the finiteness of the limit. Subtracting this equation from (25), we obtain

$$G_1(E) = \frac{i}{48} \lim_{\delta \to 0+} \Big[\int_{x_{\text{left}}(E)+\delta}^{x_{\text{rt}}(E)-\delta} V''(t)(E - V(t))^{-3/2}\, dt - \sum_{\ell=1}^{3} q_\ell^{\text{final}}(E) \delta^{-\ell/2} \Big],$$

with $q_\ell^{\text{final}}(E)$ uniquely specified by demanding the finiteness of the limit. The integral is easily seen to be $0(\delta^{-1/2})$, so $q_\ell^{\text{final}}(E) = 0$ for $\ell = 2, 3$. Thus we obtain our basic formula for $G_1(E)$, namely

$$(27) \qquad G_1(E) = \frac{i}{48} \lim_{\delta \to 0+} \Big[\int_{x_{\text{left}}(E)+\delta}^{x_{\text{rt}}(E)-\delta} V''(t)(E - V(t))^{-3/2} \, dt - q(E)\delta^{-1/2} \Big],$$

with $q(E)$ specified by demanding the finiteness of the limit.

Now we apply equation (21) and our knowledge of the $G_k(E)$ to compare the approximate ODE solutions $F_+^{\text{left}}(x, E)$ and $F_+^{\text{rt}}(x, E)$ on their common domain U_{center}. We will see that they are real parts of complex approximate solutions $F_c^{\text{left}}(x, E)$, $F_c^{\text{rt}}(x, E)$ which differ only by a complex factor $\mathcal{A}(E)$ modulo a tiny error. In fact, define

$$F_c^{\text{left}}(x, E) = \frac{e^{\pm i \frac{\pi}{4}} \, e^{i \int_{x_{\text{left}}(E)}^{x} (E - V(t))^{1/2} \, dt}}{(E - V(x))^{1/4}} \sum_{k=0}^{N'} u_k^{\text{left}}(x, E) \text{ for } (x, E) \in U_{\text{center}}$$

$$F_c^{\text{rt}}(x, E) = \frac{e^{\mp i \frac{\pi}{4}} \, e^{-i \int_{x}^{x_{\text{rt}}(E)} (E - V(t))^{1/2} \, dt}}{(E - V(x))^{1/4}} \sum_{k=0}^{N'} u_k^{\text{rt}}(x, E) \text{ for } (x, E) \in U_{\text{center}}$$

$$\mathcal{A}_{\text{semiclassical}}(E) = \exp\{\pm i \frac{\pi}{2} + i \int_{x_{\text{left}}(E)}^{x_{\text{rt}}(E)} (E - V(t))^{1/2} dt\} \text{ for } |E - E_0| < c_{\#}^{(0)} \sigma$$

$$\mathcal{A}_{\text{correction}}(E) = \sum_{k=0}^{N'} G_k(E) = 1 + \sum_{k=1}^{N'} G_k(E) \text{ for } |E - E_0| < c_{\#}^{(0)} \sigma$$

$$\mathcal{A}(E) = \mathcal{A}_{\text{semiclassical}}(E) \cdot \mathcal{A}_{\text{correction}}(E) \text{ for } |E - E_0| < c_{\#}^{(0)} \sigma.$$

Note that $|\mathcal{A}_{\text{semiclassical}}(E)| = 1$ and $|1 - \mathcal{A}_{\text{correction}}(E)| < \frac{1}{10}$, by (24). Evidently, $F_+^{\text{left}} = \text{Re}(F_c^{\text{left}})$ and $F_+^{\text{rt}} = \text{Re}(F_c^{\text{rt}})$ on U_{center}. We have

$$(28) \quad \mathcal{A}_{\text{correction}}(E) \cdot \sum_{k=0}^{N'} u_k^{\text{rt}}(x, E) = \sum_{k_1, k_2 = 0}^{N'} G_{k_1}(E) u_{k_2}^{\text{rt}}(x, E) =$$

$$\sum_{k=0}^{N'} u_k^{\text{left}}(x, E) + \sum_{k_1 + k_2 > N'} G_{k_1}(E) \cdot u_{k_2}^{\text{rt}}(x, E) \equiv \sum_{k=0}^{N'} u_k^{\text{left}}(x, E) + u_{\text{error}}(x, E),$$

by virtue of (21).

To estimate $u_{\text{error}}(x,E)$ we recall that $|G_{k_1}(E)| \le C_\# \Lambda^{-k_1}$ by (23), and $|\partial_x^\alpha u_{k_2}^{\text{rt}}(x,E)| \le C_\#^\alpha \Lambda^{-k_2} B^{-\alpha}(x)$ by the analogue of Lemma 2 for (u_k^{rt}). Hence $|\partial_x^\alpha \{G_{k_1}(E) u_{k_2}^{\text{rt}}(x,E)\}| \le C_\#^\alpha \Lambda^{-N'-1} B^{-\alpha}(x)$ for $k_1 + k_2 > N'$, so

$$|\partial_x^\alpha u_{\text{error}}(x,E)| \le C_\#^\alpha \Lambda^{-N'-1} B^{-\alpha}(x) \quad \text{in } U_{\text{center}}. \tag{29}$$

Thus $u_{\text{error}}(x,E)$ is extremely small.

Also

$$\mathcal{A}_{\text{semiclassical}}(E) \cdot \frac{e^{\mp i\frac{\pi}{4}}\, e^{-i\int_x^{x_{\text{rt}}(E)} (E-V(t))^{1/2}\,dt}}{(E-V(x))^{1/4}} = \frac{e^{\pm i\frac{\pi}{4}}\, e^{+i\int_{x_{\text{left}}(E)}^{x} (E-V(t))^{1/2}\,dt}}{(E-V(x))^{1/4}}.$$

Multiplying this identity by (28) and recalling the definitions of F_c^{left}, F_c^{rt}, we get

$$\begin{aligned}\mathcal{A}(E) F_c^{\text{rt}}(x,E) &= F_c^{\text{left}}(x,E) + \Big[\mathcal{A}_{\text{semiclassical}}(E) \cdot \frac{e^{\mp i\frac{\pi}{4}}\, e^{-i\int_x^{x_{\text{rt}}(E)} (E-V(t))^{1/2}\,dt}}{(E-V(x))^{1/4}} \cdot \\ &\qquad \cdot u_{\text{error}}(x,E)\Big] \\ &\equiv F_c^{\text{left}}(x,E) + F_{\text{error}}(x,E).\end{aligned} \tag{30}$$

Since $|\mathcal{A}_{\text{semiclassical}}(E)| = 1$, estimate (29) gives

$$|F_{\text{error}}(x,E)| \le C_\# \Lambda^{-N'-1} (E-V(x))^{-1/4} \quad \text{in} \quad U_{\text{center}}. \tag{31}$$

Also

$$\begin{aligned}|\frac{\partial}{\partial x} F_{\text{error}}(x,E)| &= |i(E-V(x))^{1/4} u_{\text{error}}(x,E) + \frac{1}{4}(E-V(x))^{-5/4} V'(x) \cdot \\ &\qquad \cdot u_{\text{error}}(x,E) + (E-V(x))^{-1/4} \frac{\partial}{\partial x} u_{\text{error}}(x,E)| \\ &\le (E-V(x))^{1/4}\{|u_{\text{error}}(x,E)| \cdot (1 + (E-V(x))^{-3/2}|V'(x)|) + (E-V(x))^{-1/2}|\frac{\partial}{\partial x} \cdot \\ &\qquad \cdot u_{\text{error}}(x,E)|\} \\ &\le C_\#(E-V(x))^{1/4}\{\Lambda^{-N'-1} \cdot (1 + (E-V(x))^{-3/2}|V'(x)|) + (E-V(x))^{-1/2} \cdot \\ &\qquad \cdot \Lambda^{-N'-1} B^{-1}(x)\}\end{aligned} \tag{31}$$

by (29). Since $E - V(x) \geq c_\# S(x)$ in U_{center} and $|V'(x)| \leq C_\# S(x) B^{-1}(x)$, this gives

$$|\frac{\partial}{\partial x} F_{\text{error}}(x,E)| \leq C_\# (E - V(x))^{1/4} \{\Lambda^{-N'-1} \cdot (1 + S^{-1/2}(x) B^{-1}(x))\}.$$

Since $S^{1/2}(x)B(x) = \lambda(x) \geq c_\# \Lambda > c_\#$, this implies

$$(32) \qquad |\frac{\partial}{\partial x} F_{\text{error}}(x,E)| \leq C_\# \Lambda^{-N'-1} (E - V(x))^{+\frac{1}{4}} \quad \text{in } U_{\text{center}}.$$

Estimates (31), (32) show that F_{error} is extremely small.

Note that $F_c^{\text{left}}(x,E)$ and $F_c^{\text{rt}}(x,E)$ are excellent approximate solutions of (1) in U_{center}. In fact, the proof of the Corollary to Lemma 2 shows that

$$(33) \quad |(\frac{\partial^2}{\partial x^2} + E - V(x)) F_c^{\text{left}}|, |(\frac{\partial^2}{\partial x^2} + E - V(x)) F_c^{\text{rt}}| \leq C_\# \Lambda^{-N'} S^{-1/4}(x) B^{-2}(x)$$
$$\text{in } U_{\text{center}}.$$

We have almost completely carried out the program sketched in the introduction to this section. What remains is to write the complex factor $\mathcal{A}(E)$ in the form $R(E)e^{i\Phi(E)}$. That is, we have to define a branch of the complex logarithm of $\mathcal{A}(E)$. This is quite easy. Let $\log(1+z)$ be the branch of the logarithm defined on $\{|z| < \frac{1}{10}\}$ and equal to zero when $z = 0$. Then define the complex phase

$$\Phi_c(E) = \pm i\frac{\pi}{2} + i \int_{x_{\text{left}}(E)}^{x_{\text{rt}}(E)} (E - V(t))^{1/2}\, dt + \log(1 + \sum_{k=1}^{N'} G_k(E))$$
$$\text{for } |E - E_0| < c_\#^{(0)} \sigma.$$

This makes sense by virtue of (24). Now we have $\mathcal{A}(E) = R(E)e^{i\Phi(E)}$ with $R(E) = \exp(\text{Re } \Phi_c(E))$, $\Phi(E) = \text{Im}(\Phi_c(E))$. We make some straightforward estimates on $R(E)$, $\Phi(E)$. From (23), (27) we have $|G_k(E)| \leq C_\# \Lambda^{-k}$, $\text{Re } G_1(E) = 0$. Therefore $|\text{Re } \Phi_c(E)| \leq C_\# \Lambda^{-2}$, so $|R(E) - 1| \leq C_\# \Lambda^{-2}$.

Also,

$$\Phi(E) = \pm\frac{\pi}{2} + \int_{x_{\text{left}}(E)}^{x_{\text{rt}}(E)} (E - V(t))^{1/2}\,dt + \operatorname{Im}\Big\{G_1(E) + \Big[\log(1+G_1(E)) - G_1(E)\Big] + \Big[\log\Big(1 + \frac{\sum_{k=2}^{N'} G_k(E)}{1+G_1(E)}\Big)\Big]\Big\}$$

and the two expressions in square brackets satisfy $|\partial_E^\beta[\text{Expression}]| \le C_\#^\beta \Lambda^{-2}\sigma(\beta)$ by virtue of (23). Thus $\Phi(E) = \pm\frac{\pi}{2} + \int_{x_{\text{left}}(E)}^{x_{\text{rt}}(E)} (E - V(t))^{1/2}\,dt + \operatorname{Im}\ G_1(E) + \Phi_{\text{error}}(E)$, with $|\partial_E^\beta \Phi_{\text{error}}(E)| \le C_\#^\beta \Lambda^{-2}\sigma(\beta)$. Recalling equation (27) for $G_1(E)$, we see that $\Phi(E) = \pm\frac{\pi}{2} + \int_{x_{\text{left}}(E)}^{x_{\text{rt}}(E)} (E - V(t))^{1/2}\,dt + \frac{1}{48}\lim_{\delta\to 0+}\Big[\int_{x_{\text{left}}(E)+\delta}^{x_{\text{rt}}(E)-\delta} V''(t)(E - V(t))^{-3/2}\,dt - q(E)\delta^{-1/2}\Big] + \Phi_{\text{error}}(E)$ with $q(E)$ uniquely specified by demanding the finiteness of the limit.

The Global WKB Lemma

In this section we assume the following

Hypotheses:

(H0) We are given positive numbers ε, N, and we set $N' = [\varepsilon N/500]$. On an interval I, we are given positive functions $B(x)$, $S(x)$ and a real-valued function $V(x)$. We are given a real number E_0.

(H1) If $x, y \in I$ and $|x - y| < cB(x)$, then $c < B(y)/B(x) < C$ and $c < S(y)/S(x) < C$.

(H2) For $x \in I$ and $\alpha \ge 0$ we have $|(\frac{d}{dx})^\alpha V(x)| \le C_\alpha S(x) B^{-\alpha}(x)$.

(H3) The equation $V(x) = E_0$ has two solutions $x_{\text{left}} < x_{\text{rt}}$ in I, and they satisfy $\text{dist}(x_{\text{left}}, \partial I) > cB(x_{\text{left}})$, $\text{dist}(x_{\text{rt}}, \partial I) > cB(x_{\text{rt}})$.

(H4) For $x_{\text{left}} \le x \le x_{\text{left}} + c_1 B(x_{\text{left}})$ we have $-V'(x) > cS(x_{\text{left}})B^{-1}(x_{\text{left}})$, and for $x_{\text{rt}} - c_1 B(x_{\text{rt}}) \le x \le x_{\text{rt}}$ we have $+V'(x) > cS(x_{\text{rt}})B^{-1}(x_{\text{rt}})$.

(H5) For $x_{\text{left}} + c_1 B(x_{\text{left}}) \le x \le x_{\text{rt}} - c_1 B(x_{\text{rt}})$ we have $E_0 - V(x) \ge cS(x)$.

(H6) The number Λ, defined by $\Lambda^{-1} = \int_{x_{\text{left}}}^{x_{\text{right}}} S^{-1/2}(x) B^{-2}(x)\,dx$, satisfies $\Lambda \ge$

$C^1_\#$, where $C^1_\#$ is a positive number determined entirely by ε, N, c, c_1, C and finitely many of the C_α in (H0)...(H5).

Assuming these hypotheses, we make the following

Definitions. Set $\lambda(x) = S^{1/2}(x)B(x)$, $S_{\min} = \inf_{x\in I} S(x)$, $\sigma(\beta) = \int_{x_{\text{left}}}^{x_{\text{right}}} \frac{\Lambda\, dy}{\lambda(y)S^\beta(y)B(y)}$, $B_{\text{left}} = B(x_{\text{left}})$, $S_{\text{left}} = S(x_{\text{left}})$, $\lambda_{\text{left}} = \lambda(x_{\text{left}})$, $B_{rt} = B(x_{\text{rt}})$, $S_{rt} = S(x_{\text{ rt}})$, $\lambda_{rt} = \lambda(x_{\text{rt}})$.

For $|E - E_0| < c_\# S_{\text{left}}$, define $x_{\text{left}}(E)$ to be the solution of $V(x) = E$ near x_{left}.

For $|E - E_0| < c_\# S_{\text{rt}}$, define $x_{\text{rt}}(E)$ to be the solution of $V(x) = E$ near x_{rt}.

Define regions

$$U_{\text{far left}} = \{(x,E) \mid |E-E_0| < c_\# S_{\text{left}},\ x \in I,\ x < x_{\text{left}}(E) - \lambda_{\text{left}}^{\varepsilon-2/3} B_{\text{left}}\}$$

$$U_{\text{Airey left}} = \{(x,E) \mid |E-E_0| < c_\# S_{\text{left}},\ |x - x_{\text{left}}(E)| < \lambda_{\text{left}}^{-\varepsilon} B_{\text{left}}\}$$

$$U_{\text{medium left}} = \{(x,E) \mid |E-E_0| < c_\# S_{\text{left}},\ x_{\text{left}}(E) + \lambda_{\text{left}}^{\varepsilon-2/3} B_{\text{left}} < x < x_{\text{left}}(E) + c_\# B_{\text{left}}\}$$

$$U_{\text{center}} = \{(x,E) \mid |E-E_0| < c_\#^{(0)} S_{\min},\ x_{\text{left}} + c_\#^{(1)} B_{\text{left}} < x < x_{\text{ rt}} - c_\#^{(1)} B_{\text{rt}}\}$$

$$U_{\text{medium rt}} = \{(x,E) \mid |E-E_0| < c_\# S_{\text{rt}},\ x_{\text{rt}}(E) - c_\# B_{\text{rt}} < x < x_{\text{rt}}(E) - \lambda_{\text{rt}}^{\varepsilon-2/3} B_{\text{rt}}\}$$

$$U_{\text{Airey rt}} = \{(x,E) \mid |E-E_0| < c_\# S_{\text{rt}},\ |x - x_{\text{rt}}(E)| < \lambda_{\text{rt}}^{-\varepsilon} B_{\text{rt}}\}$$

$$U_{\text{far rt}} = \{(x,E) \mid |E-E_0| < c_\# S_{\text{rt}},\ x \in I,\ x > x_{\text{rt}}(E) + \lambda_{\text{rt}}^{\varepsilon-2/3} B_{\text{rt}}\}.$$

We pick the constants $c_\#$, $c_\#^{(0)}$, $c_\#^{(1)}$ so that for $|E_1 - E_0| < c_\#^{(0)} S_{\min}$, the intervals defined by intersecting the above regions with $\{E = E_1\}$ cover I. The $c_\#$, $c_\#^{(0)}$, $c_\#^{(1)}$ depend only on ε, N, c, c_1, C and finitely many of the C_α in hypotheses (H0)...(H5).

For $|E - E_0| < c_\# S_{\min}$, let $(u_k^{\text{left}}(x,E))_{0\le k\le N'}$ be the canonical solution of the transport equations for the potential $E - V(x)$ and the turning point $x_{\text{left}}(E)$. Similarly, in the same range of E, let $(u_k^{rt}(x,E))$ be the canonical solution of the transport equations for the potential $E - V(x)$ and the turning point $x_{rt}(E)$. Then define

$$F_c^{\text{left}}(x,E) = \frac{e^{\pm i\,\frac{\pi}{4}}\, e^{i\int_{x_{\text{left}}(E)}^{x} (E-V(t))^{1/2} dt}}{(E-V(x))^{1/4}} \sum_{k=0}^{N'} u_k^{\text{left}}(x,E)$$

and

$$F_c^{\text{rt}}(x,E) = \frac{e^{\mp i\,\frac{\pi}{4}}\, e^{-i\int_{x}^{x_{\text{rt}}(E)} (E-V(t))^{1/2} dt}}{(E-V(x))^{1/4}} \sum_{k=0}^{N'} u_k^{\text{rt}}(x,E).$$

Thus, $u_k^{\text{left}}(x,E)$, $u_k^{\text{rt}}(x,E)$, $F_c^{\text{left}}(x,E)$, $F_c^{\text{rt}}(x,E)$ are all well-defined for $|E - E_0| < c_\# S_{\min}$, $x_{\text{left}}(E) < x < x_{\text{rt}}(E)$.

In terms of these definitions, our conclusions on the construction of approximate solutions of $(\frac{\partial^2}{\partial x^2} + E - V(x))F = 0$ are as follows.

Global WKB Lemma. *Assume ε, N, E_0, $V(x)$, $S(x)$, $B(x)$ are given, and that hypotheses (H0)... (H6) are satisfied. Then there are functions $Y^{\text{left}}(x,E)$ and $Y^{\text{rt}}(x,E)$ with the following properties.*

(I) Satisfying the ODE

(A) *On $U_{\text{Airey left}}$, define $F_{\text{Airey left}}(x,E) = \lambda_{\text{left}}^{-1/3}\left(\frac{\partial Y^{\text{left}}(x,E)}{\partial x}\right)^{-1/2} \cdot$ $\cdot A(\lambda_{\text{left}}^{2/3} Y^{\text{left}}(x,E))$. Then $|(\frac{\partial^2}{\partial x^2} + E - V(x))F_{\text{Airey left}}| \le C_\# \lambda_{\text{left}}^{-N'} B_{\text{left}}^{-3/2}$ on $U_{\text{Airey left}}$.*

(B) *On $U_{\text{medium left}}$ we have $|(\frac{\partial^2}{\partial x^2} + E - V(x))F_c^{\text{left}}| \le C_\# \lambda_{\text{left}}^{10}$ $\left(\frac{\lambda_{\text{left}}^{2/3}(x - x_{\text{left}}(E))}{B_{\text{left}}}\right)^{-3/2\,N'} B_{\text{left}}^{-3/2}$.*

(C) *On U_{center} we have $|(\frac{\partial^2}{\partial x^2} + E - V(x))F_c^{\text{left, right}}| \le C_\# \Lambda^{-N'} S^{-1/4}(x) B^{-2}(x)$.*

(D) *On $U_{\text{medium rt}}$ we have $|(\frac{\partial^2}{\partial x^2} + E - V(x))F_c^{\text{rt}}| \le C_\# \lambda_{\text{rt}}^{10} \cdot$ $\cdot \left(\frac{\lambda_{\text{rt}}^{2/3}(x_{\text{rt}}(E) - x)}{B_{\text{rt}}}\right)^{-\frac{3}{2}N'} \cdot B_{\text{rt}}^{-3/2}$.*

(E) *On $U_{\text{Airey rt}}$, define $F_{\text{Airey rt}}(x,E) = \lambda_{\text{rt}}^{-1/3}(-\frac{\partial Y^{\text{rt}}(x,E)}{\partial x})^{-1/2} \cdot$ $\cdot A(\lambda_{\text{rt}}^{2/3} Y^{\text{rt}}(x,E))$. Then $|(\frac{\partial^2}{\partial x^2} + E - V(x))F_{\text{Airey rt}}| \le C_\# \lambda_{\text{rt}}^{-N'} B_{\text{rt}}^{-3/2}$ on $U_{\text{Airey rt}}$.*

(II) Behavior of the Solutions

(A) *On $U_{\text{Airey left}}$ we can express Y^{left} as $Y^{\text{left}}(x,E) = Y_0^{\text{left}}(x,E) +$*

$\lambda_{\text{left}}^{-2} Y_1^{\text{left}}(x,E)$ *with* $|\partial_x^\alpha \partial_E^\beta Y_i^{\text{left}}| \leq C_\#^{\alpha\beta} B_{\text{left}}^{-\alpha} S_{\text{left}}^{-\beta}$ *and* $\lambda_{\text{left}}^2 Y_0^{\text{left}} (\frac{\partial Y_0^{\text{left}}}{\partial x})^2 = E - V(x)$ *to order* $\geq N'$ *at* $x = x_{\text{left}}(E)$. *Also* $|\lambda_{\text{left}}^2 Y^{\text{left}} (\frac{\partial Y^{\text{left}}}{\partial x})^2 + \{Y^{\text{left}}, x\} - (E - V(x))| \leq C_\# \lambda_{\text{left}}^{-N'} S_{\text{left}}$ *on* $U_{\text{Airey left}}$.

(B) *On* $U_{\text{medium left}}$ *we can express* u_k^{left} *as* $u_k^{\text{left}}(x,E) = \lambda_{\text{left}}^{-k} \left(\frac{x - x_{\text{left}}(E)}{B_{\text{left}}}\right)^{-\frac{3}{2}k} \cdot f_k^{\text{left}}\left((\frac{x - x_{\text{left}}(E)}{B_{\text{left}}})^{1/2}, \frac{E - E_0}{S_{\text{left}}} \right)$ *with* $|\partial_y^\alpha \partial_s^\beta f_k^{\text{left}}(y,s)| \leq C_\#^{\alpha\beta}$.

(C) *On* U_{center} *we have* $|\partial_x^\alpha \partial_E^\beta u_k^{\text{left, rt}}(x,E)| \leq C_\#^{\alpha\beta} \Lambda^{-k} B^{-\alpha}(x) \sigma(\beta)$.

(D) *On* $U_{\text{medium rt}}$ *we can express* $u_k^{\text{rt}}(x,E)$ *as*

$$u_k^{\text{rt}}(x,E) = \lambda_{\text{rt}}^{-k} (\frac{x_{\text{rt}}(E) - x}{B_{\text{rt}}})^{-\frac{3}{2}k} f_k^{\text{rt}}((\tfrac{x_{\text{rt}}(E) - x}{B_{\text{rt}}})^{1/2}, \tfrac{E - E_0}{S_{\text{rt}}})$$

with $|\partial_y^\alpha \partial_s^\beta f_k^{\text{rt}}(y,s)| \leq C_\#^{\alpha\beta}$.

(E) *On* $U_{\text{Airey rt}}$ *we can express* $Y^{\text{rt}}(x,E)$ *as* $Y^{\text{rt}}(x,E) = Y_0^{\text{rt}}(x,E) + \lambda_{\text{rt}}^{-2} Y_1^{\text{rt}}(x,E)$, *with* $|\partial_x^\alpha \partial_E^\beta Y_i^{\ \text{rt}}| \leq C_\#^{\alpha\beta} B_{\text{rt}}^{-\alpha} S_{\text{rt}}^{-\beta}$ *and* $\lambda_{\text{rt}}^2 \ Y_0^{\text{rt}} (\frac{\partial Y_0^{\text{rt}}}{\partial x})^2 = E - V(x)$ *to order* $\geq N'$ *at* $x = x_{\text{rt}}(E)$. *Also* $|\lambda_{\text{rt}}^2 Y^{\text{rt}} (\frac{\partial Y^{\text{rt}}}{\partial x})^2 + \{Y^{\text{rt}}, x\} - (E - V(x))| \leq C_\# \lambda_{\text{rt}}^{-N'} S_{\text{rt}}$ *on* $U_{\text{Airey, rt}}$.

(III) Patching the Solutions Together

(A) *On* $U_{\text{far left}} \cap U_{\text{Airey left}}$ *we have* $|\partial_x^\alpha F_{\text{Airey left}}| \leq C_\# \lambda_{\text{left}}^{-N'} B_{\text{left}}^{\frac{1}{2} - \alpha}$, $0 \leq \alpha \leq 1$.

(B) *On* $U_{\text{Airey left}} \cap U_{\text{medium left}}$ *we have* $|\partial_x^\alpha \{F_{\text{Airey left}} - \mathit{Re}\ F_c^{\text{left}}\}|$
$\leq C_\# \lambda_{\text{left}}^{10} \left(\frac{\lambda_{\text{left}}^{2/3}(x - x_{\text{left}}(E))}{B_{\text{left}}} \right)^{-\frac{3}{2}N'} \cdot B_{\text{left}}^{\frac{1}{2} - \alpha}$ *for* $0 \leq \alpha \leq 1$.

(C) *On* U_{center} *we can satisfy* $u_k^{\text{left}}(x,E) = \sum_{\ell=0}^{k} G_\ell(E) u_{k-\ell}^{\text{rt}}(x,E)$ *for a unique sequence* $(G_\ell(E))_{0 \leq \ell \leq N'}$ *with* $G_0(E) = 1$. *With* $\mathcal{A}(E) = \left[\exp\ i\{\pm\frac{\pi}{2} + \int_{x_{\text{left}}(E)}^{x_{\text{rt}}(E)} (E - V(t))^{1/2}\, dt\} \right] \cdot \sum_{k=0}^{N'} G_k(E)$, *we have*

$$|F_c^{\text{left}}(x,E) - \mathcal{A}(E) F_c^{\text{rt}}(x,E)| \leq C_\# \Lambda^{-N'-1} (E - V(x))^{-1/4} \quad \textit{and}$$

$$|\frac{\partial}{\partial x}\{F_c^{\text{left}}(x,E) - \mathcal{A}(E) F_c^{\text{rt}}(x,E)\}| \leq C_\# \Lambda^{-N'-1} (E - V(x))^{+1/4}$$

for $(x,E) \in U_{\text{center}}$.

We have $|\partial_E^\beta G_\ell(E)| \leq C_\#^\beta \Lambda^{-\ell} \sigma(\beta) \quad$ *for* $\quad |E - E_0| < c_\#^{(0)} S_{\min}$.

(D) *On* $U_{\text{medium rt}} \cap U_{\text{Airey rt}}$ *we have* $|\partial_x^\alpha \{F_{\text{Airey rt}} - \mathit{Re}\ F_c^{\text{rt}}\}|$
$\leq C_\# \lambda_{\text{rt}}^{10} \big(\frac{\lambda_{\text{rt}}^{2/3}(x_{\text{rt}}(E)-x)}{B_{\text{rt}}}\big)^{-\frac{3}{2}N'} \cdot B_{\text{rt}}^{\frac{1}{2}-\alpha}$ *for* $\alpha = 0, 1$.

(E) *On* $U_{\text{Airey rt}} \cap U_{\text{far rt}}$ *we have* $|\partial_x^\alpha F_{\text{Airey rt}}| \leq C_\# \lambda_{\text{rt}}^{-N'} B_{\text{rt}}^{\frac{1}{2}-\alpha}$, $0 \leq \alpha \leq 1$.

(IV) *Explicit Formulas*

(A) *For* $|E - E_0| < c_\#^{(0)} S_{\min}$ *and* $x_{\text{left}}(E) < x < x_{\text{rt}}(E)$ *we have* $u_0^{\text{left}}(x, E) = u_0^{\text{rt}}(x, E) = 1$ *and*

$$u_1^{\text{left}}(x, E) = \lim_{\delta \to 0+} \Big[\int_{x_{\text{left}}(E)+\delta}^{x} \{ \tfrac{5i}{32} \tfrac{(V')^2}{(E-V)^{5/2}} + \tfrac{i}{8} \tfrac{V''}{(E-V)^{3/2}} \}\, dt - \sum_{\ell=1}^{3} q_\ell^{\text{left}}(E) \delta^{-\ell/2} \Big]$$

$$u_1^{\text{rt}}(x, E) = - \lim_{\delta \to 0+} \Big[\int_{x}^{x_{\text{rt}}(E)-\delta} \{ \tfrac{5i}{32} \tfrac{(V')^2}{(E-V)^{5/2}} + \tfrac{i}{8} \tfrac{V''}{(E-V)^{3/2}} \}\, dt - \sum_{\ell=1}^{3} q_\ell^{\text{rt}}(E) \delta^{-\ell/2} \Big]$$

with $q_\ell^{\text{left}}(E)$, $q_\ell^{\text{rt}}(E)$ *uniquely specified by demanding the finiteness of the limit.*

(B) *For* $|E - E_0| < c_\#^{(0)} S_{\min}$ *we have* $G_0(E) \equiv 1$ *and*

$$G_1(E) = \tfrac{i}{48} \lim_{\delta \to 0+} \Big[\int_{x_{\text{left}}(E)+\delta}^{x_{\text{rt}}(E)-\delta} V''(t)(E - V(t))^{-3/2}\, dt - q(E)\delta^{-1/2} \Big]$$

with $q(E)$ *uniquely specified by demanding the finiteness of the limit.*

(C) *For* $|E - E_0| < c_\#^{(0)} S_{\min}$ *we have* $\mathcal{A}(E) = R(E) e^{i\Phi(E)}$ *with* $|R(E) - 1| < C_\# \Lambda^{-2}$ *and* $R(E)$ *real,*

$$\Phi(E) = \pm \tfrac{\pi}{2} + \int_{x_{\text{left}}(E)}^{x_{\text{rt}}(E)} (E - V(t))^{1/2}\, dt +$$
$$+ \tfrac{1}{48} \lim_{\delta \to 0+} \cdot \Big[\int_{x_{\text{left}}(E)+\delta}^{x_{\text{rt}}(E)-\delta} V''(t)(E - V(t))^{-3/2}\, dt - q(E)\delta^{-1/2} \Big] + \Phi_{\text{extra}}(E)$$

with $\Phi_{\text{extra}}(E)$ *real and satisfying* $|\partial_E^\beta \Phi_{\text{extra}}| \leq C_\#^\beta \Lambda^{-2} \sigma(\beta)$ *for all* $\beta \geq 0$.

(V) *What the Constants May Depend On*

The constants $C_\#$, $c_\#$, $C_\#^\alpha$, $C_\#^\beta$, $C_\#^{\alpha\beta}$ *in parts (I)... (IV) above depend only on* ε, N, c_1, c, C *and finitely many of the* C_α *in the hypotheses (H0)... (H5).*

Proof. All these assertions were proven in the previous section. ■

Remark. Fix ε, N, and set $N' = [\varepsilon N/500]$. Suppose $V(x)$ is any smooth potential defined on an interval I, and let E be a real number. Assume $\{x \in I \mid V(x) < E\} = (x_{\text{left}}(E), x_{\text{rt}}(E))$ for $x_{\text{left}}(E)$, $x_{\text{rt}}(E)$ in the interior of I, and assume also that $V'(x) \neq 0$ at $x = x_{\text{left}}(E)$, $x_{\text{rt}}(E)$. Then the canonical solutions $u_k^{\text{left}}(x)$, $u_k^{\text{rt}}(x)$ to the transport equations for the potential $E - V(x)$ are well-defined for $x_{\text{left}}(E) < x < x_{\text{rt}}(E)$. Hence also the coefficients $G_k(E)$ with $u_k^{\text{left}} = \sum_{\ell=0}^{k} G_\ell u_{k-\ell}^{\text{rt}}$ are well-defined, so the complex factor $\mathcal{A}(E)$ of III(C) above is well-defined. If $|\sum_{k=1}^{N'} G_k(E)| < \frac{1}{10}$, then the functions $R(E)$, $\Phi(E)$ in IV(C) are also well-defined. The point is that the definition of $R(E)$, $\Phi(E)$ is independent of the choice of weight functions $S(x)$, $B(x)$ used to describe the problem.

Eigenvalues and Eigenfunctions of Schrödinger Operators

In this section we compare the exact eigenvalues and eigenfunctions of a Schrödinger operator with the approximations developed in the preceding sections. Recall that the global WKB lemma constructs several approximate solutions of $(\frac{d^2}{dx^2} + E - V(x))F = 0$, which are mutually consistent if the phase $\Phi(E)$ is an integer multiple of π. We will prove that the eigenvalues of $H = -\frac{d^2}{dx^2} + V(x)$ are very close to the solutions of $\Phi(E) \equiv 0 \mod \pi$, and that the eigenfunctions of H are very close to constant multiples of the approximate solutions in the WKB lemma.

Thus let $V(x)$ be defined on an interval I_{BVP} (possibly $[0, \infty)$ or $\mathbb{R}^1$). We impose Dirichlet or Neumann conditions at the endpoints, so that H becomes a self–adjoint operator on L^2. Perhaps H has continuous spectrum, so we assume

(E1) The continuous spectrum of H lies entirely in $[E_\infty, +\infty)$ for a critical energy E_∞.

We fix an energy $E_0 \le E_\infty$ and a subinterval $I \subset I_{\rm BVP}$ on which $V(x)$, E_0 satisfy the hypotheses (H0)…(H6) of the global WKB lemma. We adopt the definitions and notation of the previous section, and we make the following additional assumptions.

(E2) If $|E - E_0| \le c_\#^{(0)} S_{\min}$ and $E \le E_\infty$, then $V(x) \ge E$ outside $(x_{\rm left}(E), x_{\rm rt}(E))$.

(E3) If $|E - E_0| < c_\#^{(0)} S_{\min}$ and $E \le E_\infty$ then for some $\tau \in (0,1)$ we have

$$\tau(V(x)-E)\exp(2(1-\tau)\int_{x_{\rm right}(E)}^{x} (V(t)-E)^{1/2}\,dt) \ge \lambda_{\rm right}^N \cdot \max(E-V)$$
$$\text{for } x \in I_{\rm BVP},\ x > x_{\rm right}(E) + \lambda_{\rm right}^K B_{\rm right},$$

and

$$\tau(V(x)-E)\exp(2(1-\tau)\int_{x}^{x_{\rm left}(E)} (V(t)-E)^{1/2}\,dt) \ge \lambda_{\rm left}^N \cdot \max(E-V)$$
$$\text{for } x \in I_{\rm BVP}, x < x_{\rm left}(E) - \lambda_{\rm left}^K B_{\rm left}.$$

(E4) $\max(E_0 - V) \le \lambda_{\rm right}^K S_{\rm right}$ and $\max(E_0 - V) \le \lambda_{\rm left}^K S_{\rm left}$

(E5) For $|E - E_0| \le c_\#^{(0)} S_{\min}$ we have

$$\Big[\int_{x_{\rm left}(E)}^{x_{\rm right}(E)} \frac{dx}{S^{1/2}(x)B^4(x)}\Big] \cdot \Big[\int_{x_{\rm left}(E)}^{x_{\rm right}(E)} \frac{dx}{(E-V(x))^{1/2}}\Big] \le \Lambda^K.$$

(E6) $\int_{x_{\rm left}(E_1)}^{x_{\rm right}(E_1)} \frac{dx}{(E_1 - V(x))^{1/2}} \le \Lambda^K \cdot \min(S_{\rm left}^{-1/2} B_{\rm left}, S_{\rm right}^{-1/2} B_{\rm right})$ for $|E_1 - E_0| < c_\#^0 S_{\min}$

(E7) $|E_0| \le C_\# S_{\min}$.

Here K is a large positive number. We assume that N in the global WKB lemma has been taken larger than $K\varepsilon^{-10}$. We broaden our notion of constants $C_\#$, $c_\#$, etc. to allow them to depend on τ, K, as well as on the constants in (H0)…(H5).

Assumptions (E1), (E2) are very natural, while (E3)…(E7) are technical but are satisfied in most of the examples we care about.

Before we can study the eigenvalues and eigenfunctions of H, we need a crude lower bound for the L^2–norm of the approximate ODE solutions $\mathrm{Re}(F_c^{\rm left})$,

$\mathrm{Re}(F_c^{\mathrm{right}})$ in the global WKB lemma. (After all, knowing that $(H-E)F$ is small tells us nothing if F is small also.) Such a lower bound is easily proved using a standard stationary phase inequality.

Lemma 1 (Stationary Phase). *Suppose $\theta(x)$ is supported in $\{|x-x_0|<\delta\}$ and satisfies $|(\frac{d}{dx})^m\theta(x)| \le C_m\delta^{-m}$ for $m \ge 0$. Let $\phi(x)$ be real–valued and satisfy $|(\frac{d}{dx})^m\phi(x)| \le C_m\delta^{-m} (m \ge 0)$ and $|\frac{d\phi}{dx}| \ge c\delta^{-1}$ on $\{|x-x_0|<\delta\}$. Then for λ a large real number we have*

$$|\int_{\mathbb{R}} \theta(x)e^{i\lambda\phi(x)}\,dx| \le C_K\lambda^{-K}\delta, \quad \text{any } K>0.$$

Sketch of proof. Make repeated use of the identity $e^{i\lambda\phi} = \frac{1}{i\lambda\phi'}\frac{d}{dx}e^{i\lambda\phi}$ and integration by parts. ■

Our lower bound for $\|\mathrm{Re}(F_c^{\mathrm{left}})\|$ is as follows.

Lemma 2. *Suppose $|E-E_0| < c_{\#}^{(0)}S_{\min}$ and suppose $J=\{|x-x_0|<c_{\#}B(x_0)\}$ is contained in $(x_{\mathrm{left}}(E)+c_{\#}B_{\mathrm{left}}, x_{\mathrm{right}}(E)-c_{\#}B_{\mathrm{right}})$. Then for $|\omega|=1$ we have*

$$\int_J |Re(\omega F_c^{\mathrm{left}})|^2\,dx \ge c_{\#}\int_J \frac{dx}{(E-V(x))^{1/2}}.$$

Sketch of Proof. Let $\chi(x)$ be a smooth cutoff function supported in J. We have

$$(1) \quad \int_J \chi(\mathrm{Re}(\omega F_c^{\mathrm{left}}))^2\,dx = \frac{1}{4}\int_J \chi(\omega F_c^{\mathrm{left}})^2 + \frac{1}{4}\int_J \chi(\overline{\omega F_c^{\mathrm{left}}})^2 + \frac{1}{2}\int_J \chi|F_c^{\mathrm{left}}|^2.$$

The first integral on the right is negligibly small, by virtue of the stationary phase lemma with $\theta = \frac{\chi(\sum_{k=0}^{N'} u_k^{\mathrm{left}}(x,E))^2}{(E-V(x))^{1/2}}$, $\lambda=\lambda(x_0)$, $\phi(x)=\frac{1}{\lambda(x_0)}\int_{x_0}^x (E-V(t))^{1/2}\,dt$. The second term on the right of (1) is the complex conjugate of the first term and so is also negligibly small. The third term on the right in (1) has the correct order of magnitude. ■

Corollary. $\int_{x_{\mathrm{left}}(E)+c_{\#}B_{\mathrm{left}}}^{x_{\mathrm{right}}(E)-c_{\#}B_{\mathrm{rt}}} |Re(\omega F_c^{\mathrm{left}})|^2\,dx \ge c_{\#}\int_{x_{\mathrm{left}}(E)}^{x_{\mathrm{right}}(E)} \frac{dx}{(E-V(x))^{1/2}}$.

Proof. On the right we can change the limits of integration to $x_{\text{left}}(E) + c_{\#} B_{\text{left}}$, $x_{\text{right}}(E) - c_{\#} B_{\text{right}}$ without changing the order of magnitude of the integral. The Corollary then follows by simply dividing $[x_{\text{left}}(E) + c_{\#} B_{\text{left}}, x_{\text{right}} - c_{\#} B_{\text{right}}]$ into intervals J_ν to which we can apply Lemma 2. ■

Of course we have analogous results for $\text{Re}(F_c^{\text{right}})$.

Now we begin to study the eigenfunctions of H. Thus suppose $F \in \text{Domain}(H)$, $\int_{I_{\text{BVP}}} |F|^2\, dx = 1$, F is real–valued and

$$-\frac{d^2}{dx^2} F + VF = E_1 F \quad \text{for some } E_1 \quad \text{with } |E_1 - E_0| < c_{\#}^{(0)} S_{\min}, E_1 \leq E_{\infty}.$$

We compare F with the approximate solutions in the global WKB lemma. To do so, we look separately at each of the following intervals:

$$\begin{aligned}
I_{\text{far left}} &= \{x \in I_{\text{BVP}} \mid x < x_{\text{left}}(E_1) - \lambda_{\text{left}}^{\varepsilon - 2/3} B_{\text{left}}\} \\
I_{\text{Airey left}} &= \{x \in I \mid (x, E_1) \in U_{\text{Airey left}}\} \\
I_{\text{medium left}} &= \{x \in I \mid (x, E_1) \in U_{\text{medium left}}\} \\
I_{\text{center}} &= \{x \in I \mid (x, E_1) \in U_{\text{center}}\} \\
I_{\text{medium right}} &= \{x \in I \mid (x, E_1) \in U_{\text{medium right}}\} \\
I_{\text{Airey right}} &= \{x \in I \mid (x, E_1) \in U_{\text{Airey right}}\} \\
I_{\text{far right}} &= \{x \in I_{\text{BVP}} \mid x > x_{\text{right}}(E_1) + \lambda_{\text{right}}^{\varepsilon - 2/3} B_{\text{right}}\}
\end{aligned}$$

These intervals cover I_{BVP}. After understanding how F is forced to behave on each of these intervals, we check our information for consistency on their intersections. We begin with $I_{\text{far left}}$. Our basic tool is Agmon's decay estimate, specialized to the trivial one–dimensional case.

Lemma 3 (Agmon lemma). *Define an auxiliary function* $\varphi(x)$ *as follows.*

$$\varphi(x) = \int_x^{x_{\text{left}}(E)} (V(t) - E_1)^{1/2}\, dt \quad \text{if } x \le x_{\text{left}}(E_1),$$
$$\varphi(x) = 0 \quad \text{if } x_{\text{left}}(E_1) \le x \le x_{\text{right}}(E_1),$$
$$\varphi(x) = \int_{x_{\text{right}}(E_1)}^{x} (V(t) - E_1)^{1/2}\, dt \quad \text{if } x \ge x_{\text{right}}(E_1).$$

Then we have the estimate

$$\tau \int_{I_{\text{BVP}} \setminus (x_{\text{left}}(E_1), x_{\text{right}}(E_1))} e^{2(1-\tau)\varphi} \Big[(\frac{dF}{dx})^2 + (V - E_1)F^2 \Big]\, dx \le \int_{x_{\text{left}}(E_1)}^{x_{\text{right}}(E_1)} (E_1 - V)F^2\, dx.$$

Sketch of Proof. Integrating by parts formally we get

$$0 = \int_{I_{\text{BVP}}} (e^{2(1-\tau)\varphi} F)(-\frac{d^2F}{dx^2} + (V - E_1)F)\, dx = \int_{I_{\text{BVP}}} e^{2(1-\tau)\varphi} \Big[(\frac{dF}{dx})^2 + (V - E_1)F^2 + 2(1-\tau)\frac{d\varphi}{dx}(\frac{dF}{dx})F \Big]\, dx.$$

This formal process is easily justified: We approximate φ by a smooth function assumed constant near the endpoints of I_{BVP} and pass to the limit. Outside $(x_{\text{left}}(E_1), x_{\text{right}}(E_1))$ we have $(\frac{d\varphi}{dx})^2 = (V - E_1)$, so the expression in brackets is greater than or equal to $\tau\{(\frac{dF}{dx})^2 + (V - E_1)F^2\}$. Inside $(x_{\text{left}}(E_1), x_{\text{right}}(E_1))$ we have $\varphi = \frac{d\varphi}{dx} = 0$. Therefore

$$0 \ge \int_{I_{\text{BVP}} \setminus (x_{\text{left}}(E_1), x_{\text{right}}(E_1))} \tau e^{2(1-\tau)\varphi} \Big\{ \Big(\frac{dF}{dx}\Big)^2 + (V - E_1)F^2 \Big\}\, dx - \int_{x_{\text{left}}(E_1)}^{x_{\text{right}}(E_1)} (E_1 - V)F^2\, dx,$$

which proves the lemma. ■

Let us see how Agmon's function $\varphi(x)$ behaves to the left of $x_{\text{left}}(E_1)$. We know that

$$V(x) - E_1 \ge c_\# \Big(\frac{S_{\text{left}}}{B_{\text{left}}}\Big)(x_{\text{left}}(E_1) - x) \quad \text{for } x_{\text{left}}(E_1) - c_\# B_{\text{left}} < x < x_{\text{left}}(E_1).$$

Therefore $\varphi(x) \geq c_{\#}\lambda_{\text{left}}(\frac{x_{\text{left}}(E_1)-x}{B_{\text{left}}})^{3/2}$ in that range of x. In particular, in $J_1 = \{x_{\text{left}}(E_1) - c_{\#}B_{\text{left}} < x < x_{\text{left}}(E_1) - \lambda_{\text{left}}^{\varepsilon-2/3}B_{\text{left}}\}$ we have $\varphi(x) \geq c_{\#}\lambda_{\text{left}}^{\frac{3}{2}\varepsilon}$. Since $\varphi(x)$ is decreasing in $I_{\text{far left}}$, it follows that $\varphi(x) \geq c_{\#}\lambda_{\text{left}}^{\frac{3}{2}\varepsilon}$ throughout $I_{\text{far left}}$.

So Lemma 3 implies:

(2)

$$\tau \exp(c_{\#}\lambda_{\text{left}}^{\frac{3}{2}\varepsilon}) \int_{I_{\text{far left}}} \left(\frac{dF}{dx}\right)^2 dx \leq \int_{x_{\text{left}}(E_1)}^{x_{\text{right}}(E_1)} (E_1 - V)F^2\, dx \leq \max(E_1 - V)$$

and

$$(3) \qquad \tau \exp(c_{\#}\lambda_{\text{left}}^{\frac{3}{2}\varepsilon}) \int_{x_{\text{left}}(E_1)-\hat{c}_{\#}B_{\text{left}}}^{x_{\text{left}}(E_1)-\frac{1}{2}\hat{c}_{\#}B_{\text{left}}} S_{\text{left}}F^2\, dx \leq \int_{x_{\text{left}}(E_1)}^{x_{\text{right}}(E)} (E_1 - V)F^2\, dx \leq \max(E_1 - V)$$

Since F has norm 1.

We invoke the elementary inequality

$$(4) \qquad c\int_{I_1} F^2\, dx \leq \left(\frac{|I_1|}{|I_0|}\right)^2 \int_{I_0} F^2\, dx + |I_1|^2 \int_{I_1} \left(\frac{dF}{dx}\right)^2 dx$$

for intervals $I_0 \subset I_1$, taking $I_0 = \{x_{\text{left}}(E_1) - \hat{c}_{\#}B_{\text{left}} < x < x_{\text{left}}(E_1) - \frac{1}{2}\hat{c}_{\#}B_{\text{left}}\}$ and $I_1 = I_{\text{far left}} \cap \{x > x_{\text{left}}(E) - \lambda_{\text{left}}^K B_{\text{left}}\}$.

Estimates (2), (3), (4) together imply

$$c_{\#}\int_{I_1} F^2\, dx \leq \frac{\lambda_{\text{left}}^{2K}\max(E_1 - V)}{\tau \exp(c_{\#}\lambda_{\text{left}}^{\frac{3}{2}\varepsilon})S_{\text{left}}} + \frac{\lambda_{\text{left}}^{2K}B_{\text{left}}^2\max(E_1 - V)}{\tau \exp(c_{\#}\lambda_{\text{left}}^{\frac{3}{2}\varepsilon})}$$

Hypothesis (E4) gives $\max(E_1 - V) \leq \lambda_{\text{left}}^K S_{\text{left}}$, and by definition $B_{\text{left}}^2 \cdot S_{\text{left}} = \lambda_{\text{left}}^2$. Therefore

$$(5) \qquad \int_{I_1} F^2\, dx \leq C_{\#}\lambda_{\text{left}}^{3K+2}\exp(-c_{\#}\lambda_{\text{left}}^{\frac{3}{2}\varepsilon}) \leq C'_{\#}\lambda_{\text{left}}^{-N}.$$

On the other hand, hypothesis (E3) gives $\tau e^{2\varphi}(V - E_1) \geq \lambda_{\text{left}}^N \cdot \max(E_1 - V)$ in $I_{\text{far left}} \backslash I_1$. Therefore Lemma 3 implies

$$\max(E_1 - V) \cdot \lambda_{\text{left}}^N \int_{I_{\text{far left}}\backslash I_1} F^2\, dx \leq \int_{x_{\text{left}}(E_1)}^{x_{\text{right}}(E_1)} (E_1 - V)F^2\, dx \leq \max(E_1 - V),$$

again becuase F has norm 1. Thus, $\int_{I_{\text{far left}} \setminus I_1} F^2\,dx \leq \lambda_{\text{left}}^{-N}$. Together with (5), this gives the following conclusion.

Lemma 4. *We have* $\int_{I_{\text{far left}}} F^2\,dx \leq C_\# \lambda_{\text{left}}^{-N}$.

There is of course an analogous result for $I_{\text{far right}}$.

Next we study how F behaves on $I_{\text{Airey left}}$. We need some elementary results on Airey's equation.

Lemma 5. *Suppose* $\frac{d^2u}{dy^2} + yu = f$ *in an interval* $[-T, +T]$ *with* T *greater than a large universal constant. Assume* $\int_{-T}^{T} |f|^2\,dy \leq CT^{-2M}$ *and* $\int_{-T}^{T} |u|^2\,dy \leq C$ $(M \geq 2)$*. Then there is a constant* b *of size* $|b| \leq C$ *for which* $\int_{-\frac{1}{2}T}^{+\frac{1}{2}T} |u(y) - bA(y)|^2\,dy \leq C_M T^{2-2M}$.

Sketch of Proof. First we set up a Green's function for Airey's equation with good bounds. Let $A_1(y) = A(y)$, and let $A_2(y)$ be another real solution of $\frac{d^2A_2}{dy^2} + yA_2 = 0$, taken so that the Wronskian $A_1'A_2 - A_2'A_1 \equiv \pm 1$. Both $A_1(y)$ and $A_2(y)$ remain bounded as long as $y > -C$ for some fixed constant C. As $y \to -\infty$ we have $|A_1(y)| \leq \frac{C\,e^{-\frac{2}{3}|y|^{3/2}}}{|y|^{1/4}}$, $|A_2(y)| \leq \frac{C\,e^{+\frac{2}{3}|y|^{3/2}}}{|y|^{1/4}}$. Hence the Green's function $G(x,y) = A_1(\min(x,y)) \cdot A_2(\max(x,y))$ remains bounded on all of $\mathbb{R}^2$. Using the Green's function, we construct the special solution $u_1(x) = \int_{-T}^{T} G(x,y)f(y)\,dy$ of $\frac{d^2u_1}{dy^2} + yu_1 = f$ on $[-T, T]$. Moreover,

$$\int_{-T}^{T} |u_1(y)|^2\,dy \leq CT^2 \int_{-T}^{T} |f(y)|^2\,dy \leq CT^{2-2M}$$
$$\text{since } |G(x,y)| \leq C.$$

The given solution u may be written as $u = u_1 + b_1A_1 + b_2A_2$ on $[-T, T]$ for constants b_1, b_2. Our estimate for $\|u_1\|^2$ and hypothesis on $\|u\|^2$ together imply

$$\int_{-T}^{T} |b_1A_1(y) + b_2A_2(y)|^2\,dy \leq C. \tag{6}$$

For T greater than a large universal constant, we have

$$\Big|\int_{-T}^{T} A_1(y)A_2(y)\,dy\Big| << \Big(\int_{-T}^{T} A_1^2(y)\,dy\Big)^{1/2}\Big(\int_{-T}^{T} A_2^2(y)\,dy\Big)^{1/2}.$$

In fact, the left side grows as a power of T, while the right side grows exponentially. Thus A_1 and A_2 are nearly orthogonal in $L^2[-T,T]$, so (6) implies

$$\int_{-T}^{T} |b_1 A_1(y)|^2\,dy \le C \tag{7}$$

and

$$\int_{-T}^{T} |b_2 A_2(y)|^2\,dy \le C. \tag{8}$$

Estimate (7) implies $|b_1| \le C$, while (8) implies $\int_{-T/2}^{+T/2} |b_2 A(y)|^2\,dy \le C_M T^{2-2M}$. (Again we exploit the exponential growth of $\int_{-T}^{T} |A_2(y)|^2\,dy$.)

Now $u(y) - b_1 A_1(y) = u_1(y) + b_2 A_2(y)$, and the two terms on the right both have norm squared at most $C_M T^{2-2M}$ in $L^2[-T/2,+T/2]$. Thus

$$\int_{-T/2}^{T/2} |u(y) - b_1 A(y)|^2\,dy \le C_M T^{2-2M},\ |b_1| \le C. \qquad \blacksquare$$

Corollary 1. *Suppose* $[\frac{d^2}{dy^2} + W(y)]u = 0$ *on* $[-T,T]$ *with* T *greater than a large universal constant. Suppose also* $|W(y) - y| \le CT^{-M}$ *on* $[-T,T]$ *and* $\int_{-T}^{T} |u(y)|^2\,dy \le C$*. Then for a constant* b *of size* $|b| \le C_m$ *we have*

$$\int_{-T/2}^{+T/2} |u(y) - bA(y)|^2\,dy \le C_M\,T^{2-2M}.$$

Proof. Just write $\frac{d^2u}{dy^2} + yu = [y - W(y)]u \equiv f$ and apply Lemma 5. $\blacksquare$

Corollary 2. *Suppose* $[\frac{d^2}{dy^2} + W(y)]u = 0$ *on* $J = [-c\lambda^{-\varepsilon}, +c\lambda^{-\varepsilon}]$ *with* $0 < \varepsilon < \frac{1}{10}$ *and* λ *greater than a large constant depending on* ε*. Suppose* $|W(y) - \lambda^2 y| \le C\lambda^{2-N'}$ *on* J*, and suppose* $\int_J |u|^2\,dy \le 1$*. Then for a constant* b_0 *of size* $|b_0| \le C\lambda$ *we have* $\int_{\tilde{J}} |u(y) - b_0 A(\lambda^{2/3} y)|^2\,dy \le C\lambda^{10-2N'}$*, where* $\tilde{J}$ *denotes the middle half of* J*.*

Proof. Rescale to reduce to the previous Corollary. $\blacksquare$

We apply the preceding Corollary to study our eigenfunction F on the interval $I_{\text{Airey left}}$.

Lemma 6. *For a constant b of size $|b| \le C_{\#}\lambda_{\text{left}}^{20} B_{\text{left}}^{-1}$ we have*

$$\int_{|x-x_{\text{left}}(E_1)|<c_{\#}\lambda_{\text{left}}^{-\varepsilon}B_{\text{left}}} |F(x) - bF_{\text{Airey left}}\ (x, E_1)|^2\, dx \le C_{\#}\lambda_{\text{left}}^{10-2N'}.$$

Proof. We use the change of variable $x \mapsto Y^{\text{left}}(x, E_1) \equiv y$ to reduce matters to Corollary 2 of Lemma 5. Recall that a general ODE $[\frac{d^2}{dx^2} + (E_1 - V(x))]F(x) = 0$ is transformed by a change $x \mapsto y(x)$ of independent variable to $[\frac{d^2}{dy^2} + \hat{W}(y)]\hat{F}(y) = 0$ where $F(x) = (\frac{\partial y}{\partial x})^{-1/2}\hat{F}(y(x))$ and $E_1 - V(x) = (\frac{\partial y}{\partial x})^2\hat{W}(y) + \{y, x\}$ with $y = y(x)$. We rewrite this as $(E_1 - V(x)) = \lambda_{\text{left}}^2(\frac{\partial y}{\partial x})^2 y + \{y, x\} + (\frac{\partial y}{\partial x})^2(\hat{W}(y) - \lambda_{\text{left}}^2 y)$. The global WKB lemma tells us that

$$\Big| - (E_1 - V(x)) + \lambda_{\text{left}}^2(\frac{\partial y}{\partial x})y + \{y, x\}\Big| \le C_{\#}\lambda_{\text{left}}^{-N'} S_{\text{left}},$$

for $|x - x_{\text{left}}(E_1)| < \lambda_{\text{left}}^{-\varepsilon} B_{\text{left}}$, and also that

$$|\frac{\partial y}{\partial x}| \sim B_{\text{left}}^{-1} \quad \text{for } |x - x_{\text{left}}(E_1)| < \lambda_{\text{left}}^{-\varepsilon} B_{\text{left}}.$$

Therefore for $|x - x_{\text{left}}(E_1)| < \lambda_{\text{left}}^{-\varepsilon} B_{\text{left}}$ we have $|\hat{W}(y) - \lambda_{\text{left}}^2 y| \le C_{\#}\lambda_{\text{left}}^{-N'} S_{\text{left}} B_{\text{left}}^2 = C_{\#}\lambda_{\text{left}}^{2-N'}$, with $y = y(x)$. That is, with

$$\hat{I} = \{y(x)\ \Big|\ |x - x_{\text{left}}(E_1)| < \lambda_{\text{left}}^{-\varepsilon} B_{\text{left}}\},$$

we have $|\hat{W}(y) - \lambda_{\text{left}}^2 y| \le C_{\#}\lambda_{\text{left}}^{2-N'}$ on $\hat{I}$. Also

$$\begin{aligned}
\int_{\hat{I}} |\hat{F}(y)|^2\, dy &= \int_{|x-x_{\text{left}}(E_1)|<\lambda_{\text{left}}^{-\varepsilon}B_{\text{left}}} \Big|\Big(\frac{\partial y}{\partial x}\Big)^{-1/2} \hat{F}(y(x))\Big|^2 \Big(\frac{\partial y}{\partial x}\Big)^2 dx \\
&= \int_{|x-x_{\text{left}}(E_1)|<\lambda_{\text{left}}^{-\varepsilon}B_{\text{left}}} |F(x)|^2 \Big(\frac{\partial y}{\partial x}\Big)^2 dx \\
&\le \int_{|x-x_{\text{left}}(E_1)|<\lambda_{\text{left}}^{-\varepsilon}B_{\text{left}}} |F(x)|^2\, dx \cdot C_{\#}B_{\text{left}}^{-2} \le C_{\#}B_{\text{left}}^{-2}.
\end{aligned}$$

Hence on $\hat{I}$, the function $B_{\text{left}}\ \hat{F}(y)$ has norm at most $C_\#$ and is annihilated by $\frac{d^2}{dy^2} + \hat{W}(y)$ with $|\hat{W}(y) - \lambda_{\text{left}}^2 y| \leq C_\# \lambda^{2-N'}$. Note also that $\hat{I}$ contains an interval $\hat{J} = \{|y| < c_\# \lambda_{\text{left}}^{-\varepsilon}\}$. The preceding Corollary tells us that $\int_{\hat{J}_1} |B_{\text{left}}\hat{F}(y) - b_0 A(\lambda_{\text{left}}^{2/3} y)|^2\, dy \leq C_\# \lambda_{\text{left}}^{10-2N'}$ for a constant b_0 of size $|b_0| \leq C_\# \lambda_{\text{left}}^{10}$. Here $\hat{J}_1 = \{|y| < c_\# \lambda_{\text{left}}^{-\varepsilon}\}$ for a smaller constant $c_\#$ than that used in the definition of $\hat{J}$.

Now set $\hat{I}_1 = \{|x - x_{\text{left}}(E_1)| < c_\# \lambda_{\text{left}}^{-\varepsilon} B_{\text{left}}\}$ for yet a smaller constant $c_\#$. Then the image of $\hat{I}_1$ under $x \mapsto y(x)$ is contained in $\hat{J}_1$. Therefore

$$\int_{\hat{I}_1} \left| F(x) - b_0 B_{\text{left}}^{-1} \left(\frac{\partial y}{\partial x}\right)^{-1/2} A(\lambda_{\text{left}}^{2/3} y(x)) \right|^2 dx =$$

$$\int_{\hat{I}_1} \left| \left(\frac{\partial y}{\partial x}\right)^{-1/2} \hat{F}(y(x)) - b_0 B_{\text{left}}^{-1} \left(\frac{\partial y}{\partial x}\right)^{-1/2} A(\lambda_{\text{left}}^{2/3} y(x)) \right|^2 dx$$

$$\leq \int_{\hat{I}_1} |\hat{F}(y(x)) - b_0 B_{\text{left}}^{-1} A(\lambda_{\text{left}}^{2/3} y(x))|^2 \cdot \left(\frac{\partial y}{\partial x}\right) dx \cdot (C_\# B_{\text{left}}^2)$$

(since $(\frac{\partial y}{\partial x}) \sim B_{\text{left}}^{-1}$ on $\hat{I}_1$)

$$\leq C_\# \int_{\hat{J}_1} |B_{\text{left}} \hat{F}(y) - b_0 A(\lambda_{\text{left}}^{2/3} y)|^2\, dy \leq C_\# \lambda_{\text{left}}^{10-2N'}.$$

We have by definition $b_0 B_{\text{left}}^{-1} (\frac{\partial y}{\partial x})^{-1/2} A(\lambda_{\text{left}}^{2/3} y(x))$ $= (b_0 B_{\text{left}}^{-1} \lambda_{\text{left}}^{1/3}) F_{\text{Airey left}}(x, E_1)$. Putting $b = b_0 B_{\text{left}}^{-1} \lambda_{\text{left}}^{1/3}$, we have $|b| \leq C_\# \lambda_{\text{left}}^{20} B_{\text{left}}^{-1}$ and $\int_{|x - x_{\text{left}}(E_1)| < c_\# \lambda_{\text{left}}^{-\varepsilon} B_{\text{left}}} |F(x) - bF_{\text{Airey left}}(x, E_1)|^2\, dx \leq C_\# \lambda_{\text{left}}^{10-2N'}$. The proof of the Lemma is complete. ■

Set $\hat{I}_{\text{Airey left}} = \{|x - x_{\text{left}}(E_1)| < c_\# \lambda_{\text{left}}^{-\varepsilon} B_{\text{left}}\}$, so that Lemma 6 asserts that

$$\text{(9)} \qquad \int_{\hat{I}_{\text{Airey left}}} |F(x) - bF_{\text{Airey left}}(x, E_1)|^2\, dx \leq C_\# \lambda_{\text{left}}^{10-2N'}.$$

Of course there is an analogous result asserting that

$$\text{(10)} \qquad \int_{\hat{I}_{\text{Airey rt}}} |F(x) - b' F_{\text{Airey rt}}(x, E_1)|^2\, dx \leq C_\# \lambda_{\text{rt}}^{10-2N'}$$

with $\hat{I}_{\text{Airey rt}} = \{|x - x_{\text{rt}}(E_1)| < c_\# \lambda_{\text{rt}}^{-\varepsilon} B_{\text{rt}}\}$, and another constant b' of size $|b'| \leq C_\# \lambda_{\text{rt}}^{20} B_{\text{rt}}^{-1}$. Since F is real, we can take the constants b, b' to be real.

Next we study how F behaves on $I_{\text{medium left}}$. We make use of the following result.

Lemma 7. *The equation* $(\frac{d^2}{dx^2} + E_1 - V(x))u_1 = f$ *on* $I_{\text{medium left}}$ *has a solution* u_1 *wth*

$$\|u_1\|_{L^2(I_{\text{medium left}})} \leq C_\# S_{\text{left}}^{-1/2} B_{\text{left}} \|f\|_{L^2(I_{\text{medium left}})}.$$

Proof. Let us write $F_c^{\text{left}}(x)$ for $F_c^{\text{left}}(x, E_1)$. We define an approximate Green's function

$$G(x,y) = \begin{cases} F_c^{\text{left}}(x)\overline{F_c^{\text{left}}(y)} & \text{if } x \leq y \\ \overline{F_c^{\text{left}}(x)}F_c^{\text{left}}(y) & \text{if } x \geq y \end{cases}.$$

Thus $|G(x,y)| \leq \frac{C_\#}{[(E_1-V(x))(E_1-V(y))]^{1/4}}$ on $I_{\text{medium left}} \times I_{\text{medium left}}$, while $(\partial_x^2 + E_1 - V(x))G(x,y) = H_1(y)\delta(x-y) + H(x,y)$ with $|H(x,y)| \leq C_\# \lambda_{\text{left}}^{10} \left(\frac{\lambda_{\text{left}}^{2/3}(x - x_{\text{left}}(E_1))}{B_{\text{left}}}\right)^{-\frac{3}{2}N'} B_{\text{left}}^{-3/2} (E_1 - V(y))^{-1/4}$.

These estimates are immediate from the global WKB lemma. Here $H_1(y) = (\text{const.})\text{Im}(\partial_y F_c^{\text{left}}(y, E_1) \cdot \overline{F_c^{\text{left}}(y, E_1)})$. We have therefore

$$\int_{I_{\text{medium left}} \times I_{\text{medium left}}} |H(x,y)|^2 \, dxdy \leq C_\# \lambda_{\text{left}}^{20-3\varepsilon N'} B_{\text{left}}^{-3} |I_{I_{\text{medium left}}}| \cdot$$
$$\cdot \int_{\text{medium left}} \frac{dy}{(E_1 - V(y))^{1/2}} \leq C_\# \lambda_{\text{left}}^{20-3\varepsilon N'} B_{\text{left}}^{-1} S_{\text{left}}^{-1/2} = C_\# \lambda_{\text{left}}^{19-3\varepsilon N'},$$

and

$$\int_{I_{\text{medium left}} \times I_{\text{medium left}}} |G(x,y)|^2 \, dxdy \leq C_\# \Big(\int_{I_{\text{medium left}}} \frac{dx}{(E_1 - V(x))^{1/2}}\Big)^2$$
$$\leq C_\# \Big(S_{\text{left}}^{-1/2} B_{\text{left}}\Big)^2.$$

Thus $f \mapsto \mathcal{E}f(x) = \int_{I_{\text{medium left}}} H(x,y) f(y) \, dy$ has Hilbert–Schmidt norm at most $C_\# \lambda_{\text{left}}^{+\frac{19}{2} - \frac{3}{2}\varepsilon N'}$, while $f \mapsto Gf(x) = \int_{I_{\text{medium left}}} G(x,y) f(y) \, dy$ has Hilbert–Schmidt norm at most $C_\# S_{\text{left}}^{-1/2} B_{\text{left}}$. We next study $H_1(y)$. Recall that

$$(11) \qquad F_c^{\text{left}}(y) = \frac{e^{\pm i \frac{\pi}{4}} \, e^{i \int_{x_{\text{left}}(E_1)}^{y} (E_1 - V(t))^{1/2} \, dt}}{(E_1 - V(y))^{1/4}} \sum_{k=0}^{N'} u_k^{\text{left}}(y, E_1)$$

$$(12)\quad \frac{\partial}{\partial y}F_c^{\text{left}}(y) = e^{\pm i\frac{\pi}{4}}\, e^{i\int_{x_{\text{left}}(E_1)}^{y}(E_1-V(t))^{1/2}\,dt}\Big\{i(E_1-V(y))^{1/4}\sum_{k=0}^{N'}u_k^{\text{left}}(y,E_1)+$$
$$(E_1-V(y))^{-1/4}\Big(\sum_{k=0}^{N'}\frac{\partial}{\partial y}u_k^{\text{left}}(y,E_1)\Big)+\frac{1}{4}(E_1-V(y))^{-\frac{5}{4}}V'(y)\sum_{k=0}^{N'}u_k^{\text{left}}(y,E_1)\Big\}$$
$$= e^{\pm i\frac{\pi}{4}}\, e^{i\int_{x_{\text{left}}(E)}^{y}(E_1-V(t))^{1/2}\,dt}(E_1-V(y))^{1/4}\Big\{\sum_{k=0}^{N'}u_k^{\text{left}}(y,E_1)+\frac{\sum_{k=0}^{N'}\frac{\partial}{\partial y}u_k^{\text{left}}(y,E_1)}{(E_1-V(y))^{1/2}}$$
$$+\frac{V'(y)\sum_{k=0}^{N'}u_k^{\text{left}}(y,E_1)}{4(E_1-V(y))^{3/2}}\Big\}.$$

In $I_{\text{medium left}}$ we have $\sum_{k=0}^{N'}u_k^{\text{left}}(y,E_1)=1+O(\lambda_{\text{left}}^{-\frac{3}{2}\varepsilon})$ and

$$\sum_{k=0}^{N'}\frac{\partial u_k^{\text{left}}}{\partial y}(y,E_1)=\sum_{k=1}^{N'}O(\lambda_{\text{left}}^{-k}B_{\text{left}}^{+\frac{3}{2}k}(y-x_{\text{left}}(E_1))^{-\frac{3}{2}k-1}),$$

by the global WKB lemma, since $u_0^{\text{left}}\equiv 1$. The last estimate simplifies to

$$\Big|\sum_{k=0}^{N'}\frac{\partial u_k^{\text{left}}}{\partial y}(y,E_1)\Big|\le C_{\#}\lambda_{\text{left}}^{-1}B_{\text{left}}^{+\frac{3}{2}}(y-x_{\text{left}}(E_1))^{-5/2}.$$

Since also $E_1-V(y)\sim S_{\text{left}}B_{\text{left}}^{-1}(y-x_{\text{left}}(E_1))$ in $I_{\text{medium left}}$, we get

$$(13)\quad \left|\frac{\sum_{k=0}^{N'}\frac{\partial}{\partial y}u_k^{\text{left}}(y,E_1)}{(E_1-V(y))^{1/2}}\right|\le\frac{C_{\#}\lambda_{\text{left}}^{-1}B_{\text{left}}^{+3/2}(y-x_{\text{left}}(E_1))^{-5/2}}{S_{\text{left}}^{1/2}B_{\text{left}}^{-1/2}(y-x_{\text{left}}(E_1))^{1/2}}=\frac{C_{\#}\lambda_{\text{left}}^{-1}}{S_{\text{left}}^{1/2}B_{\text{left}}}\cdot$$
$$\cdot\Big(\frac{y-x_{\text{left}}(E_1)}{B_{\text{left}}}\Big)^{-3}$$
$$=C_{\#}\Big(\lambda_{\text{left}}^{2/3}\frac{(y-x_{\text{left}}(E_1))}{B_{\text{left}}}\Big)^{-3}\le C_{\#}\lambda_{\text{left}}^{-3\varepsilon}\quad\text{in } I_{\text{medium left}},$$

and

$$(14)\quad \left|\frac{\sum_{k=0}^{N'}u_k^{\text{left}}(y,E_1)V'(y)}{4(E_1-V(y))^{3/2}}\right|\le\frac{C_{\#}S_{\text{left}}B_{\text{left}}^{-1}}{S_{\text{left}}^{3/2}B_{\text{left}}^{-3/2}(y-x_{\text{left}}(E_1))^{3/2}}$$
$$=\frac{C_{\#}}{S_{\text{left}}^{1/2}B_{\text{left}}(\frac{y-x_{\text{left}}(E_1)}{B_{\text{left}}})^{3/2}}=\frac{C_{\#}}{\Big[\lambda_{\text{left}}^{2/3}(\frac{y-x_{\text{left}}(E_1)}{B_{\text{left}}})\Big]^{3/2}}\le C_{\#}\lambda_{\text{left}}^{-\frac{3}{2}\varepsilon}$$
$$\text{in } I_{\text{medium left}}.$$

Putting (13) and (14) into (12), we find that

$$\partial y F_c^{\text{left}}(y,E_1) = e^{\pm i\frac{\pi}{4}} e^{i\int_{x_{\text{left}}(E_1)}^{y}(E_1-V(t))^{1/2}\,dt}(E_1-V(y))^{1/4}\cdot(i+\text{error}) \tag{15}$$

with

$$|\text{error}| < C_\# \lambda^{-\frac{3}{2}\varepsilon} \quad \text{in } I_{\text{medium left}}. \tag{16}$$

From (11) we get at once

$$F_c^{\text{left}}(y,E_1) = \frac{e^{\pm i\frac{\pi}{4}}\, e^{i\int_{x_{\text{left}}(E_1)}^{y}(E_1-V(t))^{1/2}\,dt}}{(E_1-V(y))^{1/4}}\cdot(1+\text{error}') \tag{17}$$

with

$$|\text{error}'| < C_\# \lambda_{\text{left}}^{-\frac{3}{2}\varepsilon} \text{in } I_{\text{medium left}}. \tag{18}$$

Equations (15), (16), (17), (18) show that

$$H_1(y) = (\text{const.})\cdot \operatorname{Im}(\partial_y F_c^{\text{left}}(y,E_1)\cdot \overline{F_c^{\text{left}}(y,E_1)}) = (\text{const.}) + (\text{error}'')$$
$$\text{with } |\text{error}''| < C_\# \lambda_{\text{left}}^{-\frac{3}{2}\varepsilon} \quad \text{in } I_{\text{medium left}}.$$

All we need from this is $|H_1(y)| > c_\# > 0$ on $I_{\text{medium left}}$.

Now we can complete the proof of the lemma. With operators $\mathcal{E}$ and G as defined above, we know that $\|G\| < C_\# S_{\text{left}}^{-1/2} B_{\text{left}}$ and $\|\mathcal{E}\| \le C_\# \lambda_{\text{left}}^{\frac{19}{2}-\frac{3}{2}\varepsilon N'}$. Recall that $N' = [\varepsilon N/500]$ and that we pick $N >> \varepsilon^{-10}$. Thus $\|\mathcal{E}\| << \inf_{y\in I_{\text{medium left}}} |H_1(y)|$. It follows that the equation $H_1(y)\cdot f_1(y) + \mathcal{E} f_1(y) = f(y)$ on $I_{\text{medium left}}$ can be solved by a Neumann series with $\|f_1\|_{L^2(I_{\text{medium left}})} \le C_\# \|f\|_{L^2(I_{\text{medium left}})}$. Then taking $u_1 = G f_1$, we have

$$\|u_1\|_{L^2} \le C_\# S_{\text{left}}^{-1/2} B_{\text{left}} \|f_1\|_{L^2} \le C_\# S_{\text{left}}^{-1/2} B_{\text{left}} \|f\|_{L^2},$$

and

$$\Big(\frac{d^2}{dx^2} + E_1 - V(x)\Big)u_1(x) = \int \Big[(\partial_x^2 + E_1 - V(x))G(x,y)\Big] f_1(y)\,dy$$
$$= H_1\cdot f_1 + \mathcal{E} f_1 = f(x).$$

Thus we have constructed the desired solution with good bounds. ■

Corollary. *There is an exact solution* $F_c(y) = F_c^{\text{left}}(y, E_1) - F_{\text{error}}(y)$ *to* $(\frac{d^2}{dx^2} + E_1 - V(x))F_c(x) = 0$ *in* $I_{\text{medium left}}$*, with* $\|F_{\text{error}}\|_{L^2} \le C_\# \lambda_{\text{left}}^{10-\frac{3}{2}\varepsilon N'} S_{\text{left}}^{-1/2}$.

Proof. Set $f = (\partial_x^2 + E_1 - V(x))F_c^{\text{left}}(x, E_1)$ and take $F_{\text{error}} = u_1$ as in the preceding Lemma. Clearly F_c is an exact solution of the ODE, and $\|F_{\text{error}}\|_{L^2} \le C_\# S_{\text{left}}^{-1/2} B_{\text{left}} \|f\|_{L^2}$. The global WKB lemma gives

$$|f(x)| \le C_\# \lambda_{\text{left}}^{10} \Big(\frac{\lambda_{\text{left}}^{2/3}(x - x_{\text{left}}(E_1))}{B_{\text{left}}} \Big)^{-\frac{3}{2}N'} B_{\text{left}}^{-3/2} \le C_\# \lambda_{\text{left}}^{10-\frac{3}{2}\varepsilon N'} B_{\text{left}}^{-3/2} \quad \text{in } I_{\text{medium left}}.$$

so $\|f\|_{L^2(I_{\text{medium left}})} \le C_\# \lambda_{\text{left}}^{10-\frac{3}{2}\varepsilon N'} B_{\text{left}}^{-1}$. The Corollary follows at once. ■

Now we can understand our given eigenfunction F in $I_{\text{medium left}}$, by comparing it with the known solutions F_c and $\overline{F}_c$. We fix a cutoff χ supported in the middle half of $I_{\text{medium left}}$, thus safely away from $x_{\text{left}}(E_1)$. Then we work in the Hilbert space $\mathcal{H} = L^2(I_{\text{medium left}}, \chi\, dx)$. (Say $0 \le \chi \le 1$ everywhere and $\chi = 1$ in the middle third of $I_{\text{medium left}}$.)

Now $\|F_c^{\text{left}}\|_{\mathcal{H}}^2 = \|\overline{F}_c^{\text{left}}\|_{\mathcal{H}}^2 \sim S_{\text{left}}^{-1/2} B_{\text{left}}$, while the stationary phase lemma (lemma 1) shows that $< F_c^{\text{left}}, \overline{F}_c^{\text{left}} >_{\mathcal{H}} = \int \chi^2 (F_c^{\text{left}})^2 << S_{\text{left}}^{-1/2} B_{\text{left}}$.

Since $\|F_{\text{error}}\|_{\mathcal{H}} \le \|F_{\text{error}}\|_{L^2(I_{\text{medium left}})} \le C_\# \lambda_{\text{left}}^{10-\frac{3}{2}\varepsilon N'} S_{\text{left}}^{-1/2}$ and the right–hand side is small compared to $\|F_c^{\text{left}}\|_{\mathcal{H}}$, it follows that the exact solutions F_c, $\overline{F}_c$ have norms in $\mathcal{H}$ of the same order of magnitude as $\|F_c^{\text{left}}\|_{\mathcal{H}}$, and F_c, $\overline{F}_c$ are nearly orthogonal.

In particular, F_c, $\overline{F}_c$ are two linearly independent exact solutions of $(\frac{d^2}{dx^2} + E_1 - V(x))F = 0$. Hence our eigenfunction F may be expresed as a linear combination

$$F = b_1 F_c + b_2 \overline{F}_c$$

for uniquely determined complex coefficients b_1, b_2. Since F is real, we have $b_2 = \overline{b}_1$

and $F = 2\mathrm{Re}(bF_c)$. Since F_c and $\overline{F}_c$ are nearly orthogonal in $\mathcal{H}$, we have

$$1 = \|F\|^2_{L^2(I_{\mathrm{BVP}})} \geq \|F\|^2_{\mathcal{H}} \geq c_\#(|b_1|^2\|F_c\|^2_{\mathcal{H}} + |b_2|^2\|\overline{F}_c\|^2_{\mathcal{H}}).$$

So $|b_1| \leq C_\#\|F_c\|^{-1}_{\mathcal{H}} \leq C_\# S^{1/4}_{\mathrm{left}} B^{-1/2}_{\mathrm{left}}$.

Now $F - 2\mathrm{Re}(b_1 F^{\mathrm{left}}_c) = 2\mathrm{Re}(b_1 F_{\mathrm{error}})$, so

$$\begin{aligned}\int_{I_{\text{medium left}}} |F - 2\mathrm{Re}(b_1 F^{\mathrm{left}}_c)|^2\,dx &\leq C_\#|b_1|^2 \int_{I_{\text{medium left}}} |F_{\mathrm{error}}|^2\,dx \\ &\leq (C_\# S^{1/2}_{\mathrm{left}} B^{-1}_{\mathrm{left}}) \cdot (C_\# \lambda^{20-3\varepsilon N'}_{\mathrm{left}} S^{-1}_{\mathrm{left}}) = C_\# \lambda^{19-3\varepsilon N'}_{\mathrm{left}},\end{aligned}$$

since $S^{1/2}_{\mathrm{left}} B_{\mathrm{left}} = \lambda_{\mathrm{left}}$. Thus we have proven the following result.

Lemma 8. *For a constant $b_{\text{medium left}}$ of size $|b_{\text{medium left}}| < C_\# S^{1/4}_{\mathrm{left}} B^{-1/2}_{\mathrm{left}}$ we have*

$$\int_{I_{\text{medium left}}} |F(x) - Re(b_{\text{medium left}} F^{\mathrm{left}}_c(x, E_1))|^2\,dx \leq C_\# \lambda^{20-3\varepsilon N'}_{\mathrm{left}}.$$

Here the constant $b_{\text{medium left}}$ is complex.

There is of course an analogous result on $I_{\text{medium right}}$.

Next we study the given eigenfunction F on I_{center}. The ideas are analogous to the proof of Lemma 8, but easier.

Lemma 9. *On I_{center} we can solve the equation*

$$\Big(\frac{d^2}{dx^2} + E_1 - V(x)\Big)u = f \quad \textit{with } \|u\|_{L^2} \leq C_\# \int_{I_{\mathrm{center}}} \frac{dx}{(E_1 - V(x))^{1/2}} \|f\|_{L^2}.$$

Proof. Define the approximate Green's function

$$G(x,y) = \begin{cases} F^{\mathrm{left}}_c(x,E_1)\,\overline{F^{\mathrm{left}}_c(y,E_1)} & \text{if } x \leq y \\ \overline{F^{\mathrm{left}}_c(x,E_1)}\,F^{\mathrm{left}}_c(y,E_1) & \text{if } x \geq y. \end{cases}$$

As in the proof of Lemma 7, we have $(\frac{\partial^2}{\partial x^2} + E_1 - V(x))G(x,y) = H_1(y)\delta(x-y) + H(x,y)$, $H_1(y) = (\mathrm{const.})\mathrm{Im}[\frac{\partial}{\partial y}F^{\mathrm{left}}_c(y,E_1) \cdot \overline{F^{\mathrm{left}}_c(y,E_1)}]$, and $|G(x,y)| \leq C_\#(E_1 -$

$V(x))^{-1/4}(E_1 - V(y))^{-1/4}$ on $I_{\text{center}} \times I_{\text{center}}$, $|H(x,y)| \le C_\# \Lambda^{-N'} S^{-1/4}(x) B^{-2}(x)(E_1 - V(y))^{-1/4}$ on $I_{\text{center}} \times I_{\text{center}}$.

The estimates are immediate from the global WKB lemma. In particular, the Hilbert–Schmidt norms of the kernels $H(x,y)$, $G(x,y)$ are estimated as follows:

$$\begin{aligned}\int_{I_{\text{center}}\times I_{\text{center}}} |H(x,y)|^2\,dxdy &\le C_\# \Lambda^{-2N'} \Big(\int_{I_{\text{center}}} S^{-1/2}(x) B^{-4}(x)\,dx\Big) \\ &\qquad \Big(\int_{I_{\text{center}}} \frac{dy}{(E_1 - V(y))^{1/2}}\Big) \\ &\le C_\# \Lambda^{K-2N'} \quad \text{by hypothesis (E5)}\,,\end{aligned} \tag{19}$$

and

$$\Big(\int_{I_{\text{center}}\times I_{\text{center}}} |G(x,y)|^2\,dxdy\Big)^{1/2} \le C_\# \Big(\int_{I_{\text{center}}} \frac{dx}{(E_1 - V(x))^{1/2}}\Big). \tag{20}$$

We turn to $H_1(y)$. The equation (11), (12) hold also in I_{center}. Moreover we have in I_{center} the estimates:

$$\begin{aligned}\sum_{k=0}^{N'} u_k^{\text{left}}(y, E_1) &= 1 + O(\Lambda^{-1}) \\ \Big|\sum_{k=0}^{N'} \frac{\partial}{\partial y} u_k^{\text{left}}(y, E_1)\Big| &\le C_\# \Lambda^{-1} B^{-1}(y)\end{aligned} \tag{21}$$

by the global WKB lemma. Also $E_1 - V(y) \ge c_\# S(y)$ and $|V'(y)| \le C_\# S(y) B^{-1}(y)$ in I_{center}. Hence,

$$\frac{|\sum_{k=0}^{N'} \frac{\partial}{\partial y} u_k^{\text{left}}(y, E_1)|}{(E_1 - V(y))^{1/2}} \le \frac{C_\# \Lambda^{-1} B^{-1}(y)}{S^{1/2}(y)} = \frac{C_\# \Lambda^{-1}}{\lambda(y)} \le C_\# \Lambda^{-1} \quad \text{in } I_{\text{center}} \tag{22}$$

and

$$\begin{aligned}\frac{|V'(y)||\sum_{k=0}^{N'} u_k^{\text{left}}(y, E_1)|}{(E_1 - V(y))^{3/2}} &\le \frac{C_\# S(y) B^{-1}(y)}{S^{3/2}(y)} = \frac{C_\#}{S^{1/2}(y) B(y)} = \frac{C_\#}{\lambda(y)} \\ &\le C_\# \Lambda^{-1} \quad \text{in } I_{\text{center}}.\end{aligned} \tag{23}$$

Putting (21), (22), (23) into (11) and (12), we get

$$\frac{\partial}{\partial y} F_c^{\text{left}}(y, E_1) = e^{\pm i \frac{\pi}{4}} e^{i \int_{x_{\text{left}}(E_1)}^{y} (E_1 - V(t))^{1/2}\, dt} (E_1 - V(y))^{1/4} \{ i + \text{ error} \}$$

$$\text{with } |\text{error}| \leq C_{\#} \Lambda^{-1}$$

and

$$F_c^{\text{left}}(y, E_1) = \frac{e^{\pm i \frac{\pi}{4}} e^{i \int_{x_{\text{left}}(E_1)}^{y} (E_1 - V(t))^{1/2}\, dt}}{(E_1 - V(y))^{1/4}} \cdot \{ 1 + \text{error}' \}$$

$$\text{with } |\text{error}'| \leq C_{\#} \Lambda^{-1}.$$

Hence $H_1(y) = (\text{const.}) \text{Im}[\frac{\partial F_c^{\text{left}}}{\partial y} \cdot \overline{F}_c^{\text{left}}] = (\text{const.}) + (\text{error}'')$ with $|\text{error}''| < C_{\#} \Lambda^{-1}$. All we need from this is $|H_1(y)| \geq c_{\#} > 0$. Now we solve $H_1(y) f_1(y) + \int_{I_{\text{center}}} H(y, z) f_1(z)\, dz = f(y)$ by a Neumann series and put $u(x) = \int_{I_{\text{center}}} G(x, y) f_1(y)\, dy$ as in Lemma 7. Thus u solves the ODE and has norm

$$\begin{aligned} \|u\|_{L^2(I_{\text{center}})} &\leq C_{\#} \int_{I_{\text{center}}} \frac{dx}{(E_1 - V(x))^{1/2}} \cdot \|f_1\|_{L^2(I_{\text{center}})} \\ &\leq C_{\#} \int_{I_{\text{center}}} \frac{dx}{(E_1 - V(x))^{1/2}} \|f\|, \end{aligned}$$

provided $\|H(\cdot, \cdot)\|_{\text{Hilbert–Schmidt}} \leq \frac{1}{2} \inf_{y \in I_{\text{center}}} |H_1(y)|$ to make the Neumann series converge. This condition is satisfied by virtue of (19), since we picked $N > K \varepsilon^{-10}$. ∎

Corollary. *There is an exact solution* $F_0 = F_0^{\text{left}} - F_{\text{error}}$ *of* $[\frac{d^2}{dx^2} + E_1 - V(x)] F_c = 0$ *in* I_{center}*, with* $\|F_{\text{error}}\|_{L^2}^2 \leq C_{\#} \Lambda^{K - 2N'} \int_{I_{\text{center}}} \frac{dx}{(E_1 - V(x))^{1/2}}$.

Proof. Take $F_{\text{error}} = u$ arising from $f = (\frac{\partial^2}{\partial x^2} + E_1 - V(x)) F_0^{\text{left}}$ in Lemma 9. Thus the ODE is satisfied exactly, and $\|F_{\text{error}}\|_{L^2} \leq C_{\#} \int_{I_{\text{center}}} \frac{dx}{(E_1 - V(x))^{1/2}} \|f\|_{L^2}$.

The global WKB lemma gives $|f(x)| \leq C_{\#} \Lambda^{-N'} S^{-1/4}(x) B^{-2}(x)$ pointwise in I_{center}. Hence $\int_{I_{\text{center}}} |f(x)|^2\, dx \leq C_{\#} \Lambda^{-2N'} \left(\int_{I_{\text{center}}} \frac{dx}{S^{1/2}(x) B^4(x)} \right)$, so $\|F_{\text{error}}\|_{L^2(I_{\text{center}})}^2 \leq C_{\#} \Lambda^{-2N'} \left(\int_{I_{\text{center}}} \frac{dx}{S^{1/2}(x) B^4(x)} \right) \left(\int_{I_{\text{center}}} \frac{dx}{(E_1 - V(x))^{1/2}} \right)^2 \leq C_{\#} \Lambda^{K - 2N'} \int_{I_{\text{center}}} \frac{dx}{(E_1 - V(x))^{1/2}}$ by hypothesis (E5). ∎

Now we can repeat the proof of Lemma 8:

Let $\chi(x)$ be a smooth cutoff function supported in $I_{\text{center}} = [\overline{x}_1, \overline{x}_2]$ with $\chi(x) = 1$ for $\overline{x}_1 + \hat{c}_\# B_{\text{left}} \le x \le \overline{x}_2 - \hat{c}_\# B_{\text{right}}$,

$$\left|\left(\frac{d}{dx}\right)^m \chi(x)\right| \le C_\#^m B_{\text{left}}^{-m} \quad \text{in } [\overline{x}_1, \overline{x}_1 + \hat{c}_\# B_{\text{left}}],$$
$$\left|\left(\frac{d}{dx}\right)^m \chi(x)\right| \le C_\#^m B_{\text{right}}^{-m} \quad \text{in } [\overline{x}_2 - \hat{c}_\# B_{\text{right}}, \overline{x}_2],$$

$0 \le \chi \le 1$ everywhere.

We work in the Hilbert space $\mathcal{H} = L^2(I_{\text{center}}, \chi\, dx)$. The Corollary of Lemma 2 gives

$$\|F_c^{\text{left}}\|_{\mathcal{H}}^2 = \|\overline{F}_c^{\text{left}}\|_{\mathcal{H}}^2 \ge c \int_{I_{\text{center}}} \frac{dx}{(E_1 - V(x))^{1/2}}. \tag{24}$$

Moreover,

$$| < F_c^{\text{left}}, \overline{F}_c^{\text{left}} >_{\mathcal{H}} | = | \int_{I_{\text{center}}} \chi (F_c^{\text{left}})^2\, dx| \le C_\# \Lambda^{-N'} \int_{I_{\text{center}}} \frac{dx}{(E_1 - V(x))^{1/2}}. \tag{25}$$

To see this, write a partition of unity $\chi = \sum_\nu \chi_\nu$ with χ_ν supported in $\{|x - x_\nu| < c_\# B(x_\nu)\}$ and satisfying natural estimates. Then we have

$$\begin{aligned} | \int_{I_{\text{center}}} \chi (F_c^{\text{left}})^2\, dx| &\le \sum_\nu | \int_{I_{\text{center}}} \chi_\nu (F_c^{\text{left}})^2\, dx| \\ &\le C_\# \sum_\nu \int_{\text{supp } \chi_\nu} \frac{dx}{(\lambda(x))^{N'} (E_1 - V(x))^{1/2}} \end{aligned}$$

as one sees by applying the stationary phase Lemma. Since $\lambda(x) \ge \Lambda$, estimate (25) follows.

Now (24), (25) show that F_c^{left}, $\overline{F}_c^{\text{left}}$ are nearly orthogonal in $\mathcal{H}$. Also, (24) and the preceding corollary show that F_c, $\overline{F}_c$ are small perturbations of F_c^{left}, $\overline{F}_c^{\text{left}}$. We conclude that

$$\|F_c\|_{\mathcal{H}}^2 = \|\overline{F}_c\|_{\mathcal{H}}^2 \ge c_\# \int_{I_{\text{center}}} \frac{dx}{(E_1 - V(x))^{1/2}}, \tag{26}$$

and

$$|< F_c, \overline{F}_c >_{\mathcal{H}}| \leq \frac{1}{10} \|F_c\|_{\mathcal{H}} \|\overline{F}_c\|_{\mathcal{H}}. \tag{27}$$

In particular, F_c and $\overline{F}_c$ are linearly independent solutions of $(\frac{d^2}{dx^2} + E_1 - V(x))F = 0$. Hence our given solution F may be expressed as a linear combination $F = b_1 F_c + b_2 \overline{F}_c$ with uniquely determined (complex) constants b_1, b_2. Since F is real we have $b_2 = \bar{b}_1$, so $F = 2\mathrm{Re}(b_1 F_c)$. Moreover,

$$1 = \|F\|^2_{L^2(I_{\mathrm{BVP}})} \geq \|F\|^2_{\mathcal{H}} \geq c_\#(|b_1|^2 \|F_c\|^2_{\mathcal{H}} + |b_2|^2 \|\overline{F}_c\|^2_{\mathcal{H}}) \text{ (by (27))}$$
$$\geq c_\# |b_1|^2 \int_{I_{\mathrm{center}}} \frac{dx}{(E_1 - V(x))^{1/2}},$$

i.e. $|b_1| \leq C_\# \left(\int_{x_{\mathrm{left}}(E_1)}^{x_{\mathrm{right}}(E_1)} \frac{dx}{(E_1 - V(x))^{1/2}} \right)^{-1/2}$.
Here we have changed the limits of integration, but the order of magnitude of the integral remains unaffected. Now we have

$$F = 2\mathrm{Re}(b_1 F_c) = 2\mathrm{Re}(b_1 F_c^{\mathrm{left}}) - 2\mathrm{Re}(b_1 F_{\mathrm{error}}),$$

so that

$$\int_{I_{\mathrm{center}}} |F - 2\mathrm{Re}(b_1 F_c^{\mathrm{left}})|^2 \, dx \leq C_\# |b_1|^2 \int_{I_{\mathrm{center}}} |F_{\mathrm{error}}(x)|^2 \, dx$$
$$\leq C_\# \Big(\int_{I_{\mathrm{center}}} \frac{dx}{(E_1 - V(x))^{1/2}} \Big)^{-1} \|F_{\mathrm{error}}\|^2_{L^2}$$
$$\leq C_\# \Big(\int_{I_{\mathrm{center}}} \frac{dx}{(E_2 - V(x))^{1/2}} \Big)^{-1} \cdot C_\# \Lambda^{K-2N'} \int_{I_{\mathrm{center}}} \frac{dx}{(E_1 - V(x))^{1/2}}$$
$$= C_\# \Lambda^{K-2N'}.$$

Thus we have proven the following result.

Lemma 10. *For a constant* $b^{\mathrm{left}}_{\mathrm{center}}$ *of size* $|b^{\mathrm{left}}_{\mathrm{center}}| \leq C_\# \left(\int_{x_{\mathrm{left}}(E_1)}^{x_{\mathrm{right}}(E_1)} \frac{dx}{(E_1 - V(x))^{1/2}} \right)^{-1/2}$ *we have*

$$\int_{I_{\mathrm{center}}} |F(x) - Re(b^{\mathrm{left}}_{\mathrm{center}} F_c^{\mathrm{left}}(x, E_1))|^2 \, dx \leq C_\# \Lambda^{K-2N'}.$$

Of course, there is an analogous result for F_c^{right}, namely

$$\int_{I_{\text{center}}} |F(x) - \mathrm{Re}(b_{\text{center}}^{\text{right}} F_c^{\text{right}}(x, E_1))|^2 \, dx \le C_\# \Lambda^{K-2N'}. \tag{28}$$

for another constant $b_{\text{center}}^{\text{right}}$ of size $|b_{\text{center}}^{\text{right}}| \le C_\# \left(\int_{x_{\text{left}}}^{x_{\text{right}}(E_1)} \frac{dx}{(E_1 - V(x))^{1/2}} \right)^{-1/2}$.

Lemmas 4, 6, 8, 10 tell us how F looks on $I_{\text{far left}}$, $\hat{I}_{\text{Airey left}}$, $I_{\text{medium left}}$, I_{center}. The analogous results for the right–hand solutions tell us how F looks on $I_{\text{far right}}$, $\hat{I}_{\text{Airey right}}$, $I_{\text{medium right}}$, I_{center}. These intervals cover I_{BVP}, the domain of F. Since $I_{\text{medium left}}$ overlaps both $\hat{I}_{\text{Airey left}}$ and I_{center}, we have two different descriptions of F on each overlap. Naturally, these descriptions must be consistent. We will deduce from this that the constants b in Lemma 6, $b_{\text{medium left}}$ from Lemma 8, and $b_{\text{center}}^{\text{left}}$ from Lemma 10 are nearly the same. Thus we can describe F on $I_{\text{far left}} \cup \hat{I}_{\text{Airey left}} \cup I_{\text{medium left}} \cup I_{\text{center}}$ using a single unknown constant b. Analogously, we can describe F on $I_{\text{far right}} \cup \hat{I}_{\text{Airey right}} \cup I_{\text{medium right}} \cup I_{\text{center}}$ using another single unknown constant b'. On I_{center}, we have two different descriptions of F, which must be consistent. This tells us that the phase $\Phi(E_1)$ must be nearly a multiple of π, and that b is approximately $\pm b'$. We carry out this plan in the paragraphs below.

¿From lemmas 6, 8, 10 we get

$$\int_{\hat{I}_{\text{Airey left}}} |F - b_{\text{Airey left}} F_{\text{Airey left}}|^2 \, dx \le C_\# \lambda_{\text{left}}^{10-2N'} \tag{29}$$

$$|b_{\text{Airey left}}| \le C_\# \lambda_{\text{left}}^{20} B_{\text{left}}^{-1}, b_{\text{Airey left}} \ \underline{\text{real}} \tag{30}$$

$$\int_{I_{\text{medium left}}} |F - \mathrm{Re}(b_{\text{medium left}} F_c^{\text{left}})|^2 \, dx \le C_\# \lambda_{\text{left}}^{20-3\varepsilon N'} \tag{31}$$

$$|b_{\text{medium left}}| \le C_\# S_{\text{left}}^{1/4} B_{\text{left}}^{-1/2} \tag{32}$$

$$\int_{I_{\text{center}}} |F - \mathrm{Re}(b_{\text{center}}^{\text{left}} F_c^{\text{left}})|^2 \, dx \le C_\# \Lambda^{K-2N'} \tag{33}$$

$$|b^{\text{left}}_{\text{center}}| \le C_\# \Big(\int_{x_{\text{left}}(E_1)}^{x_{\text{right}}(E_1)} \frac{dx}{(E_1 - V(x))^{1/2}} \Big)^{-1/2}. \tag{34}$$

The global WKB lemma gives

$$|F_{\text{Airey left}} - \text{Re}(F^{\text{left}}_c)| \le C_\# \lambda_{\text{left}}^{10-\frac{3}{2}\varepsilon N'} B^{1/2}_{\text{left}} \text{ on } \hat{I}_{\text{Airey left}} \cap I_{\text{medium left}},$$

hence

$$\begin{aligned} &\int_{\hat{I}_{\text{Airey left}} \cap I_{\text{medium left}}} |b_{\text{Airey left}} F_{\text{Airey left}} - \text{Re}(b_{\text{Airey left}} F^{\text{left}}_c)|^2 \, dx \\ &\le \Big(C_\# \lambda_{\text{left}}^{20-3\varepsilon N'} |b_{\text{Airey left}}|^2 B_{\text{left}} \Big) |I_{\text{Airey left}} \cap I_{\text{medium left}}| \\ &\le C_\# \lambda_{\text{left}}^{20-3\varepsilon N'} B^2_{\text{left}} |b_{\text{Airey left}}|^2 \le C_\# \lambda_{\text{left}}^{20-3\varepsilon N'} B^2_{\text{left}} \cdot C_\# \lambda^{40}_{\text{left}} B^{-2}_{\text{left}} \\ &(\text{by } (30)) \ = C_\# \lambda^{60-3\varepsilon N'}. \end{aligned}$$

Combining this inequality with (29), (31), we get

$$\int_{\hat{I}_{\text{Airey left}} \cap I_{\text{medium left}}} |\text{Re}((b_{\text{Airey left}} - b_{\text{medium left}}) F^{\text{left}}_c)|^2 \, dx \le C_\# \lambda_{\text{left}}^{60-3\varepsilon N'}.$$

Introduce a smooth cutoff χ $(0 \le \chi \le 1)$ supported in the middle half of $\hat{I}_{\text{Airey left}}$ $\cap I_{\text{medium left}}$. (Thus $x - x_{\text{left}}(E_1) \sim \lambda^{-\varepsilon}_{\text{left}} B_{\text{left}}$ in supp χ). The preceding inequality implies

$$\int_{\mathbb{R}} \chi |\text{Re}((b_{\text{Airey left}} - b_{\text{medium left}}) F^{\text{left}}_c|^2 \, dx \le C_\# \lambda_{\text{left}}^{60-3\varepsilon N'}. \tag{35}$$

As in the proof of Lemma 2, the stationary phase Lemma shows that the integral on the left dominates

$$\begin{aligned} c_\# &\int_{\mathbb{R}} \frac{\chi \, dx}{(E_1 - V(x))^{1/2}} \cdot |b_{\text{Airey left}} - b_{\text{medium left}}|^2 \\ &\ge \frac{c_\# \, \lambda_{\text{left}}^{-\varepsilon/2} B_{\text{left}}}{S^{1/2}_{\text{left}}} \cdot |b_{\text{Airey left}} - b_{\text{medium left}}|^2. \end{aligned}$$

Therefore $|b_{\text{Airey left}} - b_{\text{medium left}}|^2 \le C_{\#}\lambda_{\text{left}}^{63-3\varepsilon N'} S_{\text{left}}^{1/2} B_{\text{left}}^{-1}$. Since $S_{\text{left}}^{1/2} B_{\text{left}}^{-1} = \lambda_{\text{left}} B_{\text{left}}^{-2}$, this yields $|b_{\text{Airey left}} - b_{\text{medium left}}|^2 \le C_{\#}\lambda_{\text{left}}^{66-3\varepsilon N'} B_{\text{left}}^{-2}$, hence

$$\begin{aligned}
&\int_{I_{\text{medium left}}} |b_{\text{Airey left}}\operatorname{Re}(F_c^{\text{left}}) - \operatorname{Re}(b_{\text{medium left}} F_c^{\text{left}})|^2\,dx \\
&\le C_{\#}\lambda_{\text{left}}^{66-3\varepsilon N'} B_{\text{left}}^{-2} \int_{I_{\text{medium left}}} |F_c|^2\,dx \\
&\qquad\qquad \le C_{\#}\lambda_{\text{left}}^{66-3\varepsilon N'} B_{\text{left}}^{-2} \int_{I_{\text{medium left}}} \frac{dx}{(E_1 - V(x))^{1/2}} \\
&\le C_{\#}\lambda_{\text{left}}^{66-3\varepsilon N'} B_{\text{left}}^{-2} \cdot S_{\text{left}}^{-1/2} B_{\text{left}} \le C_{\#}\lambda_{\text{left}}^{65-3\varepsilon N'}, \text{ since } S_{\text{left}}^{1/2} B_{\text{left}} = \lambda_{\text{left}}.
\end{aligned}$$

Combining this with (31), we get

$$\int_{I_{\text{medium left}}} |F - b_{\text{Airey left}}\operatorname{Re}(F_c^{\text{left}})|^2\,dx \le C_{\#}\lambda_{\text{left}}^{65-3\varepsilon N'}. \tag{36}$$

Next, from (36) and (33) we get

$$\int_{I_{\text{medium left}} \cap I_{\text{center}}} |\operatorname{Re}((b_{\text{Airey left}} - b_{\text{center}}^{\text{left}}) F_c^{\text{left}})|^2\,dx \le C_{\#}\Lambda^{65-3\varepsilon N' + K}.$$

Since $|I_{\text{medium left}} \cap I_{\text{center}}| \ge c_{\#} B_{\text{left}}$, $\operatorname{dist}(I_{\text{medium left}} \cap I_{\text{center}}, x_{\text{left}}(E_1)) \ge c_{\#} B_{\text{left}}$, lemma 2 shows that the integral on the left is at least
$c_{\#}|b_{\text{Airey left}} - b_{\text{center}}^{\text{left}}|^2 \int_{I_{\text{medium left}} \cap I_{\text{center}}} \frac{dx}{(E_1 - V(x))^{1/2}} \ge$
$c_{\#}|b_{\text{Airey left}} - b_{\text{center}}^{\text{left}}|^2 S_{\text{left}}^{-1/2} B_{\text{left}}$. Hence $|b_{\text{Airey left}} - b_{\text{center}}^{\text{left}}|^2 \le$
$C_{\#}\Lambda^{K+65-3\varepsilon N'} S_{\text{left}}^{1/2} B_{\text{left}}^{-1}$, so that

$$\begin{aligned}
&\int_{I_{\text{center}}} |\operatorname{Re}(b_{\text{Airey}}^{\text{left}} F_c^{\text{left}} - b_{\text{center}}^{\text{left}} F_c^{\text{left}})|^2\,dx \\
&\le C_{\#}\Lambda^{K+65-3\varepsilon N'} S_{\text{left}}^{1/2} B_{\text{left}}^{-1} \int_{I_{\text{center}}} |F_c^{\text{left}}|^2\,dx \\
&\le C_{\#}\Lambda^{65+K-3\varepsilon N'} S_{\text{left}}^{1/2} B_{\text{left}}^{-1} \int_{I_{\text{center}}} \frac{dx}{(E_1 - V(x))^{1/2}} \\
&\le C_{\#}\Lambda^{2K+65-3\varepsilon N'} \quad \text{by (E6)}.
\end{aligned}$$

Combining this with (33), we have

$$\int_{I_{\text{center}}} |F - b_{\text{Airey left}}\operatorname{Re}(F_c^{\text{left}})|^2\,dx \le C_{\#}\Lambda^{2K+65-3\varepsilon N'}. \tag{37}$$

Since also $\int_{I_{\text{center}}} |F|^2\, dx \le 1$, we get

$$b^2_{\text{Airey left}} \int_{I_{\text{center}}} (\text{Re } F_c^{\text{left}})^2\, dx \le C_\# \quad \text{since } N' = [\varepsilon N/500],$$
$$N >> K\varepsilon^{-10}.$$

Lemma 2 shows that this amounts to $c_\# b^2_{\text{Airey left}} \int_{I_{\text{center}}} \frac{dx}{(E_1 - V(x))^{1/2}} \le C_\#$, i.e.

$$|b_{\text{Airey left}}| \le C_\# \Big(\int_{x_{\text{left}}(E_1)}^{x_{\text{right}}(E_1)} \frac{dx}{(E_1 - V(x))^{1/2}} \Big)^{-1/2}. \tag{38}$$

In writing (38), we changed the limits of integration without affecting the order of magnitude of the integral. Combining estimates (29), (36), (37), (38), we obtain the following result.

Lemma 11. *For a real constant* b_{left} *of size* $|b_{\text{left}}| \le C_\# (\int_{V<E_1} \frac{dx}{(E_1 - V(x))^{1/2}})^{-1/2}$, *we have*

$$\int_{\hat{I}_{\text{Airey left}}} |F(x) - b_{\text{left}} F_{\text{Airey left}}(x, E_1)|^2\, dx \le C_\# \lambda_{\text{left}}^{10-2N'} \quad \textit{and} \tag{39}$$

$$\int_{I_{\text{medium left}} \cup I_{\text{center}}} |F(x) - b_{\text{left}} Re(F_c^{\text{left}}(x, E_1))|^2\, dx \le C_\# \Lambda^{2K+65-3\varepsilon N'}. \tag{40}$$

Of course there is an analogous result for F_c^{right}: For a real constant b_{right} of size $|b_{\text{right}}| \le C_\# (\int_{V<E_1} \frac{dx}{(E_1 - V(x))^{1/2}})^{-1/2}$ we have

$$\int_{\hat{I}_{\text{Airey right}}} |F(x) - b_{\text{right}} F_{\text{Airey right}}(x, E_1)|^2\, dx \le C_\# \lambda_{\text{right}}^{10-2N'} \quad \text{and} \tag{41}$$

$$\int_{I_{\text{medium right}} \cup I_{\text{center}}} |F(x) - b_{\text{right}} \text{Re}(F_c^{\text{right}}(x, E_1))|^2\, dx \le C_\# \Lambda^{2K+65-3\varepsilon N'}. \tag{42}$$

Comparing (40) and (42), we see that

$$\int_{I_{\text{center}}} |\text{Re}(b_{\text{left}} F_c^{\text{left}} - b_{\text{right}} F_c^{\text{right}})|^2\, dx \le C_\# \Lambda^{2K+65-3\varepsilon N'}. \tag{43}$$

From the global WKB lemma we recall that

$$|F_c^{\text{left}} - \mathcal{A}(E_1)F_c^{\text{right}}| \leq C_\# \Lambda^{-N'-1}(E_1 - V(x))^{-1/4} \text{ on } I_{\text{center}},$$

so

$$\begin{aligned}
&\int_{I_{\text{center}}} |\text{Re}(b_{\text{left}} F_c^{\text{left}} - b_{\text{left}} \mathcal{A}(E_1) F_c^{\text{right}})|^2 \, dx \\
&\qquad \leq C_\# \Lambda^{-2N'-2} \int_{I_{\text{center}}} \frac{dx}{(E_1 - V(x))^{1/2}} |b_{\text{left}}|^2 \\
&\qquad \leq C_\# \Lambda^{-2N'-2} \quad \text{by the estimate on } |b_{\text{left}}| \text{ in Lemma 11.}
\end{aligned}$$

Combining this with (43) gives

$$\int_{I_{\text{center}}} |\text{Re}((b_{\text{right}} - b_{\text{left}} \mathcal{A}(E_1)) F_c^{\text{right}})|^2 \, dx \leq C_\# \Lambda^{2K+65-3\varepsilon N'}.$$

The Corollary to Lemma 2 shows that the left–hand side dominates $c_\# |b_{\text{right}} - b_{\text{left}} \mathcal{A}(E_1)|^2 \int_{I_{\text{center}}} \frac{dx}{(E_1 - V(x))^{1/2}}$. Hence we obtain:

$$|b_{\text{right}} - b_{\text{left}} \mathcal{A}(E_1)|^2 \leq C_\# \Lambda^{2K+65-3\varepsilon N'} \left(\int_{V<E_1} \frac{dx}{(E_1 - V(x))^{1/2}} \right)^{-1}. \tag{44}$$

We next derive lower bounds for $|b_{\text{right}}|$, $|b_{\text{ left}}|$, to allow us to deduce from (44) that the complex number $\mathcal{A}(E)$ has small imaginary part.

Note that $\lambda_{\text{left}}^2 (\frac{\partial Y^{\text{left}}}{\partial x})^2 Y^{\text{left}} = (E_1 - V(x)) + \text{Error}$ with $|\text{Error}| \leq C_\# \lambda_{\text{left}}^{-2} S_{\text{left}} = C_\# B_{\text{left}}^{-2}$ in $\hat{I}_{\text{Airey left}}$, by the global WKB lemma.

Also $C_\# B_{\text{left}}^{-2} \geq (\frac{\partial Y^{\text{left}}}{\partial x})^2 \geq c_\# B_{\text{left}}^{-2}$, in $\hat{I}_{\text{Airey left}}$ again by the global WKB lemma.

Hence

$$\begin{aligned}
|F_{\text{Airey left}}(x, E_1)| &= \left| \lambda_{\text{left}}^{-1/3} \left(\frac{\partial Y^{\text{left}}}{\partial x} \right)^{-1/2} A(\lambda_{\text{left}}^{2/3} Y^{\text{left}}) \right| \\
&\leq C_\# \lambda_{\text{left}}^{-1/3} \left| \frac{\partial Y^{\text{left}}}{\partial x} \right|^{-1/2} (1 + \lambda_{\text{left}}^{2/3} |Y^{\text{left}}|)^{-1/4} \\
&= C_\# \left[\lambda_{\text{left}}^{4/3} \left(\frac{\partial Y^{\text{left}}}{\partial x} \right)^{+2} + \lambda_{\text{left}}^2 \left(\frac{\partial Y^{\text{left}}}{\partial x} \right)^2 |Y^{\text{left}}| \right]^{-1/4} \\
&\leq C_\# [\lambda_{\text{left}}^{4/3} B_{\text{left}}^{-2} + |E_1 - V(x) + \text{Error}|]^{-1/4} \\
&\leq C_\# [\lambda_{\text{left}}^{4/3} B_{\text{left}}^{-2} + |E_1 - V(x))|]^{-1/4}
\end{aligned}$$

(since $|\text{Error}| \leq C_\# B_{\text{left}}^{-2} << \lambda_{\text{left}}^{4/3} B_{\text{left}}^{-2}$)

$$\leq C_\# |E_1 - V(x)|^{-1/4} \quad \text{in } \hat{I}_{\text{Airey left}}.$$

This crude estimate is enough to show that

(44 bis)
$$\int_{\hat{I}_{\text{Airey left}}} |F_{\text{Airey left}}(x, E_1)|^2\, dx \le C_\# \int_{\hat{I}_{\text{Airey left}}} \frac{dx}{|E_1 - V(x)|^{1/2}} \le C_\# S_{\text{left}}^{-1/2} B_{\text{left}}.$$

Hence (39) yields

$$\int_{\hat{I}_{\text{Airey left}}} |F(x)|^2\, dx \le C_\# \lambda_{\text{left}}^{10-2N'} + |b_{\text{left}}|^2 \int_{\hat{I}_{\text{Airey left}}} |F_{\text{Airey left}}(x, E_1)|^2\, dx$$
$$\le C_\# \lambda_{\text{left}}^{10-2N'} + C_\# S_{\text{left}}^{-1/2} B_{\text{left}} |b_{\text{left}}|^2$$
$$\le C_\# \lambda_{\text{left}}^{10-2N'} + C_\# \int_{V<E_1} \frac{dx}{(E_1 - V(x))^{1/2}} \cdot |b_{\text{left}}|^2. \tag{45}$$

Similarly,

$$\int_{\hat{I}_{\text{Airey right}}} |F(x)|^2\, dx \le C_\# \lambda_{\text{right}}^{10-2N'} + C_\# \int_{V<E_1} \frac{dx}{(E_1 - V(x))^{1/2}} \cdot |b_{\text{right}}|^2. \tag{46}$$

From (40) we get

$$\int_{I_{\text{medium left}} \cup I_{\text{center}}} |F(x)|^2\, dx \le C_\# \Lambda^{2K+65-3\varepsilon N'}$$
$$+ |b_{\text{left}}|^2 \int_{I_{\text{medium left}} \cup I_{\text{center}}} |F_c^{\text{left}}|^2\, dx$$
(47)
$$\le C_\# \Lambda^{2K+65-3\varepsilon N'} + C_\# \int_{V<E_1} \frac{dx}{(E_1 - V(x))^{1/2}} |b_{\text{left}}|^2.$$

Similarly,

$$\int_{I_{\text{center}} \cup I_{\text{medium right}}} |F(x)|^2\, dx \le C_\# \Lambda^{2K+65-3\varepsilon N'} \tag{48}$$
$$+ C_\# \int_{V<E_1} \frac{dx}{(E_1 - V(x))^{1/2}} |b_{\text{right}}|^2.$$

Lemma 4 and its analogue for $I_{\text{far right}}$ give

$$\int_{I_{\text{far left}} \cup I_{\text{far right}}} |F(x)|^2\, dx \le C_\# \lambda_{\text{left}}^{-N} + C_\# \lambda_{\text{right}}^{-N}. \tag{49}$$

Since the regions of integration in (45), (46), (47), (48), (49) cover $I_{\rm BVP}$, we may add these inequalities to obtain

$$\int_{I_{\rm BVP}} |F(x)|^2\,dx \le C_\# \Lambda^{2K+65-3\varepsilon N'} + C_\# \int_{V<E_1} \frac{dx}{(E_1 - V(x))^{1/2}} \cdot (|b_{\rm left}|^2 + |b_{\rm right}|^2).$$

Here the left–hand side is 1, and the first term on the right is much smaller than 1. Therefore the second term on the right is at least 1/2, which proves that

$$|b_{\rm left}|^2 + |b_{\rm right}|^2 \ge c_\# \Big(\int_{V<E_1} \frac{dx}{(E_1 - V(x))^{1/2}} \Big)^{-1}. \tag{50}$$

From (44) and (50), we get

$$|b_{\rm right} - b_{\rm left}\mathcal{A}(E_1)|^2 \le C_\# \Lambda^{2K+65-3\varepsilon N'} (|b_{\rm left}|^2 + |b_{\rm right}|^2). \tag{51}$$

From the global WKB lemma we recall that

$$\mathcal{A}(E_1) = R(E_1) e^{i\Phi(E_1)} \quad \text{with } |R(E_1) - 1| < \frac{1}{10}. \tag{52}$$

From (51) and (52) we get

$$\frac{1}{2}|b_{\rm right}| < |b_{\rm left}| < 2|b_{\rm right}|, \quad \text{and then} \tag{53}$$

$$|\mathrm{Im}\ \mathcal{A}(E_1)|^2 \le C_\# \Lambda^{2K+65-3\varepsilon N'}.$$

Hence for a suitable integer k, we have

$$|\Phi(E_1) - \pi k_1| \le C_\# \Lambda^{K+33-\frac{3}{2}\varepsilon N'}. \tag{54}$$

From (51), (52), (53), (54) we get also

$$|b_{\rm right} - b_{\rm left} R(E_1) e^{i\pi k_1}| \le C_\# \Lambda^{K+33-\frac{3}{2}\varepsilon N'} |b_{\rm left}|. \tag{55}$$

Therefore

$$\int_{\hat{I}_{\text{Airey right}}} |b_{\rm right} F_{\text{Airey right}} - R(E_1)(-1)^{k_1} b_{\rm left} F_{\text{Airey right}}|^2\,dx \le$$

$$C_\# \Lambda^{2K+66-3\varepsilon N'} \int_{\hat{I}_{\text{Airey right}}} |F_{\text{Airey right}}|^2\,dx \cdot |b_{\rm left}|^2 \le C_\# \Lambda^{2K+66-3\varepsilon N'}$$

by (44 bis) and the upper bound for $|b_{\text{left}}|$ in Lemma 11. Combining this with (41), we get

$$\int_{\hat{I}_{\text{Airey right}}} |F(x) - (-1)^{k_1} R(E_1) b_{\text{left}} F_{\text{Airey right}}|^2 \, dx \leq C_{\#} \Lambda^{2K+66-3\varepsilon N'}. \tag{56}$$

Similarly,

$$\begin{aligned}
&\int_{I_{\text{medium right}} \cup I_{\text{center}}} |b_{\text{right}} \text{Re}(F_c^{\text{right}}) - (-1)^{k_1} R(E_1) b_{\text{left}} \text{Re}(F_c^{\text{right}})|^2 \, dx \\
&\leq C_{\#} \Lambda^{2K+66-3\varepsilon N'} \int_{I_{\text{medium right}} \cup I_{\text{center}}} |F_c^{\text{right}}|^2 \, dx \cdot |b_{\text{left}}|^2 \quad \text{(by (55))} \\
&\leq C_{\#} \Lambda^{2K+66-3\varepsilon N'} \int_{V < E_1} \frac{dx}{(E_1 - V(x))^{1/2}} |b_{\text{left}}|^2 \\
&\leq C_{\#} \Lambda^{2K+66-3\varepsilon N'} \text{ (by lemma 11).}
\end{aligned}$$

Combining this with (42) gives

$$\begin{aligned}
\int_{I_{\text{center}} \cup I_{\text{medium right}}} |F(x) - (-1)^{k_1} R(E_1) b_{\text{left}} \text{Re}(F_c^{\text{right}}(x, E_1))|^2 \, dx& \\
\leq C_{\#} \Lambda^{2K+66-3\varepsilon N'}&.
\end{aligned} \tag{57}$$

Estimates (39), (40), (49), (56), (57) give an accurate description of $F(x)$ in terms of a single unknown real constant. We record these results as follows.

Lemma 12. *For a real constant b of size $c_{\#}(\int_{V<E_1} \frac{dx}{(E_1 - V)^{1/2}})^{-1/2} \leq$*

$|b| \le C_{\#}(\int_{V<E_1} \frac{dx}{(E_1-V)^{1/2}})^{-1/2}$, *we have*

$$\int_{I_{\text{far left}}} |F(x)|^2\,dx \le C_{\#}\Lambda^{2K+66-3\varepsilon N'}$$
$$\int_{\hat{I}_{\text{Airey left}}} |F(x) - bF_{\text{Airey left}}(x,E_1)|^2\,dx \le C_{\#}\Lambda^{2K+66-3\varepsilon N'}$$
$$\int_{I_{\text{medium left}}} |F(x) - bRe(F_c^{\text{left}}(x,E_1))|^2\,dx \le C_{\#}\Lambda^{2K+66-3\varepsilon N'}$$
$$\int_{I_{\text{center}}} |F(x) - bRe(F_c^{\text{left}}(x,E_1))|^2\,dx \le C_{\#}\Lambda^{2K+66-3\varepsilon N'}$$
$$\int_{I_{\text{center}}} |F(x) - (-1)^{k_1}R(E_1)bRe(F_c^{\text{right}}(x,E_1))|^2\,dx \le C_{\#}\Lambda^{2K+66-3\varepsilon N'}$$
$$\int_{I_{\text{medium right}}} |F(x) - (-1)^{k_1}R(E_1)bRe(F_c^{\text{right}}(x,E_1))|^2\,dx \le C_{\#}\Lambda^{2K+66-3\varepsilon N'}$$
$$\int_{\hat{I}_{\text{Airey right}}} |F(x) - (-1)^{k_1}R(E_1)bF_{\text{Airey right}}(x,E_1)|^2\,dx \le C_{\#}\Lambda^{2K+66-3\varepsilon N'}$$
$$\int_{I_{\text{far right}}} |F(x)|^2\,dx \le C_{\#}\Lambda^{2K+66-3\varepsilon N'}.$$

Here k_1 *is an integer, and* $|\Phi(E_1) - \pi k_1| \le C_{\#}\Lambda^{K+33-\frac{3}{2}\varepsilon N'}$.

Of course the real constant b is determined (up to sign) modulo a tiny error by the requirement that $\int_{I_{\text{BVP}}} |F|^2\,dx = 1$. We postpone the calculation of b to a later section.

Lemma 12 gives precise control of the eigenfunction corresponding to a given eigenvalue. We must now understand the eigenvalues. Lemma 12 gives us strong information on the eigenvalues, but there is more to do.

Our next task is to show that there is at most one eigenvalue E_1 whose phase $\Phi(E_1)$ lies near a given multiple of π. To show this, we check that the functions $F_{\text{Airey left}}(x,E_1)$, $\text{Re}(F_c^{\text{left}}(x,E_1))$, $\text{Re}(F_c^{\text{right}}(x,E_1))$, $F_{\text{Airey right}}(x,E_1)$ and the factor $R(E_1)$ change very little under a change in E_1 which produces only a small shift in the phase $\Phi(E_1)$. Hence by Lemma 12, if F_1, F_2 are eigenfunctions whose eigenvalues E_1, E_2 both have phase near πk_1, then F_1 must be nearly proportional to F_2. If $E_1 \ne E_2$, then F_1 and F_2 are orthogonal and therefore far from proportional.

This contradiction shows that there can be at most one eigenvalue E_1 with $\Phi(E_1)$ near πk_1. We now carry out the details.

Lemma 13. *For* $|E - E_0| < c_\# S_{\min}$ *we have*

$$\int_{\hat{I}_{\text{Airey left}}} |\frac{\partial}{\partial E} F_{\text{Airey left}}(x,E)|^2\,dx \le C_\# \Big(\int_{V<E} \frac{dy}{(E-V(y))^{1/2}}\Big)^3 \tag{58}$$

$$\int_{I_{\text{medium left}}} |\frac{\partial}{\partial E} F_c^{\text{left}}(x,E)|^2\,dx \le C_\# \Big(\int_{V<E} \frac{dy}{(E-V(y))^{1/2}}\Big)^3 \tag{59}$$

$$\int_{I_{\text{center}}} |\frac{\partial}{\partial E} F_c^{\text{left}}(x,E)|^2\,dx \le C_\# \Big(\int_{V<E} \frac{dy}{(E-V(y))^{1/2}}\Big)^3 \tag{60}$$

$$\int_{I_{\text{medium right}}} |\frac{\partial}{\partial E} \{R(E) F_c^{\text{right}}(x,E)\}|^2\,dx \le C_\# \Big(\int_{V<E} \frac{dy}{(E-V(y))^{1/2}}\Big)^3 \tag{61}$$

$$\int_{\hat{I}_{\text{Airey right}}} |\frac{\partial}{\partial E} \{R(E) F_{\text{Airey right}}(x,E)\}|^2\,dx \le C_\# \Big(\int_{V<E} \frac{dy}{(E-V(y))^{1/2}}\Big)^3. \tag{62}$$

Here the intervals $\hat{I}_{\text{Airey left}} \dots \hat{I}_{\text{Airey right}}$ *are defined in terms of the energy* E.

Proof. First we estimate $R(E) = |1 + \sum_{k=1}^{N'} G_k(E)|$. The global WKB lemma gives $|R(E) - 1| \le C_\# \Lambda^{-1}$, and

$$c_\# |\frac{\partial}{\partial E} G_k(E)| \le \Lambda^{-k} \int_{V<E} \frac{\Lambda\,dy}{\lambda(y) S(y) B(y)} \le \Lambda^{-k} \int_{V<E} \frac{\Lambda}{\lambda(y)} \cdot \frac{1}{S^{1/2}(y) B(y)} \cdot \frac{dy}{S^{1/2}(y)}$$

(62 bis)

$$\le \Lambda^{-k-1} \int_{V<E} \frac{dy}{S^{1/2}(y)} \le \frac{C_\#}{\Lambda^{k+1}} \int_{V<E} \frac{dy}{(E-V(y))^{1/2}} \text{ by (E7).}$$

Hence $|\frac{\partial}{\partial E} R(E)| \le C_\# \Lambda^{-2} \int_{V<E} \frac{dy}{(E-V)^{1/2}}$, so

$$\Big|\frac{\partial R(E)}{\partial E}\Big|^2 \int_{\hat{I}_{\text{Airey right}}} |F_{\text{Airey right}}(x,E)|^2\,dx \le \Big[C_\# \Lambda^{-2} \int_{V<E} \frac{dy}{(E-V)^{1/2}}\Big]^2 \cdot \Big[C_\# \int_{V<E} \frac{dy}{(E-V)^{1/2}}\Big]$$

by virtue of the analogue of (44 bis) for $F_{\text{Airey right}}$. This shows that (62) follows from the analogue of (58) for $F_{\text{Airey right}}$. Similarly,

$$\Big|\frac{\partial R(E)}{\partial E}\Big|^2 \int_{I_{\text{medium right}}} |F_c^{\text{right}}(x,E)|^2\,dx \le \Big[C_\# \Lambda^{-2} \int_{V<E} \frac{dy}{(E-V)^{1/2}}\Big]^2 \int_{I_{\text{medium right}}} \frac{C_\#}{(E-V)^{1/2}}\,dy,$$

which reduces (61) to the analogue of (59) for F_c^{right}. So it is enough to prove (58), (59), (60).

To prove (58), note the estimates

$$\begin{gathered} |A(t)| \le C,\ |A'(t)| \le C(1+|t|)^{1/4} \quad \text{for } t \in \mathbb{R}, \\ \Big|\frac{\partial Y_{\text{left}}}{\partial E}\Big| \le C_\# S_{\text{left}}^{-1},\ \Big|\frac{\partial^2 Y_{\text{left}}}{\partial x\,\partial E}\Big| \le C_\# B_{\text{left}}^{-1} S_{\text{left}}^{-1} \quad \text{in } U_{\text{Airey left}}, \\ |Y_{\text{left}}| \le C_\#,\ c_\# B_{\text{left}}^{-1} < \Big(\frac{\partial Y_{\text{left}}}{\partial x}\Big) < C_\# B_{\text{left}}^{-1} \quad \text{in } U_{\text{Airey left}}. \end{gathered}$$

The estimates in $U_{\text{Airey left}}$ are consequences of the global WKB lemma.

Now recall $F_{\text{Airey left}}(x,E) = \lambda_{\text{left}}^{-1/3}\big(\frac{\partial Y_{\text{left}}}{\partial x}\big)^{-1/2} A(\lambda_{\text{left}}^{2/3} Y_{\text{left}})$, so that

$$\begin{aligned} \Big|\frac{\partial}{\partial E} F_{\text{Airey left}}(x,E)\Big| &= \Big| -\frac{1}{2}\lambda_{\text{left}}^{-1/3}\Big(\frac{\partial Y_{\text{left}}}{\partial x}\Big)^{-3/2}\Big(\frac{\partial^2 Y_{\text{left}}}{\partial x\,\partial E}\Big) A(\lambda_{\text{left}}^{2/3} Y_{\text{left}}) \\ &\qquad + \lambda_{\text{left}}^{-1/3}\cdot \lambda_{\text{left}}^{2/3} \frac{\partial Y_{\text{left}}}{\partial E}\Big(\frac{\partial Y_{\text{left}}}{\partial x}\Big)^{-1/2} A'(\lambda^{2/3} Y_{\text{left}})\Big| \\ &\le C_\#[\lambda_{\text{left}}^{-1/3} B_{\text{left}}^{+3/2}(B_{\text{left}}^{-1} S_{\text{left}}^{-1}) + \lambda_{\text{left}}^{-1/3} B_{\text{left}}^{+1/2}(1+\lambda_{\text{left}}^{2/3})^{1/4}\cdot \lambda_{\text{left}}^{2/3} S_{\text{left}}^{-1}] \\ &= C_\#[\lambda_{\text{left}}^{-1/3} B_{\text{left}}^{1/2} S_{\text{left}}^{-1} + \lambda_{\text{left}}^{1/2} B_{\text{left}}^{1/2} S_{\text{left}}^{-1}] \le C_\# \lambda_{\text{left}}^{1/2} B_{\text{left}}^{1/2} S_{\text{left}}^{-1} \quad \text{in } U_{\text{Airey left}}. \end{aligned}$$

Therefore

(63)

$$\begin{aligned} \int_{\hat I_{\text{Airey left}}} \Big|\frac{\partial}{\partial E} F_{\text{Airey left}}(x,E)\Big|^2\,dx &\le C_\# \lambda_{\text{left}} B_{\text{left}} S_{\text{left}}^{-2} |\hat I_{\text{Airey left}}| \\ &\le C_\# S_{\text{left}}^{-3/2} B_{\text{left}}^3 \quad \text{by } |\hat I_{\text{Airey left}}| < C_\# B_{\text{left}} \text{ and by definition of } \lambda_{\text{left}}. \end{aligned}$$

On the other hand, $(\int_{V<E} \frac{dx}{(E-V)^{1/2}})^3 \ge c_\#(\frac{B_{\text{left}}}{S_{\text{left}}^{1/2}})^3$, so (63) implies (58).

Next we prove (59). By definition of F_c^{left} we have

$$(64)\quad \frac{\partial}{\partial E} F_c^{\text{left}}(x,E) = \exp\Big[\pm i\frac{\pi}{4} + i\int_{x_{\text{left}}(E)}^{x} (E-V(t))^{1/2}\,dt\Big]\cdot$$
$$\Big\{ \frac{\sum_{k=0}^{N'} u_k^{\text{left}}(x,E)}{(E-V(x))^{1/4}} \cdot \frac{i}{2}\int_{x_{\text{left}}(E)}^{x} (E-V(t))^{-1/2}\,dt - \frac{1}{4}\frac{\sum_{k=0}^{N'} u_k^{\text{left}}(x,E)}{(E-V(x))^{5/4}} + \frac{\sum_{k=0}^{N'} \frac{\partial u_k^{\text{left}}}{\partial E}(x,E)}{(E-V(x))^{1/4}} \Big\}.$$

In $I_{\text{medium left}}$ we have $\Big|\frac{\sum_{k=0}^{N'} u_k^{\text{left}}}{(E-V)^{5/4}}\Big| \le \frac{C_\#}{(E-V)^{5/4}} \le \frac{C_\#}{(E-V(x))^{1/4}} \cdot \frac{C_\#}{S_{\text{left}}\lambda_{\text{left}}^{\varepsilon-2/3}}$ and $\frac{\sum_{k=1}^{N'} |\frac{\partial u_k^{\text{left}}}{\partial E}|}{(E-V(x))^{1/4}} \le \frac{C_\# S_{\text{left}}^{-1}}{(E-V(x))^{1/4}}$ by the global WKB lemma. So (64) shows that

$$\Big|\frac{\partial F_c^{\text{left}}}{\partial E}\Big| \le C_\#(E-V(x))^{-1/4}\Big\{\int_{V<E}\frac{dy}{(E-V)^{1/2}} + S_{\text{left}}^{-1}\lambda_{\text{left}}^{2/3-\varepsilon}\Big\} \quad \text{in } I_{\text{medium left}}.$$

Now the integral in curly brackets dominates $S_{\text{left}}^{-1/2}B_{\text{left}} = S_{\text{left}}^{-1}\lambda_{\text{left}}$, which dominates the 2$^{\text{nd}}$ term in curly brackets. Hence $|\frac{\partial F_c^{\text{left}}}{\partial E}| \le C_\#(E-V(x))^{-1/4}\{\int_{V<E}\frac{dy}{(E-V)^{1/2}}\}$ in $I_{\text{medium left}}$, from which (59) is obvious.

To prove (60), we return to (64), and estimate the right–hand side on I_{center}. On I_{center} the global WKB lemma shows that

$$\Big|\sum_{k=0}^{N'} u_k^{\text{left}}\Big| \le C_\#, \Big|\sum_{k=1}^{N'}\frac{\partial u_k^{\text{left}}}{\partial E}\Big| \le \sum_{k=1}^{N'} C_\#\Lambda^{-k}\int_{V<E}\frac{\Lambda\,dy}{\lambda(y)S(y)B(y)}$$
$$\le C_\#\int_{V<E}\frac{dy}{\lambda(y)S(y)B(y)} = C_\#\int_{V<E}\frac{dy}{\lambda(y)[S^{1/2}(y)B(y)]S^{1/2}(y)}$$
$$\le C_\#\Lambda^{-2}\int_{V<E}\frac{dy}{S^{1/2}(y)} \le C_\#\Lambda^{-2}\int_{V<E}\frac{dy}{(E-V(y))^{1/2}} \quad \text{by (E7)}.$$

Hence (64) implies

$$\Big|\frac{\partial F_c^{\text{left}}}{\partial E}\Big| \le \frac{C_\#}{(E-V(x))^{1/4}}\Big\{\int_{V<E}\frac{dy}{(E-V)^{1/2}} + \frac{1}{(E-V(x))}\Big\} \quad \text{on } I_{\text{center}},$$

and since

$$\int_{V<E} \frac{dy}{(E-V)^{1/2}} \geq c_{\#} \frac{B(x)}{S^{1/2}(x)} \quad \text{(by (E7))}$$
$$= \frac{c_{\#}\lambda(x)}{S(x)} \geq \frac{c_{\#}\lambda(x)}{(E-V(x))},$$

the first term in curly brackets dominates. So we have

$$\Big|\frac{\partial F_c^{\text{left}}}{\partial E}\Big| \leq \frac{C_{\#}}{(E-V(x))^{1/4}} \int_{V<E} \frac{dy}{(E-V)^{1/2}} \quad \text{on } I_{\text{center}},$$

from which (60) is obvious. ■

As E_1 varies, the intervals $I_{\text{far left}}$, $\hat{I}_{\text{Airey left}}$, etc. in Lemma 12 vary. However, for a given energy $\overline{E}_1$, we form the interval $\{|E_1 - \overline{E}_1| < \tilde{S}\} = \mathcal{T}$, with $\tilde{S} = \min(\lambda_{\text{left}}^{-\varepsilon} S_{\text{left}}, \lambda_{\text{right}}^{-\varepsilon} S_{\text{right}}, c_{\#}^{0} S_{\min})$. Then we can find fixed intervals $\overline{I}_{\text{far left}}$, $\overline{I}_{\text{Airey left}}$, $\overline{I}_{\text{medium left}}$, $\overline{I}_{\text{center}}$, $\overline{I}_{\text{medium right}}$, $\overline{I}_{\text{Airey right}}$, $\overline{I}_{\text{far right}}$ which cover I_{BVP}, do not depend on E_1, but satisfy $\overline{I}_{\text{far left}} \subset I_{\text{far left}}$, $\overline{I}_{\text{Airey left}} \subset \hat{I}_{\text{Airey left}}$, etc. for $E_1 \in \mathcal{T}$.

The conclusions of Lemmas 12 and 13 hold a fortiori with $I_{\text{far left}} \ldots I_{\text{far right}}$ replaced by $\overline{I}_{\text{far left}} \ldots \overline{I}_{\text{far right}}$.

For $E_1 \in \mathcal{T}$ we note that $\int_{V<E_1} \frac{dy}{(E_1-V)^{1/2}}$ is comparable to $\int_{x_{\text{left}}}^{x_{\text{right}}} \frac{dy}{S^{1/2}(y)}$, which does not depend on E_1.

Now suppose $E_1, E_2 \in \mathcal{T} \cap (-\infty, E_\infty]$ are distinct eigenvalues of $-\frac{d^2}{dx^2} + V(x)$. Let F_1, F_2 denote the corresponding (real) eigenfunctions of norm 1. Lemma 12 shows precisely how F_1, F_2 behave, in terms of two real constants b_1, b_2 of size $\sim (\int_{x_{\text{left}}}^{x_{\text{right}}} \frac{dx}{S^{1/2}(x)})^{-1/2}$. Lemma 12, Lemma 13 and the fundamental theorem of calculus show that

$$\Big(\int_{\overline{I}_{\text{Airey left}} \cup \overline{I}_{\text{medium left}} \cup \overline{I}_{\text{center}} \cup \overline{I}_{\text{medium right}} \cup \overline{I}_{\text{Airey right}}} |b_1^{-1}F_1 - b_2^{-1}F_2|^2\, dx\Big)^{1/2}$$
$$\leq C_{\#}|E_1 - E_2| \Big(\int_{x_{\text{left}}}^{x_{\text{right}}} \frac{dx}{S^{1/2}(x)}\Big)^{3/2}$$
$$+ C_{\#}\Lambda^{K+33-\frac{3}{2}\varepsilon N'} \Big(\int_{x_{\text{left}}}^{x_{\text{right}}} \frac{dx}{S^{1/2}(x)}\Big)^{+1/2}$$

and that

$$\Big(\int_{\overline{I}_{\text{far left}}\cup\overline{I}_{\text{far right}}}|b_1^{-1}F_1|^2+|b_2^{-1}F_2|^2\,dx\Big)^{1/2}\le\Big(\int_{x_{\text{left}}}^{x_{\text{right}}}\frac{dx}{S^{1/2}(x)}\Big)^{+1/2}\cdot C_{\#}\Lambda^{K+33-\frac{3}{2}\varepsilon N'}.$$

Since $\overline{I}_{\text{far left}},\ldots,\overline{I}_{\text{ far right}}$ cover I_{BVP}, we get:

$$\Big(\int_{I_{\text{BVP}}}|F_1-(b_1b_2^{-1})F_2|^2\,dx\Big)^{1/2}\le C_{\#}|E_1-E_2|\Big(\int_{x_{\text{left}}}^{x_{\text{right}}}\frac{dx}{S^{1/2}(x)}\Big)+C_{\#}\Lambda^{K+33-\frac{3}{2}\varepsilon N'}.$$

Since F_1 and F_2 are eigenfunctions for distinct eigenvalues, they are orthogonal on I_{BVP}, so the left–hand side is at least one. Consequently,

$$|E_1-E_2|\Big(\int_{x_{\text{left}}}^{x_{\text{right}}}\frac{dx}{S^{1/2}(x)}\Big)\ge c_{\#}\text{ if }E_1,E_2\in\mathcal{T}=\{|E-\overline{E}_1|<\tilde{S}\}\cap(-\infty,E_\infty].$$

That is, if E_1, E_2 are distinct eigenvalues in $\{|E-\overline{E}_0|<c^0_{\#}S_{\min}\}\cap(-\infty,E_\infty]$, then

$$|E_1-E_2|\ge\min\Big\{\lambda_{\text{left}}^{-\varepsilon}S_{\text{left}},\lambda_{\text{left}}^{-\varepsilon}S_{\text{right}},c_{\#}\Big(\int_{x_{\text{left}}}^{x_{\text{right}}}\frac{dx}{S^{1/2}(x)}\Big)^{-1}\Big\}\tag{65}$$

We have $\int_{x_{\text{left}}}^{x_{\text{right}}}\frac{dx}{S^{1/2}(x)}>c_{\#}\frac{B_{\text{left}}}{S_{\text{left}}^{1/2}}=c_{\#}\frac{\lambda_{\text{left}}}{S_{\text{left}}}$, so

$$\Big(\int_{x_{\text{left}}}^{x_{\text{right}}}\frac{dx}{S^{1/2}(x)}\Big)^{-1}\le C_{\#}\frac{S_{\text{left}}}{\lambda_{\text{left}}}<<c_{\#}\frac{S_{\text{left}}}{\lambda^{\varepsilon}_{\text{left}}}.$$

Similarly for $\frac{S_{\text{right}}}{\lambda^{\varepsilon}_{\text{right}}}$. Therefore the right–hand side of (65) is comparable to $(\int_{x_{\text{left}}}^{x_{\text{right}}}\frac{dx}{S^{1/2}(x)})^{-1}$, and we get

$$|E_1-E_2|\ge c_{\#}\Big(\int_{x_{\text{left}}}^{x_{\text{right}}}\frac{dx}{S^{1/2}(x)}\Big)^{-1}\text{ for }E_1,E_2\text{ distinct eigenvalues in }\{|E-E_0|<c^0_{\#}S_{\min}\}\cap(-\infty,E_\infty].\tag{66}$$

This is our basic estimate for the gap between eigenvalues. Let us check what it tells us about the phase difference. Recall that

$$\Phi(E)=\pm\frac{\pi}{2}+\int_{V<E}(E-V(t))^{1/2}\,dt+\text{Im }\log(1+\sum_{k=1}^{N'}G_k(E))\tag{66 bis}$$

for a suitable branch of the logarithm. We know that

$$\left|\sum_{k=1}^{N'} G_k(E)\right| < \frac{1}{10} \text{ and } \left|\frac{\partial G_k}{\partial E}\right| \leq \frac{C_\#}{\Lambda^2}\int_{V<E}\frac{dx}{(E-V)^{1/2}} \tag{67}$$

by (62 bis).

Hence

$$\begin{aligned}\frac{d\Phi}{dE} &\geq \frac{1}{2}\int_{V<E}(E-V(t))^{-1/2}\,dt - \frac{C_\#}{\Lambda^2}\int_{V<E}(E-V)^{-1/2}\,dt\\ &\geq c_\# \int_{x_{\text{left}}}^{x_{\text{right}}} \frac{dx}{S^{1/2}(x)} \quad \text{for } |E-E_0| \leq c^0_\# S_{\min}.\end{aligned}$$

For any two energies E_1, E_2 in that range we have therefore $|\Phi(E_1)-\Phi(E_2)| \geq (c_\# \int_{x_{\text{left}}}^{x_{\text{right}}} \frac{dx}{S^{1/2}(x)}) \cdot |E_1 - E_2|$. Estimate (66) now implies the following result.

Lemma 14. *Let E_1, E_2 be distinct eigenvalues of $-\frac{d^2}{dx^2}+V(x)$ with $|E_i - E_0| < c^0_\# S_{\min}$ $(i=1,2)$ and $E_i \leq E_\infty$. Then $|\Phi(E_1)-\Phi(E_2)| \geq c_\#$.*

In particular, there is at most one eigenvalue with a phase near πk_1 for a given integer k_1.

Lemma 12 tells us what an eigenfunction looks like, once we know the eigenvalue. It says also that eigenvalues have phase very close to multiples of π. Lemma 14 says that or given k_1 there is at most one eigenvalue E_1 with $\Phi(E_1)$ near k_1. It remains to show that there is at least one eigenvalue E_1 with $\Phi(E_1)$ near k_1.

Lemma 15. *Suppose $|E_1 - E_0| < \frac{c^0_\#}{2} S_{\min}$ with $\Phi(E_1) = \pi k_1$ for an integer k_1. Then there is an energy E_1' in the spectrum of $-\frac{d^2}{dx^2}+V(x)$ with $|E_1' - E_0| < c^0_\# S_{\min}$ and $|\Phi(E_1') - \pi k_1| \leq C_\# \Lambda^{K+33-\frac{3}{2}\varepsilon N'}$. In particular, if $E_\infty \geq E_1 + C_\# \Lambda^{K+33-\frac{3}{2}\varepsilon N'} (\int_{x_{\text{left}}}^{x_{\text{right}}} \frac{dx}{S^{1/2}(x)})^{-1}$, then E_1' is an eigenvalue and $E_1' \leq E_\infty$.*

Proof. We use the global WKB lemma to construct a non zero function F in the domain of $H = -\frac{d^2}{dx^2} + V(x)$ satisfying

$$\|(H-E_1)F\|_{L^2(I_{\text{BVP}})} \leq C_\# \Big(\int_{x_{\text{left}}}^{x_{\text{right}}} \frac{dx}{S^{1/2}(x)}\Big)^{-1} \|F\|_{L^2(I_{\text{BVP}})} \cdot \Lambda^{K+33-\frac{3}{2}\varepsilon N'}. \tag{68}$$

Elementary spectral theory will then show that some E_1' can be found in the spectrum of H, with

(69) $$|E_1' - E_1| \leq C_\#\Big(\int_{x_{\mathrm{left}}}^{x_{\mathrm{right}}} \frac{dx}{S^{1/2}(x)}\Big)^{-1} \cdot \Lambda^{K+33-\frac{3}{2}\varepsilon N'}.$$

From (66 bis) and (67) we have $\frac{d\Phi}{dE} \leq C_\#(\int_{x_{\mathrm{left}}}^{x_{\mathrm{right}}} \frac{dx}{S^{1/2}(x)})$ for $|E - E_0| \leq c_\#^0 S_{\min}$. We will check that E_1' lies in this range. That is a consequence of the estimate

(70) $$\int_{x_{\mathrm{left}}}^{x_{\mathrm{right}}} \frac{dx}{S^{1/2}(x)} \geq c_\#\Big(\frac{S_{\min}}{\Lambda}\Big)^{-1}$$

and (69).

To check (70), pick $\overline{x} \in [x_{\mathrm{left}}, x_{\mathrm{right}}]$ with $S(\overline{x}) \sim S_{\min}$. We may take $|\overline{x} - x_{\mathrm{left}}| > c_\# B_{\mathrm{left}}$ and $|\overline{x} - x_{\mathrm{right}}| > c_\# B_{\mathrm{right}}$. Then we have

$$\int_{x_{\mathrm{left}}}^{x_{\mathrm{right}}} \frac{dx}{S^{1/2}(x)} \geq \frac{B(\overline{x})}{S^{1/2}(\overline{x})} = c_\# \frac{\lambda(\overline{x})}{S(\overline{x})} \geq \frac{c_\# \Lambda}{S_{\min}},$$

proving (70).

Now we know that $|\frac{d\Phi}{dE}| \leq C_\#(\int_{x_{\mathrm{left}}}^{x_{\mathrm{right}}} \frac{dx}{S^{1/2}(x)})$ for E between E_1 and E_1'. Thus

$$\begin{aligned}|\Phi(E_1') - \Phi(E_1)| &\leq C_\#\Big(\int_{x_{\mathrm{left}}}^{x_{\mathrm{right}}} \frac{dx}{S^{1/2}(x)}\Big)|E_1' - E_1| \\ &\leq C_\# \Lambda^{K+33-\varepsilon N'} \quad \text{by (69)}.\end{aligned}$$

That is, $|\Phi(E_1') - \pi k_1| \leq C_\# \Lambda^{K+33-\frac{3}{2}\varepsilon N'}$.

Furthermore, if $E_\infty \geq E_1 + C_\# \Lambda^{K+33-\frac{3}{2}\varepsilon N'}(\int_{x_{\mathrm{left}}}^{x_{\mathrm{right}}} \frac{dx}{S^{1/2}(x)})^{-1}$, then $E_1' < E_\infty$. Since the part of the spectrum of H below E_∞ consists entirely of eigenvalues, E_1' must be an eigenvalue of H.

Thus the proof of the lemma is reduced to constructing a nonzero $F \in \mathrm{Domain}(H)$ satisfying (68).

Let $\chi_{\text{Airey left}}, \chi_{\text{medium left}}, \chi_{\text{center}}, \chi_{\text{medium right}}, \chi_{\text{Airey right}}$ be smooth functions with the following properties:

(71) $$\chi_{\text{Airey left}} + \chi_{\text{medium left}} + \chi_{\text{center}} + \chi_{\text{medium right}} + \chi_{\text{Airey right}} = 1$$
$$\text{on } \hat{I}_{\text{Airey left}} \cup I_{\text{medium left}} \cup I_{\text{center}} \cup I_{\text{medium right}} \cup \hat{I}_{\text{Airey right}}.$$

$$(72)\quad \left|\left(\frac{d}{dx}\right)^{\alpha}\chi_{\text{Airey left}}\right| \quad \leq \quad C_{\#}^{\alpha}(\lambda_{\text{left}}^{-\varepsilon}B_{\text{left}})^{-\alpha} \quad \text{and similarly for} \quad \chi_{\text{Airey right}}$$

$$(73)\quad \left|\left(\frac{d}{dx}\right)^{\alpha}\chi_{\text{medium left}}\right| \leq C_{\#}^{\alpha}(\lambda_{\text{left}}^{-\varepsilon}B_{\text{left}})^{-\alpha}\text{and similarly for } \chi_{\text{medium right}}$$

$$(74)\quad \left|\left(\frac{d}{dx}\right)^{\alpha}\chi_{\text{center}}(x)\right| \leq C_{\#}^{\alpha}B^{-\alpha}(x)$$

and with $\chi_{\text{Airey left}}$ supported in $I_{\text{Airey left}}$, $\chi_{\text{medium left}}$ supported in $I_{\text{medium left}}$, χ_{center} supported in I_{center}, $\chi_{\text{medium right}}$ supported in $I_{\text{medium right}}$, $\chi_{\text{Airey right}}$ supported in $I_{\text{Airey right}}$. We may assume $\chi_{\text{Airey left}} > 0$ in $\hat{I}_{\text{left}}$, and similarly for $\hat{I}_{\text{Airey right}}$. We define

$$F = \Big[\chi_{\text{Airey left}}F_{\text{Airey left}}(x, E_1) + (\chi_{\text{medium left}} + \chi_{\text{center}})\text{Re}(F_c^{\text{left}}(x, E_1))\Big]$$
$$+ (-1)^{k_1}R(E_1)\Big[\chi_{\text{medium right}} \cdot \text{Re}(F_c^{\text{right}}(x, E_1)) + \chi_{\text{Airey right}}F_{\text{Airey right}}(x, E_1)\Big].$$

In view of the support properties of the χ's, we know that F vanishes in a neighborhood of the endpoints of I_{BVP}, hence belongs to the domain of H. The Corollary to Lemma 2 shows us that

$$(75)\quad \int_{I_{\text{center}}} |F|^2\,dx \geq c_{\#}\int_{x_{\text{left}}}^{x_{\text{right}}} \frac{dx}{S^{1/2}(x)}.$$

We estimate $(\frac{d^2}{dx^2} + E_1 - V(x))F$ on each of the intervals $I_{\text{far left}}$ $I_{\text{Airey left}} \cdots$ $I_{\text{Airey right}}$ $I_{\text{far right}}$. In $I_{\text{far left}}$ we have

$$\begin{aligned}(\frac{d^2}{dx^2} + E_1 - V(x))F = \chi_{\text{Airey left}}\Big\{\Big(\frac{d^2}{dx^2} + E_1 - V(x)\Big)F_{\text{Airey left}}\Big\} \\ + 2\frac{d\chi_{\text{Airey left}}}{dx}\cdot\frac{dF_{\text{Airey left}}}{dx} \\ + \frac{d^2\chi_{\text{Airey left}}}{dx^2}\cdot F_{\text{Airey left}}.\end{aligned}$$

Outside $I_{\text{far left}} \cap I_{\text{Airey left}}$ the right–hand side is identically zero, while inside $I_{\text{far left}} \cap I_{\text{Airey left}}$ it is dominated by $C_{\#}\lambda_{\text{left}}^{-N'+2\varepsilon}B_{\text{left}}^{-3/2}$, by the global WKB lemma.

Hence

$$\int_{I_{\text{far left}}} \Big|\Big(\frac{d^2}{dx^2}+E_1-V(x)\Big)F\Big|^2\,dx \le C_\#\lambda_{\text{left}}^{-2N'+4\varepsilon}B_{\text{left}}^{-3}|I_{\text{Airey left}}| \le C_\#\lambda_{\text{left}}^{-2N'+4\varepsilon}B_{\text{left}}^{-2}.$$

Hypothesis (E6) yields

$$\text{(75 bis)} \qquad \int_{x_{\text{left}}}^{x_{\text{right}}} \frac{dx}{S^{1/2}(x)} \le \Lambda^K \frac{B_{\text{left}}}{S_{\text{left}}^{1/2}} = \Lambda^K \frac{B_{\text{left}}^2}{\lambda_{\text{left}}},$$

so that $\lambda_{\text{left}}^{-2N'+4\varepsilon}B_{\text{left}}^{-2} \le C_\#\Lambda^{-2N'+K+4\varepsilon}(\int_{x_{\text{left}}}^{x_{\text{right}}}\frac{dx}{S^{1/2}(x)})^{-1}$. Hence

$$\text{(76)} \qquad \int_{I_{\text{far left}}} \Big|\Big(\frac{d^2}{dx^2}+E_1-V(x)\Big)F\Big|^2\,dx \le C_\#\Lambda^{K-2N'+4\varepsilon}\Big(\int_{x_{\text{left}}}^{x_{\text{right}}}\frac{dx}{S^{1/2}(x)}\Big)^{-1}.$$

An analogous result holds for $I_{\text{far right}}$.

On $\tilde I_{\text{Airey left}} \equiv \text{supp}\ \chi_{\text{Airey left}}$ we have all the χ's identically zero except χ_{Airey} and $\chi_{\text{medium left}}$. Thus

$$\begin{aligned}\Big(\frac{d^2}{dx^2}+E_1-V(x)\Big)F &= \chi_{\text{Airey left}}\Big(\frac{d^2}{dx^2}+E_1-V(x)\Big)F_{\text{Airey left}}\\ &+\chi_{\text{medium left}}\Big(\frac{d^2}{dx^2}+E_1-V(x)\Big)\text{Re}(F_c^{\text{left}})\\ &+2\frac{d\chi_{\text{Airey left}}}{dx}\frac{d}{dx}\Big(F_{\text{Airey left}}-\text{Re}(F_c^{\text{left}})\Big)\\ &+\frac{d^2}{dx^2}\chi_{\text{Airey left}}\cdot\Big(F_{\text{Airey left}}-\text{Re}(F_c^{\text{left}})\Big) \quad \text{on } \tilde I_{\text{Airey left}}.\end{aligned} \qquad \text{(77)}$$

Here we have used $\frac{d}{dx}\chi_{\text{medium left}} = -\frac{d}{dx}\chi_{\text{Airey left}}$ and $\frac{d^2\chi_{\text{medium left}}}{dx^2} = -\frac{d^2\chi_{\text{Airey left}}}{dx^2}$, which hold on $\tilde I_{\text{Airey left}}$ since $\chi_{\text{Airey left}}+\chi_{\text{medium left}}\equiv 1$ there.

The global WKB lemma shows that the terms on the right of (77) are dominated by $C_\#\lambda_{\text{left}}^{-N'}B_{\text{left}}^{-3/2}$, $C_\#\lambda_{\text{left}}^{10-\frac32\varepsilon N'}B_{\text{left}}^{-3/2}$, $C_\#\lambda_{\text{left}}^{10-\frac32\varepsilon N'+\varepsilon}B_{\text{left}}^{-3/2}$, $C_\#\lambda_{\text{left}}^{10-\frac32\varepsilon N'+2\varepsilon}B_{\text{left}}^{-3/2}$ respectively, in view of our estimates on the derivatives of the χ's. Hence

$$\Big|\Big(\frac{d^2}{dx^2}+E_1-V(x)\Big)F\Big| \le C_\#\lambda_{\text{left}}^{10-\frac32\varepsilon N'+2\varepsilon}B_{\text{left}}^{-3/2} \quad \text{on } \tilde I_{\text{Airey left}},$$

so that as in the proof of (76) we have

$$\text{(78)} \qquad \int_{\tilde I_{\text{Airey left}}} \Big|\Big(\frac{d^2}{dx^2}+E_1-V(x)\Big)F\Big|^2 dx \le C_\# \lambda_{\text{left}}^{20-3\varepsilon N'+4\varepsilon} B_{\text{left}}^{-2}$$

$$\le C_\# \Lambda^{21-3\varepsilon N'+K}\Big(\int_{x_{\text{left}}}^{x_{\text{right}}} \frac{dx}{S^{1/2}(x)}\Big)^{-1}.$$

There is an analogous estimate for $\tilde I_{\text{Airey right}} \equiv \text{supp}\,\chi_{\text{Airey right}}$. Note that $\hat I_{\text{Airey left}} \subset \tilde I_{\text{Airey left}}$, and similarly for $\hat I_{\text{Airey right}}$. Next we investigate $I_{\text{medium left}} \backslash \tilde I_{\text{Airey left}} \equiv \tilde I_{\text{medium left}}$. Here all the χ's are identically zero except for $\chi_{\text{medium left}}$, χ_{center}. So $\chi_{\text{medium left}} + \chi_{\text{center}} = 1$ on $\tilde I_{\text{medium left}}$, and thus $F = \text{Re}(F_c^{\text{left}})$ on $\tilde I_{\text{medium left}}$. Hence the global WKB lemma gives $|(\frac{d^2}{dx^2}+E_1-V(x))F| \le C_\# \lambda_{\text{left}}^{10-\frac{3}{2}\varepsilon N'} B_{\text{left}}^{-3/2}$ on $\tilde I_{\text{medium right}}$, so that as in the proofs of (77), (78) we get

$$\text{(79)} \quad \int_{\tilde I_{\text{medium left}}} \Big|\Big(\frac{d^2}{dx^2}+E_1-V(x)\Big)F\Big|^2 dx \le C_\# \Lambda^{20-3\varepsilon N'+K}\Big(\int_{x_{\text{left}}}^{x_{\text{right}}} \frac{dx}{S^{1/2}(x)}\Big)^{-1}.$$

For $\tilde I_{\text{medium right}} \equiv I_{\text{medium right}} \backslash \tilde I_{\text{Airey right}}$, the situation is different, since the matching of F_c^{left} with F_c^{right} comes into play. We argue as follows.

Since $\mathcal{A}(E_1) = R(E_1)e^{i\Phi(E_1)}$ with $\Phi(E_1) = \pi k_1$, we have $\mathcal{A}(E_1) = (-1)^{k_1} R(E_1)$. On $\tilde I_{\text{medium right}}$, all the χ's are identically zero except for χ_{center} and $\chi_{\text{medium right}}$. Hence

$$F = \chi_{\text{center}} \text{Re}(F_c^{\text{left}}) + \chi_{\text{medium right}} \cdot (-1)^{k_1} R(E_1) \text{Re}(F_c^{\text{right}}) \quad \text{in } \tilde I_{\text{medium right}}.$$

Using also $\chi_{\text{center}} + \chi_{\text{medium right}} \equiv 1$ in $\tilde I_{\text{medium right}}$ we conclude that

$$\Big(\frac{d^2}{dx^2}+E_1-V(x)\Big)F = \chi_{\text{center}}\Big(\frac{d^2}{dx^2}+E_1-V(x)\Big)\text{Re}(F_c^{\text{left}})$$

$$+ \chi_{\text{medium right}} \cdot (-1)^{k_1} R(E_1)\Big(\frac{d^2}{dx^2}+E_1-V(x)\Big)$$

$$\text{Re}(F_c^{\text{right}})$$

$$+ 2\frac{d\,\chi_{\text{center}}}{dx} \cdot \frac{d}{dx}\{\text{Re}(F_c^{\text{left}} - \mathcal{A}(E_1)F_c^{\text{right}})\}$$

$$+ \frac{d^2\chi_{\text{center}}}{dx^2} \cdot \text{Re}(F_c^{\text{left}} - \mathcal{A}(E_1)F_c^{\text{right}}) \text{ on } \tilde I_{\text{medium right}}.$$

The global WKB lemma shows that the first two terms on the right are bounded by $C_\# \Lambda^{-N'} S_{\text{right}}^{-1/4} B_{\text{right}}^{-2}$, $C_\# \lambda_{\text{right}}^{10-\frac{3}{2}\varepsilon N'} B_{\text{right}}^{3/2}$, while the last two terms on the right are supported in $I_{\text{center}} \cap \tilde{I}_{\text{medium right}}$ and are dominated by $C_\# \Lambda^{-N'-1} S_{\text{right}}^{-1/4} B_{\text{right}}^{-2}$ and $C_\# \Lambda^{-N'-1} S_{\text{right}}^{+1/4} B_{\text{right}}^{-1}$ there. Hence $|(\frac{d^2}{dx^2} + E_1 - V(x))F| \leq$ $C_\# \Lambda^{10-\frac{3}{2}\varepsilon N'} \lambda_{\text{right}}^{1/2} B_{\text{right}}^{-3/2}$ in $\tilde{I}_{\text{medium right}}$, so that

(80)

$$\int_{\tilde{I}_{\text{medium right}}} \left| \Big(\frac{d^2}{dx^2} + E_1 - V(x) \Big) F \right|^2 dx \leq C_\# \Lambda^{20-3\varepsilon N'} \lambda_{\text{right}} B_{\text{right}}^{-3} |\tilde{I}_{\text{medium right}}|$$
$$\leq C_\# \Lambda^{20-3\varepsilon N'} \lambda_{\text{right}} B_{\text{right}}^{-2} \leq C_\# \Lambda^{20-3\varepsilon N'+K} \Big(\int_{x_{\text{left}}}^{x_{\text{right}}} \frac{dx}{S^{1/2}(x)} \Big)^{-1},$$

by virtue of the analogue of (75 bis) for λ_{right}, B_{right}.

Next we look at $\tilde{I}_{\text{center}} = I_{\text{center}} \backslash (I_{\text{medium left}} \cup I_{\text{medium right}})$. Here $\chi_{\text{center}} \equiv 1$ and the other χ's are identically zero. Hence $F = \text{Re}(F_c^{\text{left}})$ on $\tilde{I}_{\text{center}}$. The global WKB lemma therefore tells us that $|(\frac{d^2}{dx^2} + E_1 - V(x))F| \leq C_\# \Lambda^{-N'} S^{-1/4}(x) B^{-2}(x)$ on $\tilde{I}_{\text{center}}$, hence $\int_{\tilde{I}_{\text{center}}} |(\frac{d^2}{dx^2} + E_1 - V(x))F|^2 \, dx \leq C_\# \Lambda^{-2N'} \int_{I_{\text{center}}} \frac{dx}{S^{1/2}(x) B^4(x)}$. Hypothesis (E5) shows that the right–hand side is dominated by $C_\# \Lambda^{-2N'+K} (\int_{x_{\text{left}}}^{x_{\text{right}}} \frac{dx}{S^{1/2}(x)})^{-1}$, so we get

$$\text{(81)} \quad \int_{\tilde{I}_{\text{center}}} \left| \Big(\frac{d^2}{dx^2} + E_1 - V(x) \Big) F \right|^2 dx \leq C_\# \Lambda^{40+K-3\varepsilon N'} \Big(\int_{x_{\text{left}}}^{x_{\text{right}}} \frac{dx}{S^{1/2}(x)} \Big)^{-1}.$$

Estimates (76), (78) and their analogues for $I_{\text{far right}}$, $\tilde{I}_{\text{Airey right}}$, together with estimates (79), (80), (81) show that

$$\text{(82)} \quad \int_{I_{\text{BVP}}} \left| \Big(\frac{d^2}{dx^2} + E_1 - V(x) \Big) F \right|^2 dx \leq C_\# \Lambda^{40+K-3\varepsilon N'} \Big(\int_{x_{\text{left}}}^{x_{\text{right}}} \frac{dx}{S^{1/2}(x)} \Big)^{-1}.$$

The reason is that the regions of integration in those estimates cover I_{BVP}.

Now from (75) and (82), we see that

$$\|(H - E_1)F\|^2 \leq \|F\|^2 \cdot \frac{C_\# \Lambda^{40+K-3\varepsilon N'} (\int_{x_{\text{left}}}^{x_{\text{right}}} \frac{dx}{S^{1/2}(x)})^{-1}}{(\int_{x_{\text{left}}}^{x_{\text{right}}} \frac{dx}{S^{1/2}(x)})},$$

which implies (68). The proof of the lemma is complete. ■

We summarize our knowledge of the eigenvalues of E in the following result.

Lemma 16. *Let* $J = [E_0 - \frac{1}{4}c^0_{\#}S_{\min}, E_0 + \frac{1}{4}c^0_{\#}S_{\min}] \cap (-\infty, E_\infty]$, $J(k) = \{E \mid |E - E_0| < c^0_{\#}S_{\min}$ *and* $|\Phi(E) - \pi k| < C_{\#}\Lambda^{K+33-\frac{3}{2}\varepsilon N'}\}$, $\mathcal{K} = \{k \in \mathbb{Z} \mid J(k) \cap J \neq \emptyset\}$. *Then* $\mathcal{K} = \{k \in \mathbb{Z} \mid k_{\min} \leq k \leq k_{\max}\}$ *for integers* $k_{\min} < k_{\max}$. *Every eigenvalue in* J *belongs to one of the* $J(k)$ $(k \in \mathcal{K})$. *For* $k_{\min} \leq k < k_{\max}$, *there is exactly one eigenvalue in* $J(k)$. *Also there is exactly one eigenvalue* $\leq E_\infty$ *in* $J(k_{\max})$, *unless* $|E_\infty - E_0| < c^0_{\#}S_{\min}$ *and* $|\Phi(E_\infty) - \pi k_{\max}| < C_{\#}\Lambda^{K+33-\frac{3}{2}\varepsilon N'}$, *in which case there is at most one eigenvalue* $\leq E_\infty$ *in* $J(k_{\max})$.

Proof. First we check that $\mathcal{K} = \{k \in \mathbb{Z} \mid k_{\min} \leq k \leq k_{\max}\}$ with $k_{\min} < k_{\max}$. In fact, $\mathcal{K} = \{k \in \mathbb{Z} \mid \text{distance } (\pi k, \Phi(J)) < C_{\#}\Lambda^{K+33-\frac{3}{2}\varepsilon N'}\}$. Since $0 < c_{\#}\int_{x_{\text{left}}}^{x_{\text{right}}} \frac{dx}{S^{1/2}(x)} < \frac{d\Phi(E)}{dE} < C_{\#}\int_{x_{\text{left}}}^{x_{\text{right}}} \frac{dx}{S^{1/2}(x)}$ for $E \in J$, and since (length J) $\int_{x_{\text{left}}}^{x_{\text{right}}} \frac{dx}{S^{1/2}(x)} \geq \frac{1}{4}c^0_{\#}S_{\min}\int_{x_{\text{left}}}^{x_{\text{right}}} \frac{dx}{S^{1/2}(x)} \geq c_{\#}\Lambda$ by (70), we know that $\Phi(J)$ is a finite interval of length $\geq c_{\#}\Lambda$. Hence the desired form of $\mathcal{K}$ is clear.

Next we check that every eigenvalue $E \in J$ belongs to $J(k)$ for some $k \in \mathcal{K}$. In fact, lemma 12 shows that $|\Phi(E) - \pi k| < C_{\#}\Lambda^{K+33-\frac{3}{2}\varepsilon N'}$ for an integer k. Thus, $E \in J(k)$. Also, $k \in \mathcal{K}$ since $E \in J(k) \cap J$. So E belongs to $J(k)$ for some $k \in \mathcal{K}$, as asserted.

Next we note that every $J(k)$ contains at most one eigenvalue $\leq E_\infty$, as follows immediately from Lemma 14.

It remains only to show that a given $J(k)$ $(k \in \mathcal{K})$ contains at least one eigenvalue $\leq E_\infty$, unless $k = k_{\max}$, $|E_\infty - E_0| < c^0_{\#}S_{\min}$ and $|\Phi(E_\infty) - \pi k| < C_{\#}\Lambda^{K+33-\frac{3}{2}\varepsilon N'}$. Our argument is as follows. Given $k \in \mathcal{K}$ there is by definition some $E_1 \in J$ with $|\Phi(E_1) - \pi k| < C_{\#}\Lambda^{K+33-\frac{3}{2}\varepsilon N'}$.

Now $\frac{d\Phi}{dE} \geq c_{\#}\int_{x_{\text{left}}}^{x_{\text{right}}} \frac{dx}{S^{1/2}(x)}$ on $J^+ = [E_0 - \frac{c^0_{\#}}{2}S_{\min}, E_0 + \frac{c^0_{\#}}{2}S_{\min}]$, and the distance from $E_1 \in J$ be the boundary of J^+ is at least $\frac{1}{4}c^0_{\#}S_{\min} > \frac{c_{\#}\Lambda}{(\int_{x_{\text{left}}}^{x_{\text{right}}} \frac{dx}{S^{1/2}(x)})}$ by (70). Hence there is some $\hat{E}_1 \in J^+$ with $\Phi(\hat{E}_1) = \pi k$. Lemma 15 implies that there is a point $E \in J(k) \cap \text{spectrum}(H)$. If $J(k)$ lies to the left of E_∞, then E must be an eigenvalue, by virtue of assumption (E1), and $E \le E_\infty$. Hence it is enough to show that $J(k)$ lies to the left of E_∞ unless $|E_\infty - E_0| < c^0_{\#} S_{\min}$ and $|\Phi(E_\infty) - \pi k| < C_{\#}\Lambda^{K+33-\frac{3}{2}\varepsilon N'}$ and $k = k_{\max}$. This is proved as follows.

Since $k \in \mathcal{K}$, we know that $J(k)$ intersects $J \subset (-\infty, E_\infty]$. Also since $\frac{d\Phi}{dE} > 0$ on $\{|E - E_0| < c^0_{\#} S_{\min}\}$, we know that $J(k)$ is an interval. Hence either $J(k)$ lies to the left of E_∞, or else $E_\infty \in J(k)$. We need only analyze the case $E_\infty \in J(k)$. Let $k' \in \mathcal{K}$ be different from k. The same argument just used for k shows that either $J(k')$ lies to the left of E_∞, or else $E_\infty \in J(k')$. We cannot have $E_\infty \in J(k')$, since E_∞ belongs to $J(k)$, and $J(k) \cap J(k') = \emptyset$. Thus $J(k')$ lies to the left of E_∞, while $J(k)$ contains E_∞. Since $J(k')$ and $J(k)$ are disjoint intervals, it follows that $J(k')$ lies to the left of $J(k)$. Since $\frac{d\Phi}{dE} > 0$ on $\{|E - E_0| < c^0_{\#} S_{\min}\}$, it follows that $k' < k$.

In other words, k is the maximal element of $\mathcal{K}$, $k = k_{\max}$. Also since $E_\infty \in J(k)$, we have by definition $|E_\infty - E_0| < c^0_{\#} S_{\min}$ and $|\Phi(E_\infty) - \pi k| < C_{\#}\Lambda^{K+33-\frac{3}{2}\varepsilon N'}$. This is what we were supposed to show. The proof of the lemma is complete. ■

Minor Improvements

The hypotheses in the preceding section have become two baroque. We begin this section by introducing a reasonably simple hypothesis that implies the complicated assumption (E3). In fact, suppose

$$\text{(E3)}' \quad \text{For } x \in I_{\text{BVP}}, x < x_{\text{left}} - \frac{1}{2}\lambda^K_{\text{left}} B_{\text{left}}, \text{ we have } V(x) - E_\infty \ge \frac{100}{|x - x_{\text{left}}|^2}.$$

Similarly, for $x \in I_{\text{BVP}}$, $x > x_{\text{rt}} + \frac{1}{2}\lambda^K_{\text{rt}} B_{\text{rt}}$, we have $V(x) - E_\infty \ge \frac{100}{|x - x_{\text{rt}}|^2}$.

We check that this and (E4) imply (E3), with $\tau = \frac{1}{2}$. It is enough to study $x < x_{\text{left}} - \lambda^K_{\text{left}} B_{\text{left}}$, since the argument for $x > x_{\text{rt}} + \lambda^K_{\text{rt}} B_{\text{rt}}$ is analogous. For $x <$

$x_{\text{left}} - \lambda^K_{\text{left}} B_{\text{left}}$, we have

$$\int\limits_x^{x_{\text{left}}(E)} (V(t)-E)^{1/2}\, dt \geq \int\limits_x^{x_{\text{left}}-\frac{1}{2}\lambda^K_{\text{left}}B_{\text{left}}} \frac{10}{|t-x_{\text{left}}|}\, dt +$$

$$\int\limits_{x_{\text{left}}(E)-c_{\#}B_{\text{left}}}^{x_{\text{left}}(E)} \cdot \Big(c_{\#}\frac{S_{\text{left}}}{B_{\text{left}}}(x_{\text{left}}(E)-t)\Big)^{1/2}\, dt$$

$$\geq 10\ln\Big(\frac{|x-x_{\text{left}}|}{\frac{1}{2}\lambda^K_{\text{left}}B_{\text{left}}}\Big) + c_{\#}S^{1/2}_{\text{left}}B_{\text{left}} = 10 ln\Big(\frac{|x-x_{\text{left}}|}{\frac{1}{2}\lambda^K_{\text{left}}B_{\text{left}}}\Big) + c_{\#}\lambda_{\text{left}}.$$

Hence with $\tau = \frac{1}{2}$ we have $\tau(V(x)-E)\exp(2(1-\tau)\int\limits_x^{x_{\text{left}}(E)}(V(t)-E)^{1/2}\, dt) \geq \frac{50}{|x-x_{\text{left}}|^2}\Big(\frac{|x-x_{\text{left}}|}{\frac{1}{2}\lambda^K_{\text{left}}B_{\text{left}}}\Big)^{10} e^{c_{\#}\lambda_{\text{left}}}$. For $x \leq x_{\text{left}} - \lambda^K_{\text{left}}B_{\text{left}}$, the right–hand side is smallest when $|x-x_{\text{left}}| = \lambda^K_{\text{left}}B_{\text{left}}$. Hence $\tau(V(x)-E)\exp(2(1-\tau)\int\limits_x^{x_{\text{left}}(E)}(V(t)-E)^{1/2}\, dt) \geq \frac{c_{\#}e^{c_{\#}\lambda_{\text{left}}}}{\lambda^K_{\text{left}}B^2_{\text{left}}} = \frac{c_{\#}e^{c_{\#}\lambda_{\text{left}}}S_{\text{left}}}{\lambda^{K+2}_{\text{left}}} = \Big[\frac{c_{\#}e^{c_{\#}\lambda_{\text{left}}}}{\lambda^{2K+N+2}_{\text{left}}}\Big]\lambda^N_{\text{left}}\cdot(\lambda^K_{\text{left}}S_{\text{left}}) \geq$
$\lambda^N_{\text{left}}\Big[\frac{c_{\#}e^{c_{\#}\lambda_{\text{left}}}}{\lambda^{2K+N+2}_{\text{left}}}\Big] \cdot$ MAX $(E-V)$ by (E4).

For $\lambda_{\text{left}} \geq C_{\#}$, the factor in brackets is more than 1, so the preceding estimate implies (E3). We have $\lambda_{\text{left}} \geq \Lambda \geq C_{\#}$, so (E3) is proven.

Also, assumptions (E3)$'$ and (E2) imply (E1) on the continuous spectrum of H. In fact, (E2) shows that V is bounded below on I_{BVP}, while (E3$'$) gives $V - E_\infty \geq 0$ outside a bounded subinterval $I' \subset I_{\text{BVP}}$. Letting $-M$ be a lower bound for $V - E_\infty$ on I_{BVP}, we cut I' into finitely many subintervals I_ν of length $< 10^{-10}M^{-1/2}$. If F has average zero on each of the I_ν, then $\int\limits_{I_\nu}(F')^2 + (V-E_\infty)F^2 \geq \int\limits_{I_\nu}(F')^2 - MF^2 \geq 0$, so $\int\limits_{I'}(F')^2 + (V-E_\infty)F^2 \geq 0$.

This shows that the quadratic form $<(H-E_\infty)F, F>$ is non–negative on a finite–codimension subspace of $L^2(I_{\text{BVP}})$, which implies that H_∞ has no continuous spectrum in $(-\infty, E_\infty)$. Thus (E1) follows from (E2), (E3$'$).

One last task for this section is to estimate the derivatives of the semiclassical phase

$$\phi_{\rm sc}(E) = \int\limits_{x_{\rm left}(E)}^{x_{\rm rt}(E)} (E - V(t))^{1/2}\ dt.$$

Under the hypotheses of the global WKB lemma, we have the following result.

Lemma. $|(\frac{d}{dE})^\beta \phi_{\rm sc}(E)| \le C_\# \int\limits_{x_{\rm left}}^{x_{\rm rt}} S^{\frac{1}{2}-\beta}(x)\ dx$ *for* $|E - E_0| < c^0_\# S_{\rm min}$.

Sketch of Proof. Write a partition of unity $1 = \theta_{\rm left}(x) + \theta_{\rm rt}(x) + \sum\limits_\nu \theta_\nu(x)$ in $[x_{\rm left} - c_\# B_{\rm left}, x_{\rm rt} + c_\# B_{\rm rt}]$, with

$$\theta_{\rm left} \text{ supported in } |x - x_{\rm left}| < c_\# B_{\rm left},\ |(\frac{d}{dx})^\alpha \theta_{\rm left}| \le C^\alpha_\# B^{-\alpha}_{\rm left},$$
$$\theta_{\rm rt} \text{ supported in } |x - x_{\rm rt}| < c_\# B_{\rm rt},\ |(\frac{d}{dx})^\alpha \theta_{\rm rt}| \le C_\# B^{-\alpha}_{\rm rt},$$
$$\theta_\nu \text{ supported in } |x - x_\nu| < c_\# B(x_\nu),\ |(\frac{d}{dx})^\alpha \theta_\nu| \le C^\alpha_\# B^{-\alpha}(x_\nu),$$
$$|x_{\rm left} - x_\nu|, |x_{\rm rt} - x_\nu| > c_\# \text{ diam(supp } \theta_\nu).$$

Then

$$\phi_{\rm sc}(E) = \int\limits_{-\infty}^{\infty} \theta_{\rm left}(t) \cdot (E - V(t))^{1/2}_+\ dt + \int\limits_{-\infty}^{\infty} \theta_{\rm rt}(t) \cdot (E - V(t))^{1/2}_+\ dt$$
$$+ \sum_\nu \int\limits_{-\infty}^{\infty} \theta_\nu(t) \cdot (E - V(t))^{1/2}\ dt \equiv \phi_{\rm left}(E) + \phi_{\rm rt}(E) + \sum_\nu \phi_\nu(E).$$

We have $E - V(t) = (G_{\rm left}(t,E))^2 \cdot (t - x_{\rm left}(E))$ for $|E - E_0| < c^0_\# S_{\rm min}$, $t \in \text{supp } \theta_{\rm left}$, with $|\partial^\alpha_t \partial^\beta_E G_{\rm left}| \le C^{\alpha\beta}_\# S^{\frac{1}{2}-\beta}_{\rm left} B^{-\frac{1}{2}-\alpha}_{\rm left}$, in view of our hypotheses on non–degeneracy of $V'(x)$ at $x_{\rm left}$. Hence, setting $t = x_{\rm left}(E) + s$, we have $\phi_{\rm left}(E) = \int\limits_0^\infty \theta_{\rm left}(x_{\rm left}(E) + s) \cdot G_{\rm left}(x_{\rm left}(E) + s, E) \cdot s^{1/2}\ ds \equiv \int\limits_0^\infty F_{\rm left}(E,s) \cdot s^{1/2}\ ds$, with $F_{\rm left}(E,s)$ supported in $0 \le s \le c_\# B_{\rm left}$, $|\partial^\beta_E F_{\rm left}| \le C^\beta_\# S^{\frac{1}{2}-\beta}_{\rm left} B^{-\frac{1}{2}}_{\rm left}$. Hence $|(\frac{d}{dE})^\beta \phi_{\rm left}(E)| \le \int\limits_0^{c_\# B_{\rm left}} C^\beta_\# S^{\frac{1}{2}-\beta}_{\rm left} B^{-1/2}_{\rm left} s^{1/2}\ ds \le C^\beta_\# S^{\frac{1}{2}-\beta}_{\rm left} B_{\rm left} \le$
$C^\beta_\# \int\limits_{|x - x_{\rm left}(E)| < c_\# B_{\rm left}} S^{1/2-\beta}(x)\ dx$. Similarly $|(\frac{d}{dE})^\beta \phi_{\rm rt}(E)| \le$

$C_{\#}^{\beta} \int\limits_{|x-x_{\rm rt}(E)|<c_{\#}B_{\rm rt}} S^{1/2-\beta}(x)\ dx$. Trivially, $(\frac{d}{dE})^{\beta}\phi_{\nu}(E) = c_{\beta}\int \theta_{\nu}(x)(E-V(x))^{\frac{1}{2}-\beta}\ dx$ so $|(\frac{d}{dE})^{\beta}\phi_{\nu}(E)| \leq C_{\#}^{\beta} \int\limits_{\text{supp }\theta_{\nu}} S^{\frac{1}{2}-\beta}(x)\ dx$. Summing our estimates for $(\frac{d}{dE})^{\beta}\phi_{\nu}(E)$, $(\frac{d}{dE})^{\beta}\phi_{\rm left}(E)$, $(\frac{d}{dE})^{\beta}\phi_{\rm rt}(E)$, we get the conclusion of the Lemma. ■

In the next section, we summarize what we know so far about eigenvalues and eigenfunctions of $-\frac{d^2}{dx^2}+V(x)$.

The WKB Theorems

Set-up. We are given the following: A potential $V(x)$ defined on a (possibly unbounded) interval $I_{\rm BVP}$; two positive functions $S(x)$, $B(x)$ defined on a subinterval $I \subset I_{\rm BVP}$; two real numbers $E_0 \leq E_{\infty}$; positive numbers $\varepsilon < \frac{1}{100}$, $K > 1$ and $N > K\varepsilon^{-10}$. We define $N' = [\varepsilon N/500]$ and $N'' = \frac{3}{2}\varepsilon N' - K - 33$.

Our goal is to understand the eigenvalues and eigenfunctions of the self–adjoint operator $H = \frac{-d^2}{dx^2} + V(x)$ on $L^2(I_{\rm BVP})$, with Dirichlet or Neumann conditions at the endpoints.

Hypotheses

Assumptions on $V(x)$, $S(x)$, $B(x)$ in I

(Hyp0) If $x, y \in I$ and $|x-y| < cB(x)$, then $c < \frac{B(y)}{B(x)} < C$ and $c < \frac{S(y)}{S(x)} < C$.

(Hyp1) For $x \in I$ and $\alpha \geq 0$ we have $|(\frac{d}{dx})^{\alpha}V(x)| \leq C_{\alpha}S(x)B^{-\alpha}(x)$.

(Hyp2) The equation $V(x) = E_0$ has two solutions $x_{\rm left} < x_{\rm rt}$ in I, and they satisfy $\text{dist}(x_{\rm left}, \partial I) > cB(x_{\rm left}), \text{dist}(x_{\rm rt}, \partial I) > cB(x_{\rm rt})$.

(Hyp3) For $x_{\rm left} \leq x \leq x_{\rm left} + c_1B(x_{\rm left})$ we have $-V'(x) > cS(x_{\rm left})B^{-1}(x_{\rm left})$, and for $x_{\rm rt} - c_1B(x_{\rm rt}) \leq x \leq x_{\rm rt}$ we have $+V'(x) > cS(x_{\rm rt})B^{-1}(x_{\rm rt})$.

(Hyp4) For $x_{\rm left} + c_1B(x_{\rm left}) \leq x \leq x_{\rm rt} - c_1B(x_{\rm rt})$ we have $cS(x) < E_0 - V(x) < CS(x)$.

To state the remaining hypotheses, we establish some notation. Set $\lambda(x) =$

$S^{1/2}(x)B(x)$ for $x \in I$. Then set

$$B_{\text{left}} = B(x_{\text{left}}), \quad S_{\text{left}} = S(x_{\text{left}}), \quad \lambda_{\text{left}} = \lambda(x_{\text{left}}).$$

$$B_{\text{rt}} = B(x_{\text{rt}}), \quad S_{\text{rt}} = S(x_{\text{rt}}), \quad \lambda_{\text{rt}} = \lambda(x_{\text{rt}}).$$

For $|E - E_0| < cS_{\text{left}}$, let $x_{\text{left}}(E)$ be the solution of $V(x) = E$ nearest to x_{left}, and for $|E - E_0| < cS_{\text{rt}}$, let $x_{\text{rt}}(E)$ be the solution of $V(x) = E$ nearest to x_{rt}. Define $S_{\min} = \inf_{x_{\text{left}}<x<x_{\text{rt}}} S(x)$ and $\Lambda = \left(\int_{x_{\text{left}}}^{x_{\text{rt}}} \frac{dx}{S^{1/2}(x)B^2(x)}\right)^{-1}$.

Our remaining hypotheses are as follows.

Assumptions on $V(x)$ in all of I_{BVP}

(Hyp5) If $|E - E_0| < c_2 S_{\min}$ and $E \le E_\infty$, then $V(x) > E$ for all $x \in I_{\text{BVP}} \backslash [x_{\text{left}}(E), x_{\text{rt}}(E)]$.

(Hyp6) If $x \in I_{\text{BVP}}$ satisfies $x < x_{\text{left}} - \frac{1}{2}\lambda_{\text{left}}^K B_{\text{left}}$ then $V(x) \ge E_\infty + \frac{100}{|x-x_{\text{left}}|^2}$, and if $x \in I_{\text{BVP}}$ satisfies $x > x_{\text{rt}} + \frac{1}{2}\lambda_{\text{rt}}^K B_{\text{rt}}$, then $V(x) \ge E_\infty + \frac{100}{|x-x_{\text{rt}}|^2}$.

Technical Assumptions

(Hyp7) $\max_{x\in I} S(x) \le \lambda_{\text{left}}^K S_{\text{left}}$ and $\max_{x\in I} S(x) \le \lambda_{\text{rt}}^K S_{\text{rt}}$

(Hyp8) $\int_{x_{\text{left}}}^{x_{\text{rt}}} \frac{dx}{S^{1/2}(x)} \le \Lambda^K \cdot \min(S_{\text{left}}^{-1/2} B_{\text{left}}, S_{\text{rt}}^{-1/2} B_{\text{rt}})$

(Hyp9) $[\int_{x_{\text{left}}}^{x_{\text{rt}}} \frac{dx}{S^{1/2}(x)B^4(x)}] \cdot [\int_{x_{\text{left}}}^{x_{\text{rt}}} \frac{dx}{S^{1/2}(x)}] \le \Lambda^K$

WKB Condition

(Hyp10) Λ is bounded below by a positive constant depending only on ε, K, N, and on c, C, c_1, c_2, C_α in (Hyp0)...(Hyp4).

Definitions and Basic Properties of Phases

Assume hypotheses (Hyp0)...(Hyp10). For $|E - E_0| < c_\# S_{\min}$, define

$$\phi(E) = \int_{x_{\text{left}}(E)}^{x_{\text{rt}}(E)} (E - V(x))^{1/2}\, dx \quad \text{and}$$

$$\psi(E) = \lim_{\delta\to 0+} \Big[\int_{x_{\text{left}}(E)+\delta}^{x_{\text{rt}}(E)-\delta} V''(x)(E - V(x))^{-\frac{3}{2}}\, dx - q(E)\delta^{-1/2} \Big]$$

with $q(E)$ uniquely specified by demanding the finiteness of the limit.

Lemma 1. *For* $|E-E_0| < c_\# S_{\min}$ *we have*

$$\left|\left(\frac{d}{dE}\right)^\beta \phi(E)\right| \le C_\#^\beta \int_{x_{\rm left}}^{x_{\rm rt}} S^{\frac{1}{2}-\beta}(x)\,dx \quad \textit{and}$$

$$\left|\left(\frac{d}{dE}\right)^\beta \psi(E)\right| \le C_\#^\beta \int_{x_{\rm left}}^{x_{\rm rt}} \frac{dx}{S^{\frac{1}{2}+\beta}(x) B^2(x)}.$$

Also $c_\# \int_{x_{\rm left}}^{x_{\rm rt}} \frac{dx}{S^{1/2}(x)} < \frac{d\phi(E)}{dE} < C_\# \int_{x_{\rm left}}^{x_{\rm rt}} \frac{dx}{S^{1/2}(x)}$.

The constants $c_\#$, $C_\#$, $C_\#^\beta$ *depend only on* ε, K, N, c, C, c_1, c_2, C_α *in the hypotheses (Hyp 0)...(Hyp 4).*

WKB Eigenvalue Theorem. *If (Hyp0)...(Hyp10) hold, then there is a function* $\Phi(E)$ *on* $[E_0 - c_\# S_{\min}, E_0 + c_\# S_{\min}]$ *and there are numbers* $E_{k_{\min}}, E_{k_{\min}+1}, \dots, E_{k_{\max}}$ $\le E_\infty$ *with the following properties.*

(A) $\Phi(E) = \pm\frac{\pi}{2} + \phi(E) + \frac{1}{48}\psi(E) + \phi_{\rm error}(E)$, *with*

$$\left|\left(\frac{d}{dE}\right)^\beta \phi_{\rm error}\right| \le C_\#^\beta \Lambda^{-1} \int_{x_{\rm left}}^{x_{\rm rt}} \frac{dx}{S^{\frac{1}{2}+\beta}(x) B^2(x)}, \quad \textit{all } \beta \ge 0.$$

(B) $\{k_{\min}, k_{\min}+1, \dots, k_{\max}\}$ *is exactly the set of integers* k *for which* $|\Phi(E) - \pi k| < C_\# \Lambda^{-N''}$ *for some* $E \in [E_0 - \frac{1}{4}c_\# S_{\min}, E_0 + \frac{1}{4}c_\# S_{\min}] \cap (-\infty, E_\infty]$.

(C) *If* $k_{\min} \le k < k_{\max}$, *then* E_k *is an eigenvalue of* H.

(D) *If* $k = k_{\max}$, *then* E_k *is either an eigenvalue of* H *or equal to* E_∞.

(E) *For* $k_{\min} \le k \le k_{\max}$ *we have* $|E_k - E_0| < c_\# S_{\min}$ *and* $|\Phi(E_k) - \pi k| < C_\# \Lambda^{-N''}$.

(F) *Every eigenvalue of* H *in the interval* $[E_0 - \frac{1}{4}c_\# S_{\min}, E_0 + \frac{1}{4}c_\# S_{\min}]$ $\cap (-\infty, E_\infty]$ *is one of the* E_k $(k_{\min} \le k \le k_{\max})$.

The constants $c_\#$, $C_\#^\beta$ depend only on ε, K, N, c, C, c_1 c_2, C_α in (Hyp0)...(Hyp5).

Remark. Perhaps $E_{k_{\min}}$ or $E_{k_{\max}}$ or both lie slightly outside

$$[E_0 - \frac{1}{4}c_\# S_{\min}, E_0 + \frac{1}{4}c_\# S_{\min}].$$

We prepare to describe the eigenfunctions of H. Given an energy E with $|E - E_0| < c_\# S_{\min}$, define intervals:

$$I^E_{\text{far left}} = I_{\text{BVP}} \cap (-\infty, x_{\text{left}}(E) - \lambda_{\text{left}}^{\varepsilon - 2/3} B_{\text{left}}]$$

$$I^E_{\text{Airey left}} = [x_{\text{left}}(E) - \lambda^{-\varepsilon}_{\text{left}} B_{\text{left}}, x_{\text{left}}(E) + \lambda^{-\varepsilon}_{\text{left}} B_{\text{left}}]$$

$$I^E_{\text{medium left}} = [x_{\text{left}}(E) + \lambda_{\text{left}}^{\varepsilon - \frac{2}{3}} B_{\text{left}}, x_{\text{left}}(E) + c_\# B_{\text{left}}]$$

$$I^E_{\text{center}} = [x_{\text{left}}(E) + \frac{1}{2} c_\# B_{\text{left}}, x_{\text{rt}}(E) - \frac{1}{2} c_\# B_{\text{rt}}]$$

$$I^E_{\text{medium rt}} = [x_{\text{rt}}(E) - c_\# B_{\text{rt}}, x_{\text{rt}}(E) - \lambda_{\text{rt}}^{\varepsilon - 2/3} B_{\text{rt}}]$$

$$I^E_{\text{Airey rt}} = [x_{\text{rt}}(E) - \lambda^{-\varepsilon}_{\text{rt}} B_{\text{rt}}, x_{\text{rt}}(E) + \lambda^{-\varepsilon}_{\text{rt}} B_{\text{rt}}]$$

$$I^E_{\text{far rt}} = [x_{\text{rt}}(E) + \lambda_{\text{rt}}^{\varepsilon - 2/3} B_{\text{rt}}, \infty) \cap I_{\text{BVP}}.$$

Also define regions

$$U_{\text{Airey left}} = \left\{(x, E) \,\middle|\, |E - E_0| < c_\# S_{\min}, |x - x_{\text{left}}(E)| < \lambda^{-\varepsilon}_{\text{left}} B_{\text{left}}\right\},$$

$$U_{\text{Airey rt}} = \left\{(x, E) \,\middle|\, |E - E_0| < c_\# S_{\min}, |x - x_{\text{rt}}(E)| < \lambda^{-\varepsilon}_{\text{rt}} B_{\text{rt}}\right\}$$

$$U_{\text{oscil}} = \left\{(x, E) \,\middle|\, |E - E_0| < c_\# S_{\min}, x_{\text{left}}(E) < x < x_{\text{rt}}(E)\right\}.$$

WKB Eigenfunction Theorem. *Assume (Hyp0)... (Hyp10). Then there exist real–valued functions* $Y_{\text{left}}(x, E)$ *on* $U_{\text{Airey left}}$*,* $Y_{\text{rt}}(x, E)$ *on* $U_{\text{Airey rt}}$*, and complex–valued functions* $u_{\text{left}}(x, E)$*,* $u_{\text{rt}}(x, E)$ *on* U_{oscil}*, for which the following hold.*

<u>Properties of the Auxiliary Functions</u>

(1) On $U_{\text{Airey left}}$ we can write $Y_{\text{left}} = Y_0^{\text{left}} + \lambda^{-2}_{\text{left}} Y_1^{\text{left}}$, with $|\partial_x^\alpha \partial_E^\beta Y_i^{\text{left}}| \leq C_\#^{\alpha\beta} B_{\text{left}}^{-\alpha} S_{\text{left}}^{-\beta}$ and $\lambda^2_{\text{left}} (\frac{\partial Y_0^{\text{left}}}{\partial x})^2 Y_0^{\text{left}} = E - V(x)$ to order at least N' at $x = x_{\text{left}}(E)$. More precisely, $|\lambda^2_{\text{left}} (\frac{\partial Y_{\text{left}}}{\partial x})^2 \cdot Y_{\text{left}} + \{Y_{\text{left}}, x\} - (E - V(x))| \leq C_\# \lambda_{\text{left}}^{-N'} S_{\text{left}}$ on $U^{\text{left}}_{\text{Airey}}$. Similarly, on $U_{\text{Airey rt}}$ we can write $Y_{\text{rt}} = Y_0^{\text{rt}} +$

$\lambda_{\rm rt}^{-2}Y_1^{\rm rt}$, with $|\partial_x^\alpha \partial_E^\beta Y_i^{\rm rt}| \le C_\#^{\alpha\beta} B_{\rm rt}^{-\alpha} S_{\rm rt}^{-\beta}$ and $\lambda_{\rm rt}^2(\frac{\partial Y_0^{\rm rt}}{\partial x})^2 Y_0^{\rm rt} = E - V(x)$ to order at least N' at $x = x_{\rm rt}(E)$. More precisely, $|\lambda_{\rm rt}^2(\frac{\partial Y_{\rm rt}}{\partial x})^2 Y_{\rm rt} + \{Y_{\rm rt}, x\} - (E - V(x))| \le C_\# \lambda_{\rm rt}^{-N'} S_{\rm rt}$ on $U_{\rm Airey}^{\rm rt}$.

(2) For $|E - E_0| < c_\# S_{\min}$ and $x \in I_{\text{medium left}}^E$ we have $|\partial_x^\alpha u_{\rm left}| \le C_\#^\alpha \lambda_{\rm left}^{-1} (\frac{x - x_{\rm left}(E)}{B_{\rm left}})^{-\frac{3}{2}} \cdot (x - x_{\rm left}(E))^{-\alpha}$, and $|\mathrm{Re}\ u_{\rm left}| \le C_\# \lambda_{\rm left}^{-2} (\frac{x - x_{\rm left}(E)}{B_{\rm left}})^{-3}$. Similarly, for $|E - E_0| < c_\# S_{\min}$ and $x \in I_{\text{medium rt}}^E$ we have $|\partial_x^\alpha u_{\rm rt}| \le C_\#^\alpha \lambda_{\rm rt}^{-1} (\frac{x_{\rm rt}(E) - x}{B_{\rm rt}})^{-\frac{3}{2}} \cdot (x_{\rm rt}(E) - x)^{-\alpha}$ and $|\mathrm{Re}\ u_{\rm rt}| \le C_\# \lambda_{\rm rt}^{-2} (\frac{x_{\rm rt}(E) - x}{B_{\rm rt}})^{-3}$.

(3) For $|E - E_0| < c_\# S_{\min}$ and $x \in I_{\rm center}^E$ we have

$$|\partial_x^\alpha u_{\rm left}| \le C_\#^\alpha \Lambda^{-1} B^{-\alpha}(x) \quad \text{and} \quad |\mathrm{Re}\ u_{\rm left}| \le C_\# \Lambda^{-2}.$$

Similarly, on the same region we have $|\partial_x^\alpha u_{\rm rt}| \le C_\#^\alpha \Lambda^{-1} B^{-\alpha}(x)$ and $|\mathrm{Re}\ u_{\rm rt}| \le C_\# \Lambda^{-2}$.

Description of Eigenfunctions

Let $F(x)$ be a real–valued eigenfunction of H with eigenvalue E and norm 1. Assume $|E - E_0| < c_\# S_{\min}$ and $E \le E_\infty$. Then there exist real constants $b_{\rm left}$, $b_{\rm rt}$ for which we have:

(4) $\int_{I_{\text{far left}}^E} |F(x)|^2\, dx \le \Lambda^{-N''}$ and $\int_{I_{\text{far rt}}^E} |F(x)|^2\, dx \le \Lambda^{-N''}$

(5) $\int_{I_{\text{Airey left}}^E} |F(x) - b_{\rm left} \lambda_{\rm left}^{-1/3} \Big(\frac{\partial Y_{\rm left}(x,E)}{\partial x}\Big)^{-1/2} A(\lambda_{\rm left}^{2/3} Y_{\rm left}(x, E))|^2\, dx \le \Lambda^{-N''}$ and $\int_{I_{\text{Airey rt}}^E} |F(x) - b_{\rm rt} \lambda_{\rm rt}^{-1/3} \Big(\frac{-\partial Y_{\rm rt}(x,E)}{\partial x}\Big)^{-1/2} A(\lambda_{\rm rt}^{2/3} Y_{\rm rt}(x, E))|^2\, dx \le \Lambda^{-N''}$

(6)

$$\int_{I_{\text{medium left}}^E \cup I_{\rm center}^E} \Big| F(x) - b_{\rm left} \mathrm{Re}\Big[\frac{e^{\pm i \frac{\pi}{4}}\, e^{i \int_{x_{\rm left}(E)}^x (E - V(t))^{1/2}\, dt}}{(E - V(x))^{1/4}} \cdot (1 + u_{\rm left}(x)) \Big] \Big|^2\, dx \le \Lambda^{-N''}$$

and

$$\int_{I^E_{\text{medium rt}} \cup I^E_{\text{center}}} \Big| F(x) - b_{\text{rt}} \text{Re}\Big[\frac{e^{\mp i \frac{\pi}{4}} \, e^{-i \int_x^{x_{\text{rt}}(E)} (E - V(t))^{1/2} \, dt}}{(E - V(x))^{1/4}} \cdot (1 + u_{\text{rt}}(x)) \Big] \Big|^2 \, dx \le \Lambda^{-N''}$$

(7) $c_{\#} < (|b_{\text{left}}|^2 + |b_{\text{rt}}|^2) \int_{x_{\text{left}}}^{x_{\text{rt}}} \frac{dx}{S^{1/2}(x)} < C_{\#}$, and

(8) $\Big| |b_{\text{left}}| \cdot |b_{\text{rt}}|^{-1} - 1 \Big| \le C_{\#} \Lambda^{-2}$.

What the Constants may Depend on

The constants $c_\#$, $C_\#$, $C_\#^\alpha$ above depend only on ε, K, N, c, C, c_1, c_2, C_α in (Hyp0)...(Hyp5).

Proofs. (Hyp0)...(Hyp10) imply the assumptions used in the section on eigenfunctions and eigenvalues, namely (H0)...(H6) and (E1)...(E7). This follows from the discussion in the preceding section. The estimates for $\frac{d^\beta \phi(E)}{dE^\beta}$ in Lemma 1 were proved in the preceding section, and the estimates for $\frac{d^\beta}{dE^\beta}\psi(E)$ are immediate consequences of the global WKB lemma and the fact that $\psi(E) = 48\ \mathrm{Im}(G_1(E))$.

The WKB Eigenvalue Theorem is immediate from Lemma 16 in the section on eigenvalues and eigenfunctions. We define E_k to be the unique eigenvalue $\leq E_\infty$ in $J(k)$, assuming an eigenvalue exists there; and we set $E_k = E_\infty$ otherwise. (See Lemma 16 for the definition of $J(k)$). This proves (B)...(F) of the WKB Eigenvalue Theorem. Part (A) is already contained in the global WKB lemma.

The WKB Eigenfunction Theorem is immediate from Lemma 12 in the section on eigenfunctions and eigenvalues. We set $u_{\text{left}} = \sum_{k=1}^{N'} u_k^{\text{left}}(x,E)$, $u_{\text{rt}} = \sum_{k=1}^{N'} u_k^{\text{rt}}(x,E)$, $b_{\text{left}} = b$ and $b_{\text{rt}} = (-1)^{k_1} R(E_1) b$. (See Lemma 12 for the notation.)

Parts (1), (2), (3) of the Eigenfunction Theorem are contained in the global WKB lemma, and parts (4), (5), (6), (7) come from Lemma 12. Part (8) asserts that $|R(E_1) - 1| \leq C_\# \Lambda^{-2}$, which is contained in the global WKB lemma. ∎

Normalizing the WKB Eigenfunctions

The goal of this section is to compute the constants b_{left}, b_{rt} in the statement of the WKB Eigenfunction Theorem. The first step is to give a close approximation for integrals of the form $\int_{-\infty}^{\infty} \theta(t) A^2(t)\, dt$, which we accomplish in the next few lemmas.

Lemma 1. *Suppose* $|(\frac{d}{dy})^m \theta(y)| \leq C_m R^{-m}$ $(m \geq 0)$ *and* θ *of compact support and*

$|K(y)| \le \frac{C'}{|y|^M}$, *where* M, $R > 10$. *Then*

$$\int_1^\infty \theta(y)K(y)\,dy = \sum_{m=0}^{M-3} c(m)\theta^{(m)}(0) + \text{Error}$$

with $|\text{Error}| \le \text{Const.}(C', C_m, M) \cdot R^{-(M-2)}$, $c(m)$ *depending only on* $K(y)$, *and* $|c(m)| \le \text{Const.}(C', M)$.

Proof. Taylor's theorem gives $\theta(y) = \sum\limits_{m=0}^{M-1} \frac{1}{m!}\theta^{(m)}(0)y^m + \chi(y) \cdot y^M$ with $|\chi(y)| \le \text{Const.}(C_M)R^{-M}$, so

$$\int_1^\infty \theta(y)K(y)\,dy = \sum_{m=0}^{M-1} \theta^{(m)}(0) \cdot \left[\frac{1}{m!}\int_1^R K(y)y^m\,dy\right] + \int_1^R \chi(y)y^M K(y)\,dy + \int_R^\infty \theta(y)K(y)\,dy.$$

The last two terms on the right are dominated by $\text{Const.}(C', C_m, M)R^{-(M-1)}$, and the term $\theta^{(M-1)}(0) \cdot \left[\frac{1}{(M-1)!}\int_1^R K(y)y^{M-1}\,dy\right]$ is dominated by $\text{Const.}(C_{M-1}, C')R^{-(M-1)}\ell n R \le \text{Const.}(C_{M-1}, C')R^{-(M-2)}$. Hence

$$\int_1^\infty \theta(y)K(y)\,dy = \sum_{m=0}^{M-2} \theta^{(m)}(0) \cdot \left[\frac{1}{m!}\int_1^\infty K(y)y^m\,dy\right] - \sum_{m=0}^{M-2} \theta^{(m)}(0)\left[\frac{1}{m!}\int_R^\infty K(y)y^m\,dy\right] + \text{Error}$$

with $|\text{Error}| \le \text{Const.}(C', C_m, M) \cdot R^{-(M-2)}$. The second sum on the right is also dominated by $\text{Const.}(C', C_m, M)R^{-(M-2)}$, as is the term $m = M - 2$ in the first sum. Hence the lemma is proven, with $c(m) = \frac{1}{m!}\int_1^\infty y^m K(y)\,dy$. ∎

Lemma 2. *If* $\theta(y)$ *has compact support and* $|(\frac{d}{dy})^m\theta(y)| \le C_m R^{-m}$ *(all* m*) with* $R \ge 10$, *then*

$$\int_1^\infty e^{\frac{4}{3}iy^{\frac{3}{2}}} y^{-s}\theta(y)\,dy = \sum_{m=0}^{M-1} c(s,m)\theta^{(m)}(0) + \text{Error}$$

with $c(s,m)$ *universal constants,* $|\text{Error}| < \text{Const.}(C_m, M)\cdot R^{-M}$ *and* M *as large as we please.*

Proof. Integration by parts gives the formula

$$\int_1^\infty e^{\frac{4}{3}iy^{3/2}} y^{-s}\theta(y)\,dy = c_1(s)\theta(1) + c_2(s)\int_1^\infty e^{\frac{4}{3}iy^{3/2}} y^{-(s+\frac{3}{2})}\theta(y)\,dy$$
$$+c_3(s)\int_1^\infty e^{\frac{4}{3}iy^{\frac{3}{2}}} y^{-(s+1/2)}\theta'(y)\,dy.$$

The integrals on the right are analogous to the left–hand side, with s increased by at least 1/2. Hence repeated use of this formula yields an identity of the form

$$\int_1^\infty e^{\frac{4}{3}iy^{3/2}} y^{-s}\theta(y)\,dy = \sum_{j=1}^{j_{\max}} c(s,j)\theta^{(m_j)}(1)$$
$$+\sum_{j=1}^{j'_{\max}} c'(s,j)\int_1^\infty e^{\frac{4}{3}iy^{3/2}} y^{-s_j}\theta^{(m'_j)}(y)\,dy$$

with all the $s_j \geq M+2$. Lemma 1 (with $M+2$ in place of M) applies to the integrals on the right, so we get

$$\int_1^\infty e^{\frac{4}{3}iy^{3/2}} y^{-s}\theta(y)\,dy = \sum_{j=1}^{j_{\max}} c(s,j)\theta^{(m_j)}(1) + \sum_{m=0}^{M-1}\tilde{c}(s,m)\theta^{(m)}(0) + \text{Error},$$

with $|\text{Error}| \leq \text{Const.}(C_m, M)R^{-M}$.

The terms $c(s,j)\theta^{(m_j)}(1)$ with $m_j \geq M$ are dominated by $\text{Const.}(C_m, M)R^{-M}$, and for $m_j \leq M-1$ Taylor's theorem gives $\theta^{(m_j)}(1) = \sum_{m_j\leq m\leq M-1} \frac{\theta^{(m)}(0)}{(m-m_j)!} +$ error, with $|\text{error}| \leq \text{Const.}(C_m, M)R^{-M}$. Therefore, the previous equation for our integral implies

$$\int_1^\infty e^{\frac{4}{2}iy^{3/2}} y^{-s}\theta(y)\,dy = \sum_{m=0}^{M-1} c(s,m)\theta^{(m)}(0) + \text{Error}$$
$$\text{with}\quad |\text{Error}| \leq \text{Const.}(C_m, M)R^{-M}.$$

■

Of course, Lemma 2 has an analogue with $e^{\frac{4}{3}iy^{3/2}}$ replaced by $e^{-\frac{4}{3}iy^{3/2}}$.

Lemma 3. *Suppose $\theta(y)$ has compact support and satisfies $|(\frac{d}{dy})^m\theta(y)| \leq C_m R^{-m}$ (all m) with $R \geq 10$. Suppose $\mathcal{A}(y)$ satisfies*

$$\left|\mathcal{A}(y) - Re\Big[\frac{e^{\pm i\frac{\pi}{4}}\, e^{\frac{2}{3}iy^{3/2}}}{y^{1/4}}\,(1+\sum_{k=1}^{M} c_k\, y^{-\frac{3}{2}k})\Big]\chi_{y>1}\right| \leq C'_M(1+|y|)^{-M}$$

$$(any\ M).$$

Then $\int_{-\infty}^{\infty}\mathcal{A}^2(y)\theta(y)\,dy = \frac{1}{2}\int_1^\infty\left|\frac{1+\sum_{k=1}^{M} c_k y^{-\frac{3}{2}k}}{y^{1/4}}\right|^2\theta(y)\,dy + \sum_{m=0}^{M-1} c(m)\theta^{(m)}(0) + \text{Error},$ *where the $c(m)$ are independent of θ, M is as large as we please, and* $|\text{Error}| \leq \text{Const.}(M, C'_M, C_m)\cdot R^{-M}$.

Proof. Set $H_M(y) = \mathcal{A}^2(y) - \left(\text{Re}\Big[\frac{e^{\pm i\frac{\pi}{4}}\, e^{\frac{2}{3}iy^{3/2}}}{y^{1/4}}\,(1+\sum_{k=1}^{M} c_k y^{-\frac{3}{2}k})\Big]\chi_{y>1}\right)^2$.
Our hypothesis implies

$$|H_M(y)| \leq C_M(1+|y|)^{-M} \quad , M \text{ as large as we please.}$$

Lemma 1 shows that

$$\int_1^\infty H_M(y)\theta(y)\,dy = \sum_{m=0}^{M-3} c_1(m)\theta^{(m)}(0) + \text{Error}_1,$$
$$|\text{Error}_1| \leq \text{Const.}(M, C'_M, C_m)R^{-(M-2)}$$

and that

$$\int_{-\infty}^{-1} H_M(y)\theta(y)\,dy = \sum_{m=0}^{M-3} c_2(m)\theta^{(m)}(0) + \text{Error}_2,$$
$$|\text{Error}_2| \leq \text{Const.}(M, C'_M, C_m)R^{-(M-2)}.$$

Taylor's theorem gives

$$\int_{-1}^{1} H_M(y)\theta(y)\,dy = \sum_{m=0}^{M-3} c_3(m)\theta^{(m)}(0) + \text{Error}_3,$$
$$|\text{Error}_3| \leq \text{Const.}(M, C'_M, C_m)R^{-(M-2)}.$$

Adding the last three equations, we get

$$\int_{-\infty}^{\infty} H_M(y)\theta(y)\,dy = \sum_{m=0}^{M-3} c_4(m)\theta^{(m)}(0) + \text{Error}_4,$$
$$|\text{Error}_4| \le \text{Const.}(M, C'_M, C_m)R^{-(M-2)}.$$

Therefore, the lemma follows if we can show that

$$(1)\quad Q \equiv \int_1^{\infty} \Big(\text{Re}\Big[\frac{e^{\pm i\frac{\pi}{4}}\, e^{\frac{2}{3}iy^{3/2}}}{y^{1/4}}(1+\sum_{k=1}^{M} c_k y^{-\frac{3}{2}k})\Big]\Big)^2 \theta(y)\,dy$$
$$= \frac{1}{2}\int_1^{\infty} \Big|\frac{1+\sum_{k=1}^{M} c_k y^{-\frac{3}{2}k}}{y^{1/4}}\Big|^2 \theta(y)\,dy + \sum_{m=0}^{M-3} c_5(m)\theta^{(m)}(0) + \text{Error}_5$$

with $|\text{Error}_5| \le \text{Const.}(M, C'_M, C_m)\cdot R^{-(M-2)}$.

We expand Q using $(\text{Re}(\zeta))^2 = (\frac{1}{2}\zeta + \frac{1}{2}\overline{\zeta})^2 = \frac{1}{2}|\zeta|^2 + \frac{1}{4}\zeta^2 + \frac{1}{4}\overline{\zeta}^2$. Thus

$$Q = \frac{1}{2}\int_1^{\infty} \Big|\frac{1+\sum_{k=1}^{M} c_k y^{-\frac{3}{2}k}}{y^{1/4}}\Big|^2 \theta(y)\,dy + \frac{1}{4}e^{\pm i\frac{\pi}{2}} \int_1^{\infty} \frac{e^{\frac{4}{3}iy^{3/2}}}{y^{1/2}}$$
$$\Big(1+\sum_{k=1}^{M} c_k y^{-\frac{3}{2}k}\Big)^2 \theta(y)\,dy$$
$$+\frac{1}{4}e^{\mp i\frac{\pi}{2}} \int_1^{\infty} \frac{e^{-\frac{4}{3}iy^{3/2}}}{y^{1/2}}\Big(1+\sum_{k=1}^{M} \overline{c}_k y^{-\frac{3}{2}k}\Big)^2 \theta(y)\,dy.$$

The last two terms on the right are sums of integrals to which we can apply Lemma 2 and its analogue for $e^{-\frac{4}{3}iy^{3/2}}$, and the desired equation (1) follows easily.

■

We apply Lemma 3 to the Airey function.

Lemma 4. *Suppose $\theta(y)$ has compact support and satisfies $|(\frac{d}{dy})^m\theta(y)| \le C_m R^{-m}$ $(m \ge 0)$ with $R \ge 10$.*

For universal constants $\tilde{c}(0)$, $\tilde{c}(1)$, $\tilde{c}(2)$ we have

$$\int_{-\infty}^{\infty} A^2(y)\theta(y)\,dy = \frac{1}{2}\int_0^{\infty} \frac{\theta(y)\,dy}{y^{1/2}} + \sum_{m=0}^{2} \tilde{c}(m)\theta^{(m)}(0) + Error,$$

with $|Error| \le Const.(C_m) \cdot R^{-5/2}$.

Proof. The Airey function $A(y)$ satisfies the hypothesis of the previous lemma, with $c_1 = \pm\frac{5}{48}i$. Since c_1 is purely imaginary, we have $|1 + \sum_{k=1}^{M} c_k y^{-\frac{3}{2}k}|^2 = 1 + O(y^{-3})$ for $y \ge 1$. Hence

$$\frac{1}{2}\int_1^\infty \left|\frac{1+\sum_{k=1}^{M} c_k y^{-\frac{3}{2}k}}{y^{1/4}}\right|^2 \theta(y)\,dy = \frac{1}{2}\int_1^\infty \frac{\theta(y)\,dy}{y^{1/2}} + \int_1^\infty K(y)\theta(y)\,dy$$

with $|K(y)| \le Cy^{-7/2}$. Taylor's theorem gives

$$\begin{aligned}\int_1^\infty K(y)\theta(y)\,dy &= \int_1^R K(y)\Big[\theta(0) + y\theta'(0) + \frac{1}{2}y^2\theta''(0) + O(R^{-3}y^3)\Big]\,dy \\ &+ \int_R^\infty K(y)O(1)\,dy = \sum_{m=0}^{2} \theta^{(m)}(0)\cdot\frac{1}{m!}\int_1^\infty y^m K(y)\,dy \\ &+ \Big\{ -\sum_{m=0}^{2}\theta^{(m)}(0) \\ &\cdot \frac{1}{m!}\int_R^\infty y^m K(y)\,dy + \int_1^R K(y)\cdot O(R^{-3}y^3)\,dy + \int_R^\infty K(y)O(1)\,dy\Big\}.\end{aligned}$$

The terms in curly brackets are all dominated by $R^{-5/2}$, so that

$$\frac{1}{2}\int_1^\infty \left|\frac{1+\sum_{k=1}^{M} c_k y^{-\frac{3}{2}k}}{y^{1/4}}\right|^2 \theta(y)\,dy = \frac{1}{2}\int_1^\infty \frac{\theta(y)\,dy}{y^{1/2}} + \sum_{m=0}^{2} c_1(m)\theta^{(m)}(0) + \text{Error}_1$$

with $|\text{Error}_1| \le \text{Const.}(C_m)\cdot R^{-5/2}$. The previous lemma therefore gives $\int_{-\infty}^{\infty} A^2(y)\theta(y)\,dy = \frac{1}{2}\int_1^\infty \frac{\theta(y)\,dy}{y^{1/2}} + \sum_{m=0}^{2} c_2(m)\theta^{(m)}(0) + \text{Error}_2$, with $|\text{Error}_2| \le \text{Const.}(C_m)\cdot R^{-5/2}$. Taylor's theorem shows that

$$\frac{1}{2}\int_0^1 \frac{\theta(y)\,dy}{y^{1/2}} = \sum_{m=0}^{2}\theta^{(m)}(0)\Big[\frac{1}{2}\int_0^1 \frac{y^{m-\frac{1}{2}}}{m!}\,dy\Big] + \text{Error}_3,$$
$$\text{with} |\text{Error}_3| \le \text{Const.}(C_m)R^{-3}.$$

Combining this with the preceding equation, we get the conclusion of the lemma. ■

Remark. Later, we will see that the universal constants $\tilde{c}(0)$, $\tilde{c}(1)$, $\tilde{c}(2)$ are all zero. This will be important for the normalization of WKB eigenfunctions. The ideas in the proofs of Lemmas 1…4 have used only the asymptotic properties of the Airey function, which are not enough to specify the $\tilde{c}(m)$.

Now suppose we are in the setting of the WKB Eigenfunction Theorem, and let F be a real–valued eigenfunction of H, with L^2–norm 1 and eigenvalue E. We assume $|E-E_0| < c_\# S_{\min}$ and $E \le E_\infty$, so that F is described by the WKB Eigenfunction Theorem. Our goal is to compute $\int_{I_{\mathrm{BVP}}} \theta(x) F^2(x)\,dx$ for suitable smooth functions θ.

Let us start with the case of θ supported near a turning point. Thus we assume supp $\theta \subset I^E_{\text{Airey left}} = \{|x - x_{\text{left}}(E)| \le \lambda_{\text{left}}^{-\varepsilon} B_{\text{left}}\}$, and we assume the estimates $|(\frac{d}{dx})^\alpha \theta(x)| \le C^\alpha (\lambda_{\text{left}}^{-\varepsilon} B_{\text{left}})^{-\alpha}$.

For the next few paragraphs, we delete the subscript "left". According to the WKB Eigenfunction Theorem, we have in supp θ $F(x) =$
$b\lambda^{-1/3}(\frac{\partial Y}{\partial x})^{-1/2} A(\lambda^{2/3} Y) + \text{Error}$, with $\int |\text{Error}|^2\,dx < \Lambda^{-N''}$.

Thus

$$F^2(x) \le (1 + \Lambda^{-\frac{1}{10}N''})(b\lambda^{-1/3}(\frac{\partial Y}{\partial x})^{-1/2} A(\lambda^{2/3} Y))^2 + \Lambda^{\frac{1}{2}N''}|\text{Error}|^2$$

and

$$F^2(x) \ge (1 - \Lambda^{-\frac{1}{10}N''})(b\lambda^{-1/3}(\frac{\partial Y}{\partial x})^{-1/2} A(\lambda^{2/3} Y))^2 - \Lambda^{\frac{1}{2}N''}|\text{Error}|^2.$$

Estimate (44 bis) in the section on Schrödinger operators, together with the results on the order of magnitude of $|b|$ in the WKB Eigenfunction Theorem, shows that

$$|\int |\theta(x)| \Lambda^{-\frac{1}{10}N''} (b\lambda^{-1/3}(\frac{\partial Y}{\partial x})^{-1/2} A(\lambda^{2/3} Y))^2\,dx| \le C_\# \Lambda^{-\frac{1}{10}N''}.$$

Hence

$$\int |\theta(x)|\,|F^2(x) - (b\lambda^{-1/3}(\frac{\partial Y}{\partial x})^{-1/2} A(\lambda^{2/3} Y))^2|\,dx \le C'_\# \Lambda^{-N''/10}, \quad \text{so}$$

$$(2) \qquad \int_{I_{\mathrm{BVP}}} \theta(x) F^2(x)\,dx = b^2 \int_{I_{\mathrm{BVP}}} \theta(x) \lambda^{-2/3} (\frac{\partial Y}{\partial x})^{-1} A^2(\lambda^{2/3} Y)\,dx + \text{Error}_1,$$

$$(3) \qquad |\mathrm{Error}_1| \le C_{\#} \Lambda^{-\frac{1}{10}N''}.$$

In supp θ, we have $\frac{\partial Y}{\partial x} > cB^{-1}$. Hence we can make the change of variable $t = \lambda^{2/3} Y(x,E)$ to reduce matters to Lemma 4.

Define $\hat{I} =$ Image of $\{|x - x(E)| < \lambda^{-\varepsilon} B\}$ under the map $x \mapsto t = \lambda^{2/3} Y(x,E)$, and set

$$\theta^{\#}(t) = \theta(x) \lambda^{-4/3} (\frac{\partial Y}{\partial x})^{-2} \qquad \text{for} \quad t = \lambda^{2/3} Y(x,E),\ |x - x(E)| < \lambda^{-\varepsilon} B;$$

$$\theta^{\#}(t) = 0 \qquad \text{for} \quad t \notin \hat{I}.$$

We have

$$(4) \quad \int_{I_{\mathrm{BVP}}} \theta(x) \lambda^{-2/3} (\frac{\partial Y}{\partial x})^{-1} A^2(\lambda^{2/3} Y)\, dx = \int\limits_{|x - x(E)| < \lambda^{-\varepsilon} B} \theta(x) \lambda^{-4/3} (\frac{\partial Y}{\partial x})^{-2} A^2(\lambda^{2/3} Y) \cdot (\lambda^{2/3} \frac{\partial Y}{\partial x}\, dx) = \int_{-\infty}^{\infty} \theta^{\#}(t) A^2(t)\, dt.$$

Note that supp $\theta^{\#} \subset\subset \hat{I}$, hence $\theta^{\#}$ is smooth on the whole real line. To estimate the derivatives of $\theta^{\#}$ we argue as follows. From the definitions of t, $\theta^{\#}$, and from the WKB Eigenfunction Theorem, we see that

(5) $\lambda^{-2/3} t$ is a smooth function of $\dfrac{x - x(E)}{B}$, with first derivative bounded below.

(6) $\theta(x) \cdot \lambda^{-4/3} (\frac{\partial Y}{\partial x})^{-2}$ has the form $\lambda^{-4/3} B^2 \cdot$ (smooth function of $\dfrac{x - x(E)}{B \lambda^{-\varepsilon}}$).

Hence $\theta^{\#}(t)$ has the form $\lambda^{-4/3} B^2 \cdot$ (smooth function of $\lambda^{-\frac{2}{3}+\varepsilon} t$). The C^∞–seminorms of the smooth functions in (5), (6) are bounded a–priori by the WKB Eigenfunction Theorem. Hence we have $\theta^{\#}(t) = \lambda^{-4/3} B^2 \cdot \theta^{\#\#}(t)$, with $|(\frac{d}{dt})^m \theta^{\#\#}| \le C_{\#}^m \lambda^{(\varepsilon - \frac{2}{3})m}$. Applying Lemma 4 to $\theta^{\#\#}$, with $R = \lambda^{2/3 - \varepsilon}$, we get

$$(7) \quad \int_{-\infty}^{\infty} \theta^{\#}(t) A^2(t)\, dt = \frac{1}{2} \int_0^{\infty} \frac{\theta^{\#}(t)\, dt}{t^{1/2}} + \sum_{m=0}^{2} \tilde{c}(m) \, (\frac{d}{dt})^m \, \theta^{\#}(0) + \mathrm{Error}_2$$

with $|\text{Error}_2| \le C_\# \lambda^{-4/3} B^2 \cdot (\lambda^{2/3-\varepsilon})^{-5/2} = C_\# \lambda^{\frac{5}{2}\varepsilon - 3} B^2$.

On the right–hand side of (7), we want to replace

$$(\frac{d}{dt})^m \theta^\#(0) \quad \text{by} \quad (\frac{d}{dt})^m \theta^\#(t_0), \quad \text{with} \quad t_0 = \lambda^{2/3} Y(x(E), E).$$

That is, t_0 is the image of $x(E)$ under $x \mapsto t$.

The WKB Eigenfunction Theorem gives $Y = Y_0 + \lambda^{-2} Y_1$ with $|Y_1| \le C_\#$ and $Y_0 = 0$ at $x = x(E)$. Hence $|t_0| < C_\# \lambda^{2/3} \cdot \lambda^{-2} = C_\# \lambda^{-4/3}$, so

$$\begin{aligned} \Big|(\frac{d}{dt})^m \theta^\#(t_0) - (\frac{d}{dt})^m \theta^\#(0)\Big| &\le |t_0| \cdot \max\Big|(\frac{d}{dt})^{m+1}\theta^\#\Big| \\ &\le C_\#^m \lambda^{-4/3} \cdot \lambda^{-4/3} B^2 \cdot \lambda^{(\varepsilon - \frac{2}{3})(m+1)} \le C_\#^m \lambda^{-\frac{10}{3} + (m+1)\varepsilon} B^2. \end{aligned}$$

Putting this into (7), we get

$$(8) \qquad \int_{-\infty}^{\infty} \theta^\#(t) A^2(t)\, dt = \frac{1}{2}\int_0^\infty \frac{\theta^\#(t)\, dt}{t^{1/2}} + \sum_{m=0}^{2} \tilde{c}(m) (\frac{d}{dt})^m \theta^\#(t_0) + \text{Error}_3$$

with $|\text{Error}_3| \le C_\# \lambda^{\frac{5}{2}\varepsilon - 3} B^2$.

We change variable from t back to x, obtaining:

$$\begin{aligned} \int_{-\infty}^{\infty} \theta^\#(t) A^2(t)\, dt &= \frac{1}{2}\int_{Y>0} \Big\{ \frac{\theta(x)\lambda^{-4/3}(\frac{\partial Y}{\partial x})^{-2}}{(\lambda^{1/3} Y^{1/2})} \Big\} \lambda^{2/3} (\frac{\partial Y}{\partial x})\, dx + \\ &+ \sum_{m=0}^{2} \tilde{c}(m) (\lambda^{-2/3} (\frac{\partial Y}{\partial x})^{-1} \frac{d}{dx})^m \Big\{\theta(x) \lambda^{-4/3} (\frac{\partial Y}{\partial x})^{-2}\Big\}\Big|_{x = x(E)} + \text{Error}_3, \end{aligned}$$

that is,

$$\begin{aligned} (9) \qquad \int_{-\infty}^{\infty} \theta^\#(t) A^2(t)\, dt &= \frac{1}{2}\int_{Y>0} \theta(x) \Big(\lambda^2 (\frac{\partial Y}{\partial x})^2 Y\Big)^{-1/2} dx \\ &+ \sum_{m=0}^{2} \tilde{c}(m) \lambda^{-(\frac{2m+4}{3})} \Big((\frac{\partial Y}{\partial x})^{-1} \frac{d}{dx}\Big)^m \Big\{\theta(x) (\frac{\partial Y}{\partial x})^{-2}\Big\}\Big|_{x = x(E)} + \text{Error}_3, \end{aligned}$$

with $|\text{Error}_3| \le C_\# \lambda^{\frac{5}{2}\varepsilon - 3} B^2$.

In (9) we would like to change Y to Y_0, where we recall from the WKB Eigenfunction Theorem that $Y = Y_0 + \lambda^{-2} Y_1$. To see the effect of this change, define

$$\mathcal{T}(\tau) = \int_{Y_0 + \tau Y_1 > 0} \theta(x) \lambda^{-1} \Big(\frac{\partial Y_0}{\partial x} + \tau \frac{\partial Y_1}{\partial x}\Big)^{-1} (Y_0 + \tau Y_1)^{-1/2}\, dx,$$

for $|\tau| \leq c_\#$. The integral on the right–hand side of (9) is $\mathcal{T}(\lambda^{-2})$, and we want to replace it by $\mathcal{T}(0)$. So we study the dependence of $\mathcal{T}(\tau)$ on τ. This is most easily done by introducing a new independent variable $\xi = Y_0(x,E) + \tau Y_1(x,E)$. We regard ξ as a function of x and τ. ¿From the properties of Y_0, Y_1 asserted in the WKB Eigenfunction Theorem, we see that

$$|(\frac{\partial}{\partial x})^\alpha (\frac{\partial}{\partial \tau})^\gamma \xi| \leq C_\#^{\alpha\gamma} B^{-\alpha}$$

and

$$\frac{\partial \xi}{\partial x} > c_\# B^{-1} \quad \text{for} \quad |x - x(E)|, |\tau| < c_\#,$$

and also that $\xi = 0$ when $x = x(E)$, $\tau = 0$. The implicit function theorem lets us write $x = x(E) + Bg(\xi,\tau)$ for the solution of $Y_0 + \tau Y_1 = \xi$, with $|\partial_\xi^\alpha \partial_\tau^\gamma g(\xi,\tau)| \leq C_\#^{\alpha\gamma}$ for $|\xi|$, $|\tau| \leq c'_\#$.

Writing $\mathcal{T}(\tau)$ in terms of the new integration variable ξ, we get the formula

$$\text{(10)} \qquad \mathcal{T}(\tau) = \lambda^{-1} \int_{\xi > 0} \left\{ \theta(x) \Big(\frac{\partial Y_0}{\partial x} + \tau \frac{\partial Y_1}{\partial x} \Big)^{-2} \right\} \Big|_{x = x(E) + Bg(\xi,\tau)} \xi^{-1/2}\, d\xi.$$

Now $|\partial_x^\alpha \partial_\tau^\gamma (\frac{\partial Y_0}{\partial x} + \tau \frac{\partial Y_1}{\partial x})^{-2}| \leq C_\#^{\alpha\gamma} B^{2-\alpha}$ and $|\partial_x \theta(x)| \leq C_\# \lambda^{+\varepsilon} B^{-1}$.

Hence

$$\Big| \partial_x \Big\{ \theta(x) \Big(\frac{\partial Y_0}{\partial x} + \tau \frac{\partial Y_1}{\partial x} \Big)^{-2} \Big\} \Big| \leq C_\# \lambda^\varepsilon B$$

and

$$\Big| \partial_\tau \Big\{ \theta(x) \Big(\frac{\partial Y_0}{\partial x} + \tau \frac{\partial Y_0}{\partial x} \Big)^{-2} \Big\} \Big| \leq C_\# B^2,$$

$$\text{for } |x - x(E)| < c_\# B, \quad |\tau| < c_\#.$$

Therefore

$$\frac{\partial}{\partial \tau} \Big[\Big\{ \theta(x) \Big(\frac{\partial Y_0}{\partial x} + \tau \frac{\partial Y_1}{\partial x} \Big)^{-2} \Big\} \Big|_{x = x(E) + Bg(\xi,\tau)} \Big]$$
$$= \Big[\partial_x \Big\{ \theta(x) \Big(\frac{\partial Y_0}{\partial x} + \tau \frac{\partial Y_1}{\partial x} \Big)^{-2} \Big\} \Big] \cdot B \frac{\partial g(\xi,\tau)}{\partial \tau}$$
$$+ \Big[\partial_\tau \Big\{ \theta(x) \Big(\frac{\partial Y_0}{\partial x} + \tau \frac{\partial Y_1}{\partial x} \Big)^{-2} \Big\} \Big] \quad \text{is dominated by} \quad C_\# \lambda^\varepsilon B^2.$$

The integrand in (10) thus has τ–derivative at most $\lambda^{\varepsilon} B^2 \xi^{-1/2}$. That integrand is supported in $\{|\xi| < c_{\#}\}$ because of the θ–factor. Therefore from (10) we get $|\frac{d}{dt}\mathcal{T}(\tau)| \le C_{\#}\lambda^{\varepsilon-1}B^2$, for $|\tau| < c_{\#}$.

Hence $|\mathcal{T}(\lambda^{-2}) - \mathcal{T}(0)| \le C_{\#}\lambda^{\varepsilon-3}B^2$, so (9) implies

$$(11)\quad \int_{-\infty}^{\infty} \theta^{\#}(t)A^2(t)\,dt = \frac{1}{2}\int_{Y_0>0} \theta(x)\Big(\lambda^2(\frac{\partial Y_0}{\partial x})^2 Y_0\Big)^{-1/2} dx + \sum_{m=0}^{2} \tilde{c}(m)\lambda^{-(\frac{2m+4}{3})}\Big((\frac{\partial Y}{\partial x})^{-1}\frac{d}{dx}\Big)^m \Big\{\theta(x)(\frac{\partial Y}{\partial x})^{-2}\Big\}\Big|_{x=x(E)} + \text{Error}_4, \quad \text{with} \quad |\text{Error}_4| \le C_{\#}\lambda^{\frac{5}{2}\varepsilon-3}B^2.$$

Recall from the WKB Eigenfunction Theorem that $\lambda^2(\frac{\partial Y_0}{\partial x})^2 Y_0 = E - V(x)$ to order $\ge N'$ at $x = x(E)$.

Hence in supp θ, $Y_0 > 0$ is equivalent to $E - V(x) > 0$. Also, Taylor's theorem with remainder and our estimates for the x–derivatives of Y_0, $V(x)$ yield

$$\Big|\lambda^2(\frac{\partial Y_0}{\partial x})^2 Y_0 - (E - V(x))\Big| \le C_{\#}S\cdot\Big(\frac{x - x(E)}{B}\Big)^{N'} \le C_{\#}S\cdot\Big(\frac{x - x(E)}{B}\Big)^{N'-1}\cdot\Big(\frac{E - V(x)}{S}\Big) \le C_{\#}\lambda^{-\varepsilon(N'-1)}(E - V(x))$$

for $|x - x(E)| < \lambda^{-\varepsilon}B$. Thus

$$\Big|\frac{\lambda^2(\frac{\partial Y_0}{\partial x})^2 Y_0}{E - V(x)} - 1\Big| \le C_{\#}\lambda^{-\varepsilon(N'-1)} \quad \text{in} \quad \text{supp } \theta.$$

Hence

(12)

$$\frac{1}{2}\int_{Y_0>0} \theta(x)\Big(\lambda^2(\frac{\partial Y_0}{\partial x})^2 Y_0\Big)^{-1/2} dx = \frac{1}{2}\int_{E-V(x)>0} \theta(x)(E - V(x))^{-1/2}\,dx + \text{Error}_5,$$

$$|\text{Error}_5| \le C_{\#}\lambda^{-\varepsilon(N'-1)}\int_{E-V(x)>0} |\theta(x)|(E - V(x))^{-1/2}\,dx$$

$$(13)\quad \le C_{\#}\lambda^{-\varepsilon(N'-1)}S^{-1/2}B.$$

Since $S^{-1/2}B = \lambda^{-1}B^2$, (13) may be rewritten as $|\text{Error}_5| \le C_\# \lambda^{-1-\varepsilon(N'-1)}B^2 \le C_\# \lambda^{-100}B^2$, since we take $N > 100\varepsilon^{-10}$, $N' = [\varepsilon N/500]$, $0 \le \varepsilon < 1/10$. Therefore, (11) and (12) yield

$$\int_{-\infty}^{\infty} \theta^\#(t)A^2(t)\,dt = \frac{1}{2}\int_{E-V(x)>0} \theta(x)(E-V(x))^{-1/2}\,dx$$
$$+\sum_{m=0}^{2} \tilde{c}(m)\lambda^{-(\frac{2m+4}{3})}\Big((\frac{\partial Y}{\partial x})^{-1}\frac{d}{dx}\Big)^m\Big\{\theta(x)(\frac{\partial Y}{\partial x})^{-2}\Big\}\Big|_{x=x(E)} + \text{Error}_6$$

with $|\text{Error}_6| \le C_\#\lambda^{\frac{5}{2}\varepsilon-3}B^2$. Putting this into (2), (3), (4) we get

$$(14)\quad b^{-2}\int_{I_{\text{BVP}}} \theta(x)F^2(x)\,dx = \frac{1}{2}\int_{E-V(x)>0} \theta(x)(E-V(x))^{-1/2}\,dx$$
$$+\sum_{m=0}^{2} \tilde{c}(m)\lambda^{-(\frac{2m+4}{3})}\Big((\frac{\partial Y}{\partial x})^{-1}\frac{d}{dx}\Big)^m\Big\{\theta(x)(\frac{\partial Y}{\partial x})^{-2}\Big\}\Big|_{x=x(E)} + \text{Error}_7$$

where $|\text{Error}_7| \le C_\#\lambda^{\frac{5}{2}\varepsilon-3}B^2 + C_\#\Lambda^{-\frac{1}{10}N''}b^{-2}$.

The WKB Eigenfunction Theorem gives $b^{-2} \le C_\# \int_{x_{\text{left}}}^{x_{\text{rt}}} \frac{dx}{S^{1/2}(x)}$, and hypothesis (Hyp 8) dominates the right–hand side by $C_\#\Lambda^K S^{-1/2}B = C_\#\Lambda^K\lambda^{-1}B^2$. Hence

$$|\text{Error}_7| \le C_\# B^2\cdot(\lambda^{\frac{5}{2}\varepsilon-3} + \Lambda^{K-\frac{1}{10}N''}\lambda^{-1}).$$

We record our results in the following lemma.

Lemma 5. *In the setting of the WKB Eigenfunction Theorem, let F be a real–valued eigenfunction of H, with L^2–norm 1 and eigenvalue E satisfying $|E-E_0| < c_\# S_{\min}$, $E \le E_\infty$. Suppose $\theta(x)$ is supported in $\{|x-x_{\text{left}}(E)| < \lambda_{\text{left}}^{-\varepsilon}B_{\text{left}}\}$ and satisfies $\left|(\frac{d}{dx})^m\theta(x)\right| \le C_\#^m(\lambda_{\text{left}}^{-\varepsilon}B_{\text{left}})^{-m}$. Then we have*

$$b_{\text{left}}^{-2}\int_{I_{\text{BVP}}} \theta(x)F^2(x)\,dx = \frac{1}{2}\int_{E-V(x)>0}\frac{\theta(x)\,dx}{(E-V(x))^{1/2}} + \sum_{m=0}^{2}\tilde{c}(m)\lambda_{\text{left}}^{-(\frac{2m+4}{3})}\cdot$$
$$\cdot\Big((\frac{\partial Y_{\text{left}}}{\partial x})^{-1}\frac{d}{dx}\Big)^m\Big\{\theta(x)(\frac{\partial Y_{\text{left}}}{\partial x})^{-2}\Big\}\Big|_{x=x_{\text{left}}(E)} + \text{Error},$$

with

$$|\text{Error}| \le C_\# B_{\text{left}}^2\cdot(\lambda_{\text{left}}^{\frac{5}{2}\varepsilon-3} + \Lambda^{K-\frac{N''}{10}}\lambda_{\text{left}}^{-1}).$$

The coefficients $\tilde{c}(m)$ are as in Lemma 4.

Of course, there is an analogue of Lemma 5 for $\theta(x)$ supported near $x_{\mathrm{rt}}(E)$. Once we know that the $\tilde{c}(m)$ are all zero, Lemma 5 will simplify.

Next suppose $\theta(x)$ is supported in $I_\ell = [x_{\mathrm{left}} + c_\# \lambda_{\mathrm{left}}^{-\varepsilon} B_{\mathrm{left}}, x_{\mathrm{rt}} - c_\# B_{\mathrm{rt}}]$ and satisfies

$$\left|(\frac{d}{dx})^m \theta(x)\right| \leq C_\#^m (\lambda^{-\varepsilon}(x) B(x))^{-m}. \tag{15}$$

Again we want to compute $\int_{I_{\mathrm{BVP}}} \theta(x) F^2(x)\, dx$ for eigenfunctions F as in Lemma 5.

The WKB Eigenfunction Theorem shows that

$$F = b_{\mathrm{left}} \mathrm{Re}\Big[\frac{e^{\pm i\frac{\pi}{4}}\, e^{i\int_{x_{\mathrm{left}}(E)}^{x} (E - V(t))^{1/2}\, dt}}{(E - V(x))^{1/4}} (1 + u_{\mathrm{left}}(x,E))\Big] + \mathrm{Error}_8 \tag{16}$$

in supp θ, with

$$|\mathrm{Re}\ u_{\mathrm{left}}(x,E)| \leq C_\# \Lambda^{3\varepsilon - 2} \quad \text{in} \quad \text{supp } \theta, \tag{17}$$

$$\left|(\frac{d}{dx})^m u_{\mathrm{left}}(x,E)\right| \leq C_\#^m \lambda_{\mathrm{left}}^{\frac{3}{2}\varepsilon - 1} (\lambda_{\mathrm{left}}^{-\varepsilon} B_{\mathrm{left}})^{-m} \quad \text{in} \quad \text{supp } \theta \cap \{x < x_{\mathrm{left}}(E) + c_\# B_{\mathrm{left}}\} \tag{18}$$

$$\left|(\frac{d}{dx})^m u_{\mathrm{left}}(x,E)\right| \leq C_\#^m \Lambda^{-1} (B(x))^{-m} \quad \text{in} \quad \text{supp } \theta \cap \{x > x_{\mathrm{left}}(E) + c_\# B_{\mathrm{left}}\}. \tag{18$^{\mathrm{bis}}$}$$

and

$$\int_{\text{supp } \theta} |\mathrm{Error}_8|^2\, dx \leq \Lambda^{-N''}. \tag{19}$$

From (16) we get

$$F^2 \leq b_{\mathrm{left}}^2 (\mathrm{Re}[\mathrm{etc}])^2 (1 + \Lambda^{-\frac{1}{10}N''}) + \Lambda^{\frac{1}{2}N''} |\mathrm{Error}_8|^2$$

and

$$F^2 \geq b_{\text{left}}^2 (\text{Re}[\text{etc}])^2 (1 - \Lambda^{-\frac{1}{10}N''}) - \Lambda^{\frac{1}{2}N''} |\text{Error}_8|^2,$$

so that

$$\text{(20)} \quad \int_{I_{\text{BVP}}} \theta(x) F^2(x)\, dx = \\ b_{\text{left}}^2 \int_{I_{\text{BVP}}} \theta(x) \Big(\text{Re}\Big[\frac{e^{\pm i \frac{\pi}{4}}\, e^{i \int_{\text{left}(E)}^{x} (E-V(t))^{1/2}\, dt}}{(E - V(x))^{1/4}} (1 + u_{\text{left}}(x,E)) \Big] \Big)^2 dx \\ + \text{Error}_9$$

with

$$\text{(21)} \qquad |\text{Error}_9| \leq C_{\#} b_{\text{left}}^2 \Lambda^{-\frac{1}{10}N''} \int_{I_{\text{BVP}}} |\theta(x)| \frac{dx}{(E - V(x))^{1/2}} + \Lambda^{-\frac{1}{2}N''} \\ \leq C_{\#} \Lambda^{-\frac{1}{10}N''}.$$

The last inequality uses our estimate of $|b_{\text{left}}|$ from the WKB Eigenfunction Theorem.

Now we expand $(\text{Re}(\zeta))^2 = (\frac{\zeta}{2} + \frac{\overline{\zeta}}{2})^2 = \frac{\zeta^2}{4} + \frac{\overline{\zeta}^2}{4} + \frac{|\zeta|^2}{2}$ to write:

$$b_{\text{left}}^2 \int_{I_{\text{BVP}}} \theta(x) \Big(\text{Re}\Big[\frac{e^{\pm i \frac{\pi}{4}}\, e^{i \int_{x_{\text{left}}(E)}^{x} (E-V(t))^{1/2}\, dt}}{(E - V(x))^{1/4}}\ (1 + u_{\text{left}}(x,E)) \Big] \Big)^2 dx = \\ \frac{1}{4} b_{\text{left}}^2 \int_{I_{\text{BVP}}} \frac{\theta(x)\, e^{\pm i \frac{\pi}{2}}\ e^{+2i \int_{x_{\text{left}}(E)}^{x} (E-V(t))^{1/2}\, dt}}{(E - V(x))^{1/2}}\ (1 + u_{\text{left}}(x,E))^2\, dx \\ + \frac{1}{4} b_{\text{left}}^2 \int_{I_{\text{BVP}}} \frac{\theta(x) e^{\mp i \frac{\pi}{2}}\ e^{-2i \int_{x_{\text{left}}(E)}^{x} (E-V(t))^{1/2}\, dt}}{(E - V(x))^{1/2}}\ (1 + \overline{u_{\text{left}}(x,E)})^2\, dx \\ + \frac{1}{2} b_{\text{left}}^2 \int_{I_{\text{BVP}}} \frac{\theta(x)}{(E - V(x))^{1/2}} |1 + u_{\text{left}}(x,E)|^2\, dx \\ \text{(22)} \qquad \equiv \text{Term } 1(\theta) + \text{Term } 2(\theta) + \text{Term } 3(\theta).$$

We shall show that Term $1(\theta)$, Term $2(\theta)$ are negligibly small. In fact, using a partition of unity we can write $\theta = \sum_{\nu} \theta_\nu$ with each θ_ν supported in $\{|x - x_\nu| < c_{\#} \lambda^{-\varepsilon}(x_\nu) B(x_\nu)\}$, satisfying $\left| \left(\frac{d}{dx}\right)^m \theta_\nu \right| \leq C_{\#}^m (\lambda^{-\varepsilon}(x_\nu) B(x_\nu))^{-m}$, $x_{\nu+1} - x_\nu > c_{\#} \lambda^{-\varepsilon}(x_\nu) B(x_\nu)$, and supp $\theta_\nu \subset$ supp θ. We then have Term $1(\theta)$ =

$\sum_\nu$ Term $1(\theta_\nu)$, Term $2(\theta) = \sum_\nu$ Term $2(\theta_\nu)$, and $|$Term $1(\theta_\nu)|$, $|$Term $2(\theta_\nu)| \le C_\#(\lambda(x_\nu))^{-2N}\frac{B(x_\nu)}{S^{1/2}(x_\nu)}b^2_{\text{left}}$ by the stationary phase Lemma (i.e. Lemma 1 in the section on eigenfunctions and eigenvalues of Schrödinger operators.) Hence,

(23)
$$|\text{Term } 1(\theta)|, |\text{Term } 2(\theta)| \le b^2_{\text{left}}\Lambda^{-N}\sum_\nu \frac{B(x_\nu)\lambda^{-\varepsilon}(x_\nu)}{(S(x_\nu))^{1/2}} \le C_\#\Lambda^{-N}b^2_{\text{left}}\int_{x_{\text{left}}}^{x_{\text{rt}}}\frac{dx}{S^{1/2}(x)} \le C_\#\Lambda^{-N}$$

by the bound for $|b_{\text{left}}|$ in the WKB Eigenfunction Theorem.

Regarding Term $3(\theta)$, we have $\Big||1-u_{\text{left}}(x,E)|^2-1\Big| \le C_\#\Lambda^{-2}$ in supp $\theta \cap \{x > x_{\text{left}}(E)+c_\# B_{\text{left}}\}$, by (17), ($18^{\text{bis}}$). Similarly $\Big||1-u_{\text{left}}(x,E)|^2-1\Big| \le C_\#\Lambda^{3\varepsilon-2}$ in supp $\theta \cap \{x < x_{\text{left}}(E)+c_\# B_{\text{left}}\}$ by (17), (18) and the fact that $\lambda_{\text{left}} \ge c_\#\Lambda$. Hence,

$$\text{Term } 3(\theta) = \frac{1}{2}\int_{I_{\text{BVP}}} b^2_{\text{left}}\frac{\theta(x)\,dx}{(E-V(x))^{1/2}} + \text{Error}_{10}, \tag{24}$$

with

$$\begin{aligned}|\text{Error}_{10}| &\le C_\#\Lambda^{-2}b^2_{\text{left}}\int_{\text{supp }\theta}\frac{dx}{(E-V(x))^{1/2}} \\ &\quad + C_\#\Lambda^{3\varepsilon-2}b^2_{\text{left}}\int_{\text{supp }\theta\cap\{x<x_{\text{left}}(E)+c_\# B_{\text{left}}\}}\frac{dx}{(E-V(x))^{1/2}} \\ &\le C_\#\Lambda^{3\varepsilon-2}b^2_{\text{left}}\int_{E-V(x)>0}\frac{dx}{(E-V(x))^{1/2}} \le C_\#\Lambda^{3\varepsilon-2},\end{aligned} \tag{25}$$

again using our bound on $|b_{\text{left}}|$.

Putting (23), (24), (25) into (22) and then comparing the result with (20), (21), we obtain:

$$\int_{I_{\text{BVP}}}\theta(x)F^2(x)\,dx = \frac{1}{2}b^2_{\text{left}}\int_{E-V(x)>0}\frac{\theta(x)\,dx}{(E-V(x))^{1/2}} + \text{Error}_{11},$$

with $|\text{Error}_{11}| \le C_\#\Lambda^{3\varepsilon-2}$. Here we wrote the region of integration as $\{E-V(x)>0\}$, which is harmless since $E-V>0$ on supp θ.

We record this result as a Lemma.

Lemma 6. *Let F be as in Lemma 5, and suppose $\theta(x)$ is supported in $[x_{\rm left}(E) + c_{\#}\lambda_{\rm left}^{-\varepsilon} B_{\rm left}, x_{\rm rt}(E) - c_{\#}B_{\rm rt}]$, satisfying $\left|(\frac{d}{dx})^m \theta(x)\right| \leq C_{\#}^m (\lambda^{-\varepsilon}(x)B(x))^{-m}$. Then*

$$\int_{I_{\rm BVP}} \theta(x) F^2(x)\, dx = \frac{1}{2} b_{\rm left}^2 \int_{E-V(x)>0} \frac{\theta(x)\, dx}{(E - V(x))^{1/2}} + \text{Error}$$

with $|Error| \leq C_{\#}\Lambda^{3\varepsilon - 2}$.

Of course there is an analogue of Lemma 6 with the roles of $x_{\rm left}(E)$, $x_{\rm rt}(E)$ interchanged. At last we are ready to dispose of $\tilde{c}(m)$.

Lemma 7. *The constants $\tilde{c}(0)$, $\tilde{c}(1)$, $\tilde{c}(2)$ in Lemma 4 are all zero.*

Proof. We apply Lemmas 5 and 6 to the harmonic oscillator, and compare the results with the known behavior of the Hermite functions. Specifically, let $H = -\frac{d^2}{dx^2} + \lambda^2 x^2$ on the whole line, and put $E_0 = \lambda^2$, $E_\infty = 10\lambda^2$, $I = [-10, +10]$, $B(x) \equiv 1$, $S(x) \equiv \lambda^2$. Then our assumptions (Hyp0)...(Hyp10) hold if λ is bigger than a (large) universal constant. The eigenfunctions of H are the Hermite functions, and there are many eigenvalues E with $|E - E_0| < c_{\#}^0 S_{\rm min}$, i.e. $|E - \lambda^2| << \lambda^2$. Let F and E be as in Lemmas 5 and 6. Suppose $\theta(x)$ is an even C^∞ function of polynomial growth on the line. Using a partition of unity we can write

$$\theta(x) = \theta_{\text{far left}} + \theta_{\text{Airey left}} + \theta_{\text{med. left}} + \theta_{\text{center}} + \theta_{\text{med. rt}} + \theta_{\text{Airey rt}} + \theta_{\text{far rt}}$$

with:

$$\text{supp } \theta_{\text{far left}} \subset (-\infty, x_{\rm left}(E) - c_{\#}\lambda^{-\varepsilon}], \ \theta_{\text{far rt}}(x) = \theta_{\text{far left}}(-x)$$

$$\text{supp } \theta_{\text{Airey left}} \subset \{|x - x_{\rm left}(E)| < \lambda^{-\varepsilon}\}, \ \theta_{\text{Airey rt}}(x) = \theta_{\text{Airey left}}(-x)$$

$$\text{supp } \theta_{\text{med. left}} \subset [x_{\rm left}(E) + c_{\#}\lambda^{-\varepsilon}, x_{\rm left}(E) + c_{\#}], \ \theta_{\text{med rt}}(x) = \theta_{\text{med left}}(-x)$$

$$\text{supp } \theta_{\text{center}} \subset [x_{\rm left}(E) + \frac{c_{\#}}{2}, x_{\rm rt}(E) - \frac{c\#}{2}], \ \theta_{\text{center}}(x) = \theta_{\text{center}}(-x).$$

Since F is either even or odd, we have

$$\int_{-\infty}^{\infty} \theta(x) F^2(x)\, dx = 2\int_{-\infty}^{\infty} \theta_{\text{far left}} F^2\, dx + 2\int_{-\infty}^{\infty} \theta_{\text{Airey left}} F^2\, dx + 2\int_{-\infty}^{\infty} \theta_{\text{med left}} F^2\, dx + \int_{-\infty}^{\infty} \theta_{\text{center}} F^2\, dx.$$

The last two integrals on the right are computed by Lemma 6, and the integral involving $\theta_{\text{Airey left}}$ is computed by Lemma 5.

Since $\int_{-\infty}^{\infty} F^2\, dx = 1$, the rapid decrease of the Hermite functions in $(-\infty, x_{\text{left}}(E) - c_{\#}\lambda^{-\varepsilon}]$ implies that $|\int_{-\infty}^{\infty} \theta_{\text{far left}} F^2\, dx| \leq \lambda^{-10}$. Consequently, we have the formula

$$\int_{-\infty}^{\infty} \theta(x) F^2(x)\, dx = b_{\text{left}}^2 \cdot \frac{1}{2}\int_{E-\lambda^2x^2>0} \frac{\theta(x)\, dx}{(E-\lambda^2x^2)^{1/2}} + 2\sum_{m=0}^{2} \tilde{c}(m)\lambda^{-\frac{(2m+4)}{3}} \Big((\frac{\partial Y_{\text{left}}}{\partial x})^{-1} \frac{d}{dx}\Big)^m \Big\{\theta(x)(\frac{\partial Y_{\text{left}}}{\partial x})^{-2}\Big\}\Big|_{x=x_{\text{left}}(E)} \cdot b_{\text{left}}^2 + \text{Error} \tag{26}$$

with

$$|\text{Error}| \leq C_{\#}\lambda^{3\varepsilon-2} + C_{\#} b_{\text{left}}^2 \lambda^{\frac{5}{2}\varepsilon - 3}. \tag{27}$$

The WKB Eigenfunction Theorem gives $b_{\text{left}}^{-2} \sim \int\limits_{E-\lambda^2x^2>0} \frac{dx}{(E-\lambda^2x^2)^{1/2}} \sim \lambda^{-1}$, so (27) shows that

$$|\text{Error}| \leq C_{\#}\lambda^{3\varepsilon-2}. \tag{28}$$

Equations (26), (28) hold for $\theta(x)$ an even C^∞ function of polynomial growth. Let us test (26), (28) in the case $\theta(x) =$ polynomial. The computation of the moments $\int_{-\infty}^{\infty} x^{2m} F^2(x)\, dx$ is an elementary exercise using raising and lowering operators. We provide details for the convenience of the reader. We can express $H = \frac{1}{2}(a_0a_1 + a_1a_0)$ with $[a_0, a_1]$ a scalar multiple of the identity, using $a_0 = +\frac{d}{dx} + \lambda x$, $a_1 = -\frac{d}{dx} + \lambda x$, or by using $a_0 = -\frac{d}{dx} + \lambda x$, $a_1 = +\frac{d}{dx} + \lambda x$. As operators we have $x = \frac{1}{2\lambda}(a_0 + a_1)$, so $x^{2m} = (2\lambda)^{-2m} \sum_{i_1 \dots i_{2m}=0}^{1} a_{i_1} a_{i_2} \dots a_{i_{2m}}$.

Each monomial $a_{i_1} a_{i_2} \dots a_{i_{2m}}$ may be written as a linear combination of terms of the form:

$$a_1^{2m_0}[a_0, a_1]^{m_1} H^{m_2} \quad \text{with} \quad m_0 + m_1 + m_2 = m, \quad m_0 \neq 0 \tag{29}$$

$$a_0^{2m_0}[a_0, a_1]^{m_1} H^{m_2} \quad \text{with} \quad m_0 + m_1 + m_2 = m, \quad m_0 \neq 0 \tag{30}$$

$$[a_0, a_1]^{m_1} H^{m_2} \quad \text{with} \quad m_1 + m_2 = m \tag{31}$$

In fact, (29) arises from monomials with more a_1's than the a_0's; (30) arises from monomials with an excess of a_0's; and (31) arises from monomials with as many a_1's as a_0's.

Hence as operators we have

$$\begin{aligned} x^{2m} =&(2\lambda)^{-2m} \sum_{\substack{m_0+m_1+m_2=m \\ m_0 \geq 1}} \text{coeff}'(m_0, m_1, m_2) a_1^{2m_0}[a_0, a_1]^{m_1} H^{m_2} \\ &+ (2\lambda)^{-2m} \sum_{\substack{m_0+m_1+m_2=m \\ m_0 \geq 1}} \text{coeff}''(m_0, m_1, m_2) a_0^{2m_0}[a_0, a_1]^{m_1} H^{m_2} \\ &+ (2\lambda)^{-2m} \sum_{m_1+m_2=m} \text{coeff}(m_1, m_2)[a_0, a_1]^{m_1} H^{m_2}. \end{aligned} \tag{32}$$

This holds with the same coefficients whether we take $a_0 = \frac{d}{dx} + \lambda x$ and $a_1 = -\frac{d}{dx} + \lambda x$ or instead we set $a_0 = -\frac{d}{dx} + \lambda x$ and $a_1 = +\frac{d}{dx} - \lambda x$. Thus there are two versions of (32), one with $[a_0, a_1] = 2\lambda$, and the other with $[a_0, a_1] = -2\lambda$.

Averaging together the two versions of (32) cancels the terms $[a_0, a_1]^{m_1} H^{m_2}$ with m_1 odd, yielding

$$\begin{aligned} x^{2m} =& \sum_{\substack{m_0+m_1+m_2=m \\ m_0 \geq 1}} \text{coeff}'(m_0, m_1, m_2) a_1^{2m_0} \lambda^{m_1 - 2m} H^{m_2} \\ &+ \sum_{\substack{m_0+m_1+m_2=m \\ m_0 \geq 1}} \text{coeff}''(m_1, m_1, m_2) a_0^{2m_0} \lambda^{m_1 - 2m} H^{m_2} \\ &+ \sum_{\substack{m_1+m_2=m \\ m_1 \text{ even}}} \text{coeff}(m_1, m_2) \lambda^{m_1 - 2m} H^{m_2}. \end{aligned} \tag{33}$$

We apply (33) to compute the inner product $< x^{2m}F, F >$ with F a Hermite function, $HF = EF$.

The first two sums on the right side of (33) do not affect the inner product, since

$$< a_1^{2m_0}\, \lambda^{m_1-2m} H^{m_2} F, F >= \lambda^{m_1-2m} E^{m_2} < a_1^{2m_0} F, F >= 0 \quad \text{for} \quad m_0 \geq 1,$$

and similarly

$$< a_0^{2m_0}\, \lambda^{m_1-2m} H^{m_2} F, F > = \lambda^{m_1-2m} E^{m_2} < a_0^{2m_0} F, F >= 0 \quad \text{for} \quad m_0 \geq 1.$$

Therefore (33) gives

$$\begin{aligned} < x^{2m} F, F > &= \sum_{\substack{m_1+m_2=m \\ m_1 \text{ even}}} \text{coeff}(m_1, m_2) \lambda^{m_1-2m} E^{m_2} \\ &= \sum_{\substack{m_1+m_2=m \\ m_1 \text{ even}}} \text{coeff}(m_1, m_2) \lambda^{-m_1} (\frac{E}{\lambda^2})^{m_2}. \end{aligned}$$

In particular, for $\frac{E}{\lambda^2} \sim 1$ we get

$$\int_{-\infty}^{\infty} x^{2m} F^2(x)\, dx = \text{coeff}(m) (\frac{E}{\lambda^2})^m + O(\lambda^{-2}) \quad \text{for each } m. \tag{34}$$

On the other hand, from (26), (28) we get trivially

$$\int_{-\infty}^{\infty} x^{2m} F^2(x)\, dx = \int_{E-\lambda^2 x^2 > 0} (\frac{1}{2} b_{\text{left}}^2) \frac{x^{2m}\, dx}{(E - \lambda^2 x^2)^{1/2}} + O(\lambda^{-1/3}),$$

since $b_{\text{left}}^2 \sim \lambda$ and $\frac{\partial Y_{\text{left}}}{\partial x}$ satisfies $|\partial_x^\alpha Y_{\text{left}}| \leq C_\#^\alpha$, $\frac{\partial Y_{\text{left}}}{\partial x} \geq c_\#$ by the WKB Eigenfunction Theorem.

The change of variable $x = \frac{E^{1/2}}{\lambda} \overline{x}$ gives

$$\int_{E-\lambda^2 x^2 > 0} \frac{x^{2m}\, dx}{(E - \lambda^2 x^2)^{1/2}} = \hat{\text{coeff}}(m) \lambda^{-1} \cdot \Big(\frac{E}{\lambda^2}\Big)^m, \tag{36}$$

so the previous equation implies

$$\int_{-\infty}^{\infty} x^{2m} F^2(x)\, dx = \Big(\frac{1}{2} b_{\text{left}}^2 \lambda^{-1}\Big) \Big(\frac{E}{\lambda^2}\Big)^m \hat{\text{coeff}}(m) + O(\lambda^{-1/3}) \quad \text{for each } m.$$

Comparing this with (34) and recalling that $\frac{E}{\lambda^2} \sim 1$, we get

$$\text{(37)} \qquad \Big(\frac{1}{2} b_{\text{left}}^2 \lambda^{-1}\Big) \hat{\text{coeff}}(m) = \text{coeff}(m) + O(\lambda^{-1/3}) \quad \text{for each } m.$$

Taking $m = 0$ and recalling that $\int_{-\infty}^{\infty} F^2\, dx = 1$, we get $\hat{\text{coeff}}(0) = \pi$ by (36), and $\text{coeff}(0) = 1 + O(\lambda^{-2})$ by (34). Hence (37) for $m = 0$ gives $(\frac{1}{2} b_{\text{left}}^2 \lambda^{-1}) = \frac{1}{\pi} + O(\lambda^{-1/3})$. So (37) may be rewritten in the form

$$\text{coeff}(m) = \frac{1}{\pi} \hat{\text{coeff}}(m) + O(\lambda^{-1/3}).$$

Here, $\text{coeff}(m)$ and $\hat{\text{coeff}}(m)$ are universal constants, while λ may be taken arbitrarily large. Consequently,

$$\text{coeff}(m) = \frac{1}{\pi} \hat{\text{coeff}}(m).$$

This equation and (34) yield

$$\int_{-\infty}^{\infty} x^{2m} F^2(x)\, dx = \frac{1}{\pi} \hat{\text{coeff}}(m) \cdot \Big(\frac{E}{\lambda^2}\Big)^m + O(\lambda^{-2}),$$

which is equivalent to

$$\int_{-\infty}^{\infty} x^{2m} F^2(x)\, dx = \frac{\lambda}{\pi} \int_{E-\lambda^2 x^2 > 0} \frac{x^{2m}\, dx}{(E - \lambda^2 x^2)^{1/2}} + O(\lambda^{-2}) \quad \text{for each } m,$$

by virtue of (36). This implies

$$\int_{-\infty}^{\infty} \theta(x) F^2(x)\, dx = \frac{\lambda}{\pi} \int_{E-\lambda^2 x^2 > 0} \frac{\theta(x)\, dx}{(E - \lambda^2 x^2)^{1/2}} + O(\lambda^{-2})$$

for $\theta(x) = \sum_{k=0}^{10} a_k(E, \lambda) x^{2k}$, provided $|a_k(E, \lambda)| = O(1)$. Comparing this equation with (26), (28), we conclude that

$$\text{(37a)} \quad \frac{\lambda}{\pi} \int_{E-\lambda^2 x^2 > 0} \frac{\theta(x)\, dx}{(E - \lambda^2 x^2)^{1/2}} = \frac{1}{2} b_{\text{left}}^2 \int_{E-\lambda^2 x^2 > 0} \frac{\theta(x)\, dx}{(E - \lambda^2 x^2)^{1/2}}$$
$$+ 2 \sum_{m=0}^{2} \tilde{c}(m) \lambda^{-\frac{(2m+4)}{3}} b_{\text{left}}^2 \Big(\Big(\frac{\partial Y_{\text{left}}}{\partial x}\Big)^{-1} \frac{d}{dx}\Big)^m \Big\{ \theta(x) \Big(\frac{\partial Y_{\text{left}}}{\partial x}\Big)^{-2} \Big\} \Big|_{x = x_{\text{left}}(E)}$$
$$+ O(\lambda^{3\varepsilon - 2}).$$

We first apply (37a) to $\theta_0(x) = (x^2 - \frac{E}{\lambda^2})^4$, which vanishes to 4^{th} order at $x = x_{\text{left}}(E) = -(\frac{E}{\lambda^2})^{1/2}$, so that the second term on the right in (37a) equals zero. Thus, (37a) for θ_0 yields

$$\text{(37b)} \qquad b_{\text{left}}^2 = \frac{2\lambda}{\pi} + O(\lambda^{3\varepsilon-1}), \quad \text{since} \int_{E-\lambda^2x^2>0} \frac{\theta_0(x)\,dx}{(E-\lambda^2x^2)^{1/2}} \sim \lambda^{-1}.$$

Putting (37b) into (37a), we get

$$2b_{\text{left}}^2 \sum_{m=0}^{2} \tilde{c}(m)\lambda^{-\frac{(2m+4)}{3}} \Big(\Big(\frac{\partial Y_{\text{left}}}{\partial x}\Big)^{-1}\frac{d}{dx}\Big)^m \Big\{\theta(x)\Big(\frac{\partial Y_{\text{left}}}{\partial x}\Big)^{-2}\Big\}\Big|_{x=x_{\text{left}}(E))} = O(\lambda^{3\varepsilon-2})$$

for $\theta(x) = \sum\limits_{k=0}^{10} a_k(E,\lambda)x^{2k}$ with $|a_k(E,\lambda)| = O(1)$. That is,

$$\text{(38)} \qquad \sum_{m=0}^{2} \tilde{c}(m)\lambda^{-(\frac{2m+4}{3})} \Big((\frac{\partial Y_{\text{left}}}{\partial x})^{-1}\frac{d}{dx}\Big)^m \Big\{\theta(x)\Big(\frac{\partial Y_{\text{left}}}{\partial x}\Big)^{-2}\Big\}\Big|_{x=x_{\text{left}}(E)} = O(\lambda^{3\varepsilon-3}).$$

We use (38) for special $\theta(x)$ picked to make the computation of $Y_{\text{left}}(x,E)$ unnecessary. First take $\theta(x) \equiv 1$ in (38) to obtain

$$\tilde{c}(0)\lambda^{-4/3}(\frac{\partial Y_{\text{left}}}{\partial x})^{-2}\Big|_{x=x_{\text{left}}(E)} = O(\lambda^{-2}).$$

Since $\frac{\partial Y_{\text{left}}}{\partial x} \le C_\#$ near $x_{\text{left}}(E)$, this implies $\tilde{c}(0) = 0$. Next take $\theta(x) = (\frac{E}{\lambda^2} - x^2)$. At $x = x_{\text{left}}(E)$ we have $\theta = 0$, hence $(\frac{\partial Y_{\text{left}}}{\partial x})^{-1}\frac{d}{dx}\Big\{\theta(x)\Big(\frac{\partial Y_{\text{left}}}{\partial x}\Big)^{-2}\Big\}\Big|_{x=x_{\text{left}}(E)} = \Big(\frac{\partial Y_{\text{left}}}{\partial x}\Big)^{-3}\frac{d\theta}{dx}\Big|_{x=x_{\text{left}}(E)} = -2x\Big(\frac{\partial Y_{\text{left}}}{\partial x}\Big)^{-3}\Big|_{x=x_{\text{left}}(E)} =$
$+2(\frac{E}{\lambda^2})^{1/2}\cdot\Big(\frac{\partial Y_{\text{left}}}{\partial x}\Big)^{-3}\Big|_{x=x_{\text{left}}(E)}$. For this θ, (38) yields

$$\text{(39)} \qquad \tilde{c}(1)\lambda^{-2}\cdot\Big[2(\frac{E}{\lambda^2})^{1/2}\cdot\Big(\frac{\partial Y_{\text{left}}}{\partial x}\Big|_{x=x_{\text{left}}(E)}\Big)^{-3}\Big] = O(\lambda^{-8/3}),$$

since we already know that $\tilde{c}(0) = 0$. The quantity in brackets has order of magnitude 1, so (39) implies $\tilde{c}(1) = 0$. Finally, take $\theta(x) = (\frac{E}{\lambda^2} - x^2)^2$. At $x = x_{\text{left}}(E)$ we have $\theta = 0$, $\theta' = 0$, $\theta'' = \frac{8E}{\lambda^2}$, and therefore

$$\Big((\frac{\partial Y_{\text{left}}}{\partial x})^{-1}\frac{d}{dx}\Big)^2\Big\{\theta(x)(\frac{\partial Y_{\text{left}}}{\partial x})^{-2}\Big\}\Big|_{x=x_{\text{left}}(E)} = \Big\{(\frac{\partial Y_{\text{left}}}{\partial x})^{-4}\cdot\frac{8E}{\lambda^2}\Big\}\Big|_{x=x_{\text{left}}(E)}.$$

Since $\tilde{c}(0) = \tilde{c}(1) = 0$, equation (38) for this θ yields

$$\left[\frac{8E}{\lambda^2}\left(\frac{\partial Y_{\text{left}}}{\partial x}\right)^{-4}\Big|_{x=x_{\text{left}}(E)}\right]\cdot \tilde{c}(2)\lambda^{-8/3} = O(\lambda^{2\varepsilon-3}). \tag{40}$$

The quantity in brackets has order of magnitude 1, so (40) implies that $\tilde{c}(2) = 0$ as well. The proof of the lemma is complete. ■

Corollary. *Under the hypotheses of Lemma 5 we have*

$$\int_{I_{\text{BVP}}} \theta(x)F^2(x)\,dx = \frac{1}{2}b_{\text{left}}^2 \int_{E-V(x)>0} \frac{\theta(x)\,dx}{(E-V(x))^{1/2}} + \text{Error}, \tag{41}$$

with $|\mathit{Error}| \le C_\# \Lambda^{\frac{5}{2}\varepsilon-2}$.

Proof. Since $\tilde{c}(0) = \tilde{c}(1) = \tilde{c}(2) = 0$, Lemma 5 gives (41), with $|\text{Error}| \le C_\# b_{\text{left}}^2 B_{\text{left}}^2 \lambda_{\text{left}}^{-1}(\lambda_{\text{left}}^{\frac{5}{2}\varepsilon-2} + \Lambda^{K-\frac{N''}{10}})$.

¿From the WKB Eigenfunction Theorem we get

$$b_{\text{left}}^{-2} \ge c_\# \int_{x_{\text{left}}}^{x_{\text{rt}}} \frac{dx}{S^{1/2}(x)} \ge c_\# \frac{B_{\text{left}}}{S_{\text{left}}^{1/2}} = c_\# \frac{B_{\text{left}}^2}{\lambda_{\text{left}}},$$

and therefore $b_{\text{left}}^2 B_{\text{left}}^2 \lambda_{\text{left}}^{-1} \le C_\#$. Putting this into our previous estimate for the error, we get $|\text{Error}| \le C_\#(\lambda_{\text{left}}^{\frac{5}{2}\varepsilon-2} + \Lambda^{K-\frac{N''}{10}})$. Since $\lambda_{\text{left}} \ge \Lambda$ and $K - \frac{N''}{10} < -2$, the Corollary follows. ■

Now we can compute the normalization constants b_{left}, b_{rt} in the WKB Eigenfunction Theorem. We write a partition of unity

$$1 = \theta_{\text{far left}} + \theta_{\text{Airey left}} + \theta_{\text{medium left}} + \theta_{\text{center}} + \theta_{\text{medium rt}} + \theta_{\text{Airey rt}} + \theta_{\text{far rt}},$$

with:

$$\text{supp}\ \theta_{\text{far left}} \subset (-\infty, x_{\text{left}}(E) - c_\# \lambda_{\text{left}}^{-\varepsilon} B_{\text{left}}],\ |\theta_{\text{far left}}| \le C_\#;$$

$\theta_{\text{Airey left}}$ satisfying the hypotheses of Lemma 5;

$$\text{supp}\ \theta_{\text{medium left}} \subset [x_{\text{left}}(E) + c_\# \lambda_{\text{left}}^{-\varepsilon} B_{\text{left}}, x_{\text{left}}(E) + c_\# B_{\text{left}}],$$

$\theta_{\text{medium left}}$ satisfying the hypotheses of Lemma 6;

$$\text{supp } \theta_{\text{center}} \subset [x_{\text{left}}(E) + \frac{c_\#}{2} B_{\text{left}}, x_{\text{rt}}(E) - \frac{c_\#}{2} B_{\text{rt}}],$$

θ_{center} satisfying the hypotheses of Lemma 6;

$$\theta_{\text{medium rt}} \quad \text{analogous to } \theta_{\text{medium left}} \;;$$

$$\theta_{\text{Airey rt}} \quad \text{analogous to } \theta_{\text{Airey left}} \;;$$

$$\theta_{\text{far rt}} \quad \text{analogous to } \theta_{\text{far left}} \;.$$

The WKB Eigenfunction Theorem gives

$$(42) \qquad |\int_{I_{\text{BVP}}} \theta_{\text{far left}} \; F^2 \, dx|, |\int_{I_{\text{BVP}}} \theta_{\text{far rt}} \; F^2 \, dx| \le C_\# \Lambda^{-100}.$$

The Corollary to Lemma 7 gives

$$(43) \qquad \int_{I_{\text{BVP}}} \theta_{\text{Airey left}} \; F^2 \, dx = \frac{1}{2} b_{\text{left}}^2 \int_{E-V(x)>0} \frac{\theta_{\text{Airey left}} \; (x) \, dx}{(E - V(x))^{1/2}} + \text{ Error}$$

with $|\text{Error}| \le C_\# \Lambda^{\frac{5}{2}\varepsilon - 2}$.

Analogously we have

$$(44) \qquad \int_{I_{\text{BVP}}} \theta_{\text{Airey rt}} \; F^2 \, dx = \frac{1}{2} b_{\text{rt}}^2 \int_{E-V(x)>0} \frac{\theta_{\text{Airey rt}} \; (x) \, dx}{(E - V(x))^{1/2}} + \text{ Error },$$

with $|\text{Error}| \le C_\# \Lambda^{\frac{5}{2}\varepsilon - 2}$. We want to change b_{rt} to b_{left}. The WKB Eigenfunction Theorem gives $|b_{\text{rt}}| = |b_{\text{left}}| \cdot (1 + \text{error})$ with $|\text{error}| \le C_\# \Lambda^{-2}$. Hence

$$\begin{aligned} &\Big|\frac{1}{2} b_{\text{rt}}^2 \int_{E-V(x)>0} \frac{\theta_{\text{Airey rt}} \; (x) \, dx}{(E - V(x))^{1/2}} - \frac{1}{2} b_{\text{left}}^2 \int_{E-V(x)>0} \frac{\theta_{\text{Airey rt}} \, (x) \, dx}{(E - V(x))^{1/2}}\Big| \\ &\le C_\# \Lambda^{-2} b_{\text{left}}^2 \int_{x_{\text{left}}(E)}^{x_{\text{rt}}(E)} \frac{dx}{(E - V(x))^{1/2}} \le C_\# \Lambda^{-2} b_{\text{left}}^2 \int_{x_{\text{left}}}^{x_{\text{rt}}} \frac{dx}{S^{1/2}(x)} \\ &\le C_\# \Lambda^{-2}, \end{aligned}$$

by the bound on b_{left} given by the WKB Eigenfunction Theorem. Putting this into (44), we get

$$(45) \qquad \int_{I_{\text{BVP}}} \theta_{\text{Airey rt}} \; F^2 \, dx = \frac{1}{2} b_{\text{left}}^2 \int_{E-V(x)>0} \frac{\theta_{\text{Airey rt}} \, (x) \, dx}{(E - V(x))^{1/2}} + \text{Error},$$

$$\text{with } |\text{Error}| \le C_\# \Lambda^{\frac{5}{2}\varepsilon - 2}.$$

Lemma 6 yields

(46)
$$\int_{I_{\mathrm{BVP}}} \theta_{\text{medium left}} F^2\,dx = \frac{1}{2} b_{\text{left}}^2 \int_{E-V(x)>0} \frac{\theta_{\text{medium left}}\,(x)\,dx}{(E-V(x))^{1/2}} + \text{Error},$$
$$\text{with } |\text{Error}| \le C_\# \Lambda^{3\varepsilon-2}.$$

Analogously, we have

(47)
$$\int_{I_{\mathrm{BVP}}} \theta_{\text{medium rt}}\, F^2\,dx = \frac{1}{2} b_{\text{rt}}^2 \int_{E-V(x)>0} \frac{\theta_{\text{medium rt}}\,(x)\,dx}{(E-V(x))^{1/2}} + \text{Error}$$
$$\text{with } |\text{Error}| \le C_\# \Lambda^{3\varepsilon-2}.$$

As in the proof of (45), we have

$$\left|\frac{1}{2} b_{\text{rt}}^2 \int_{E-V(x)>0} \frac{\theta_{\text{medium rt}}\,(x)\,dx}{(E-V(x))^{1/2}} - \frac{1}{2} b_{\text{left}}^2 \int_{E-V(x)>0} \frac{\theta_{\text{medium rt}}\,(x)\,dx}{(E-V(x))^{1/2}}\right|$$
$$\le C_\# \Lambda^{-2} b_{\text{left}}^2 \int_{x_{\text{left}}(E)}^{x_{\text{rt}}(E)} \frac{dx}{(E-V(x))^{1/2}} \le C_\# \Lambda^{-2} b_{\text{left}}^2 \int_{x_{\text{left}}}^{x_{\text{rt}}} \frac{dx}{S^{1/2}(x)} \le C_\# \Lambda^{-2},$$

so that (47) implies

(48)
$$\int_{I_{\mathrm{BVP}}} \theta_{\text{medium rt}}\, F^2\,dx = \frac{1}{2} b_{\text{left}}^2 \int_{E-V(x)>0} \frac{\theta_{\text{medium rt}}\,(x)\,dx}{(E-V(x))^{1/2}} + \text{ Error},$$
$$\text{with } |\text{Error}| \le C_\# \Lambda^{3\varepsilon-2}.$$

Finally, Lemma 6 yields

(49)
$$\int_{I_{\mathrm{BVP}}} \theta_{\text{center}}\, F^2\,dx = \frac{1}{2} b_{\text{left}}^2 \int_{E-V(x)>0} \frac{\theta_{\text{center}}\,(x)\,dx}{(E-V(x))^{1/2}} + \text{ Error},$$
$$\text{with } |\text{Error}| \le C_\# \Lambda^{3\varepsilon-2}.$$

Adding (42), (43), (45), (46), (48), (49) and recalling that $\theta_{\text{Airey left}} + \theta_{\text{medium left}} + \theta_{\text{center}} + \theta_{\text{medium rt}} + \theta_{\text{Airey rt}} = 1$ on $\{E - V(x) > 0\}$, we obtain the formula:

$$\int_{I_{\mathrm{BVP}}} F^2\,dx = \frac{1}{2} b_{\text{left}}^2 \int_{E-V(x)>0} \frac{dx}{(E-V(x))^{1/2}} + \text{ Error},$$
$$\text{with } |\text{Error}| \le C_\# \Lambda^{3\varepsilon-2}.$$

Since $\int_{I_{\mathrm{BVP}}} F^2\,dx = 1$, this means that $|b_{\text{left}}| = \left(\frac{1}{2}\int_{E-V(x)>0} \frac{dx}{(E-V(x))^{1/2}}\right)^{-1/2} \cdot (1+$ Error), with $|\text{Error}| \le C_\# \Lambda^{3\varepsilon-2}$. The WKB Eigenfunction Theorem tells us that $\big||b_{\text{rt}}/b_{\text{left}}| - 1\big| \le C_\# \Lambda^{-2}$, so we get the same result for b_{rt} as for b_{left}.

This completes our computation of the normalizing contants. We record our result in the following theorem.

WKB Normalization Theorem. *The constants* b_{left}, b_{rt} *in the WKB Eigenfunction Theorem satisfy*

$$\left| b_{\text{left}}^2 \cdot \left(\frac{1}{2} \int_{E-V(x)>0} \frac{dx}{(E-V(x))^{1/2}} \right) - 1 \right| \leq C_\# \Lambda^{3\varepsilon - 2} \quad \textit{and}$$

$$\left| b_{\text{rt}}^2 \cdot \left(\frac{1}{2} \int_{E-V(x)>0} \frac{dx}{(E-V(x))^{1/2}} \right) - 1 \right| \leq C_\# \Lambda^{3\varepsilon - 2},$$

with $C_\#$ *depending only on* ε, K, N, c, C, c_1, c_2, C_α *in (Hyp 0)... (Hyp 5).*

Eigenvalues near the Minimum of the Potential

In this section we study the eigenvalues of $H = -\frac{d^2}{dx^2} + V(x)$ near the minimum of the potential V. We shall see that the WKB formulas for eigenvalues E remain highly accurate as E nears min V, even though the WKB description of eigenfunctions loses precision. This phenomenon is familiar for the harmonic oscillator, all of whose eigenvalues are perfectly described by the semiclassical approximation. To prove our results, we transform H to the harmonic oscillator by a (formal power series) change of variable $y = y(x)$. The precise statement of the problem is as follows.

Let $K, \varepsilon, N > 0$ be given with $\varepsilon N > 100$. Let $V(x)$ be a potential defined on a (possibly unbounded) interval I_{BVP}. Let $S, B > 0$ be given numbers, and let $x_0 \in I_{\text{BVP}}$ be given. Define $\lambda = S^{1/2} B$. Let E_∞ be a given energy, with $E_\infty \geq V(x_0)$. We make the following assumptions.

(H1*) $|(\frac{d}{dx})^\alpha V(x)| \leq C_\alpha S B^{-\alpha}$ in $I = \{x \in \mathbb{R} \mid |x - x_0| < cB\} \subset I_{\text{BVP}}$

(H2*) $\frac{d^2}{dx^2} V \geq c S B^{-2}$ in I

(H3*) $V'(x_0) = 0$

(H4*) For $x \in I_{\text{BVP}} \backslash I$ we have $V(x) \geq \min\{E_\infty, V(x_0) + c\lambda^{-2\varepsilon} S\}$

(H5*) For $x \in I_{\text{BVP}}$ with $|x - x_0| > \frac{1}{2}\lambda^K B$, we have $V(x) \geq E_\infty + \frac{1000}{|x - x_0|^2}$

(H6*) λ is bounded below by a constant depending only on C_α, c, ε, K in (H1*)...(H5*), and on N.

Let $H = -\frac{d^2}{dx^2} + V(x)$ on $L^2(I_{\mathrm{BVP}})$ with Dirichlet or Neumann conditions. Our goal is to understand the eigenvalues E of H in the range $V(x_0) \le E \le \min(E_\infty, V(x_0) + c\lambda^{-2\varepsilon}S)$. We begin with two elementary observations. First of all, the continuous spectrum of H is contained in $[E_\infty, \infty)$, since $V \ge E_\infty$ outside a bounded subinterval of I_{BVP}. Secondly, for $E \le \min(E_\infty, V(x_0) + c\lambda^{-2\varepsilon}S)$ an eigenvalue of H, the eigenfunction $F(x)$ is strongly concentrated on $\{|x - x_0| < \lambda^{-\varepsilon}B\}$. This will make it possible to study the eigenvalues of H by Taylor–expanding $V(x)$ about $x = x_0$. Specifically, we have the following result.

Lemma 1. *Let F be an eigenfunction of H with L^2–norm 1 and with eigenvalue $E \le min(E_\infty, V(x_0) + c_\#\lambda^{-2\varepsilon}S)$. Then*

$$\int_{I_{\mathrm{BVP}} \cap \{|x - x_0| > \lambda^{-\varepsilon}B\}} |F|^2 \, dx \le C_\# \lambda^{-N}.$$

In this section, $c_\#$, $C_\#$, $C_\#^{\alpha\beta}$ etc. denote constants depending only on K, ε, N, C_α, c in (H1)...(H5*).*

Sketch of Proof. We follow the discussion of the Agmon lemma in the section on eigenvalues and eigenfunctions of Schrödinger operators. We have $\{x \in I_{\mathrm{BVP}} \mid V(x) < E\} = (x_-, x_+)$ with $|x_\pm - x_0| < c_\#\lambda^{-\varepsilon}B$, and the proof of the Agmon lemma gives us

$$(1) \qquad \int_{I_{\mathrm{BVP}} \setminus (x_-, x_+)} \frac{e^\varphi}{2}\left(\left|\frac{dF}{dx}\right|^2 + (V - E)|F|^2\right) dx \le \int_{x_-}^{x_+} (E - V)|F|^2 \, dx$$

with $\varphi(x) = \int_x^{x_-} (V - E)^{1/2}$ for $x \le x_-$, $\varphi(x) = \int_{x_+}^x (V - E)^{1/2}$ for $x \ge x_+$.

We establish lower bounds for φ. Say $x < x_0 - \lambda^{-\varepsilon} B$. Then

$$\varphi(x) \geq \int_{x_0-\lambda^{-\varepsilon}B}^{x_-} (V-E)^{1/2}\, dy > c_\# \int_{x_0-\lambda^{-\varepsilon}B}^{x_0-\frac{1}{2}\lambda^{-\varepsilon}B} (\lambda^{-2\varepsilon} S)^{1/2}\, dy$$
$$\geq c_\# \lambda^{-2\varepsilon} B S^{1/2} = c_\# \lambda^{1-2\varepsilon},$$

since $V - E \geq (V(x_0) + \frac{1}{2} c S B^{-2}(x-x_0)^2) - (V(x_0) + c_\# \lambda^{-2\varepsilon} S) \geq c_\# \lambda^{-2\varepsilon} S$ for $|x - x_0| \sim \lambda^{-\varepsilon} B$.

If $x < x_0 - \lambda^K B$, then

$$\varphi(x) \geq \int_x^{x_0-\lambda^K B} (V-E)^{1/2}\, dy + \int_{x_0-\lambda^{-\varepsilon}B}^{x_-} (V-E)^{1/2}\, dy$$
$$\geq \int_x^{x_0-\lambda^K B} (V-E)^{1/2}\, dy + c_\# \lambda^{1-2\varepsilon}$$

(by the previous chain of inequalities), and the integral on the right exceeds $\int_x^{x_0-\lambda^K B} \frac{10}{|y-x_0|}\, dy = 10\ \ell n(\frac{|x-x_0|}{\lambda^K B})$. Hence:

$$\varphi(x) \geq c_\# \lambda^{1-2\varepsilon} \quad \text{if} \quad x < x_0 - \lambda^{-\varepsilon} B$$
$$V(x) - E \geq c_\# \lambda^{-2\varepsilon} S \quad \text{if} \quad |x - x_0| \sim \lambda^{-\varepsilon} B$$
$$\varphi(x) \geq c_\# \lambda^{1-2\varepsilon} + 10\ \ell n(\frac{|x-x_0|}{\lambda^K B}) \quad \text{if} \quad x < x_0 - \lambda^K B.$$

In (x_-, x_+) we have $E - V \leq C_\# S$, so (1) yields the following estimates:

$$\text{(2)} \qquad \int_{x_0-2\lambda^{-\varepsilon}B}^{x_0-\lambda^{-\varepsilon}B} |F|^2\, dy \leq C_\# \lambda^{2\varepsilon} \exp(-c_\# \lambda^{1-2\varepsilon}) \int_{x_-}^{x_+} |F|^2\, dy$$

$$\text{(3)} \qquad \int_{I_{\mathrm{BVP}} \cap (-\infty, x_0-\lambda^{-\varepsilon}B)} \left|\frac{dF}{dy}\right|^2 dy \leq C_\# S \exp(-c_\# \lambda^{1-2\varepsilon}) \int_{x_-}^{x_+} |F|^2\, dy$$

$$\text{(4)} \qquad \int_{I_{\mathrm{BVP}} \cap (-\infty, x_0-\lambda^K B)} \left[\left(\frac{|y-x_0|}{\lambda^K B}\right)^{10} \exp(c_\# \lambda^{1-2\varepsilon})\right] (V-E)|F|^2\, dy$$
$$\leq C_\# S \int_{x_-}^{x_+} |f|^2\, dy.$$

From (2) and (3) we deduce

$$(5)\qquad \int_{I_{\mathrm{BVP}}\cap[x_0-\lambda^K B, x_0-\lambda^{-\varepsilon}B]} |F|^2\,dy \le C_\# \lambda^{-N} \int_{x_-}^{x_+} |F|^2\,dy.$$

From (4) and the inequalities

$$\Big(\frac{|y-x_0|}{\lambda^k B}\Big)^{10}(V(y)-E) \ge \Big(\frac{|y-x_0|}{\lambda^K B}\Big)^{10}(V(y)-E_\infty) \ge \Big(\frac{|y-x_0|}{\lambda^K B}\Big)^{10}\cdot\frac{100}{|y-x_0|^2}$$
$$\ge \frac{100}{(\lambda^K B)^2}$$

valid in $I_{\mathrm{BVP}}\cap(-\infty, x_0-\lambda^K B]$, we conclude that

$$\int_{I_{\mathrm{BVP}}\cap(-\infty,x_0-\lambda^{-K}B]} |F|^2\,dy \le C_\# \exp(-c_\# \lambda^{1-2\varepsilon})\lambda^{2K} S B^2 \int_{x_-}^{x_+} |F|^2\,dy$$
$$\le C_\# \lambda^{-N}\int_{x_-}^{x_+} |F|^2\,dy, \quad \text{since } SB^2 = \lambda^2.$$

Putting this together with (5) gives

$$\int_{I_{\mathrm{BVP}}\cap(-\infty,x_0-\lambda^{-\varepsilon}B]} |F|^2\,dy \le C_\#\lambda^{-N}, \quad \text{since } \int_{x_-}^{x_+} |F|^2\,dy \le \int_{I_{\mathrm{BVP}}} |F|^2\,dy = 1.$$

Similarly, $\int_{I_{\mathrm{BVP}}\cap[x_0+\lambda^{-\varepsilon}B,+\infty)} |F|^2\,dy \le C_\#\lambda^{-N}$, and therefore $\int_{I_{\mathrm{BVP}}\setminus\{|x-x_0|<\lambda^{-\varepsilon}B\}} |F|^2\,dy \le C_\#\lambda^{-N}$, which is the conclusion of the Lemma.

■

Next we show that $H = -\frac{d^2}{dx^2} + V(x)$ can be transformed to the harmonic oscillator by a (formal power series) change of variables $y = y(x)$. The basic result is as follows.

Lemma 2. *Let $W(x)$ be a smooth function satisfying $|(\frac{d}{dx})^\alpha W(x)| \le C_\alpha$ and $W(0) = 0$, $W'(0) = 0$, $W''(0) \ge c_1 > 0$. Then we can find formal power series $y = \sum_{k,m,s\ge 0} y_{kms}\tau^k\lambda^{-2m}x^s$ and $\tilde\tau = \sum_{k,m\ge 0}\tilde\tau_{km}\tau^k\lambda^{-2m}$, satisfying the equation*

$$(6)\qquad (y^2-\tilde\tau)\Big(\frac{\partial y}{\partial x}\Big)^2 + \lambda^{-2}\Big\{-\frac{1}{2}\frac{\partial_x^3 y}{\partial_x y} + \frac{3}{4}\Big(\frac{\partial_x^2 y}{\partial_x y}\Big)^2\Big\} = W(x) - \tau$$

with $y_{001} > 0$ *(so that* $\frac{1}{\partial_x y}$ *makes sense as a formal power series).*

The coefficients y_{kms} *and* $\tilde{\tau}_{km}$ *are uniquely determined, and are bounded a–priori in terms of* c_1 *and the* C_α*.*

Proof. Define the <u>strength</u> of a monomial $C\tau^k\lambda^{-2m}x^s$ to be $k+m$. By induction on $A \geq 0$ we will prove

<u>Assertion A</u>: We can find coefficients $(y_{kms}^{\text{correct}})$ and $(\tilde{\tau}_{km}^{\text{correct}})$ so that (6) holds modulo terms of strength $> A$ if and only if for $k+m \leq A$ we have $y_{kms} = y_{kms}^{\text{correct}}$ (all s) and $\tilde{\tau}_{km} = \tilde{\tau}_{km}^{\text{correct}}$.

We set $y_{km}(x) = \sum\limits_{s \geq 0} y_{kms}x^s$, and begin by proving Assertion 0. The terms of strength 0 in (6) give us the following equation:

$$(y_{00}^2 - \tilde{\tau}_{00})(\frac{dy_{00}}{dx})^2 = W(x). \tag{7}$$

Since $y_{001} > 0$, $(\frac{dy_{00}}{dx})^2$ is a formal power series with non–vanishing constant term. On the other hand $W(x)$ vanishes to second order at 0, so $y_{00}^2 - \tilde{\tau}_{00}$ is a formal power series starting with ax^2. Therefore $2y_{00}\frac{dy_{00}}{dx}$ has no constant term. Since $\frac{dy_{00}}{dx}$ has nonzero constant term, it follows that y_{00} has no constant term. Thus y_{00}^2 is a formal power series starting with ax^2, and since $y_{00}^2 - \tilde{\tau}_{00}$ also starts with an x^2–term, we get $\tilde{\tau}_{00} = 0$. Equation (7) therefore becomes

$$y_{00}^2\left(\frac{dy_{00}}{dx}\right)^2 = W(x). \tag{8}$$

Our assumptions on W let us write $W(x) = (F(x))^2$ with F smooth, $F(0) = 0$, $F'(0) > 0$, and thus (8) amounts to $y_{00}\frac{dy_{00}}{dx} = \pm F(x)$. Since $y_{001} > 0$ and $F'(0) > 0$, we must have the "+" sign, so (7) amounts to $\tilde{\tau}_{00} = 0$ and $y_{00}\frac{dy_{00}}{dx} = F(x)$, which in turn means that $\frac{1}{2}(y_{00})^2 = \int_0^x F(t)\,dt$. The right–hand side has the form $ax^2 +$ higher terms with $a > 0$, and therefore has a smooth square root $g(x)$. Thus (7) means that $\tilde{\tau}_{00} = 0$ and $y_{00} = \pm\sqrt{2}g(x)$. The sign is uniquely specified by requiring $y_{001} > 0$. Thus (7) holds if and only if $\tilde{\tau}_{00} = 0$ and $y_{00s} = y_{00s}^{\text{correct}}$ (all s). We have proven Assertion 0.

Next we assume Assertion (A-1) and deduce Assertion A ($A \geq 1$). With $y_{kms} = y_{kms}^{\text{correct}}$, $\tilde{\tau}_{km} = \tilde{\tau}_{km}^{\text{correct}}$ for $k+m \leq A-1$, but with y_{kms}, $\tilde{\tau}_{km}$ arbitrary for $k+m \geq A$, we examine the terms of strength A in (6). Thus we fix k, m with $k+m = A$. The $\tau^k \lambda^{-2m}$–terms in (6) are as follows:

$$(9)_{km} \qquad 2y_{00}\Big(\frac{dy_{00}}{dx}\Big)^2 y_{km} + 2y_{00}^2\Big(\frac{dy_{00}}{dx}\Big)\Big(\frac{dy_{km}}{dx}\Big) - \tilde{\tau}_{km}\Big(\frac{dy_{00}}{dx}\Big)^2 + f_{km} = g_{km},$$

where f_{km} is determined by $(y_{k'm's}^{\text{correct}})$, $(\tilde{\tau}_{k'm'}^{\text{correct}})$ $(k'+m' \leq A-1)$, and g_{km} is the $\tau^k \lambda^{-2m}$–term in $W(x) - \tau$. In fact, $f_{km} = \sum_{s \geq 0} f_{kms} x^s$ with f_{kms} a polynomial in $(y_{k'm's'}^{\text{correct}})$, $(\tilde{\tau}_{k'm's'}^{\text{correct}})$ $(k'+m' \leq A-1)$, y_{001}^{-1}. Hence to solve (6) modulo terms of strength $> A$, we must solve $(9)_{km}$ for $k+m = A$. Since $y_{000} = 0$, $y_{001} > 0$, the constant term in $(9)_{km}$ is

$$(9\text{bis}) \qquad -\tilde{\tau}_{km}(y_{001})^2 = \text{constant term in } (g_{km} - f_{km}),$$

which uniquely specifies $\tilde{\tau}_{km}$. Once $\tilde{\tau}_{km}$ is specified, $(9)_{km}$ takes the form

$$(10)_{km} \qquad 2y_{00}\Big(\frac{dy_{00}}{dx}\Big)^2 y_{km} + 2y_{00}^2\Big(\frac{dy_{00}}{dx}\Big)\Big(\frac{dy_{km}}{dx}\Big) = \sum_{s \geq 1} h_{kms} x^s$$

with h_{kms} specified by $(y_{k'm's'}^{\text{correct}})$, $(\tilde{\tau}_{k'm'}^{\text{correct}})$ for $k'+m' \leq A-1$. To solve $(10)_{km}$ for $y_{km} = \sum_{s \geq 0} y_{kms} x^s$, we solve successively for the coefficients y_{kms}. The terms of degree $(s+1)$ in $(10)_{km}$ are as follows

$$(y_{001})^3(2+2s)y_{kms} + (\text{polynomial in } y_{00s'} \quad \text{and } y_{kms''} \text{ with } s'' < s) = h_{km(s+1)}.$$

Since $(y_{001})^3(2+2s) > 0$ for $s \geq 0$, this allows us to solve for y_{kms} in terms of $y_{00s'}$ and $(s'' < s)$. Hence we obtain inductively the unique sequence $(y_{kms})_{s \geq 0}$ that solves $(10)_{km}$. Thus $\tilde{\tau}_{km}$ and y_{kms} $(k+m = A)$ can be specified in one and only one way to make (6) hold modulo terms of strength $> A$. We have succeeded in deducing Assertion A from Assertion (A-1). By induction, Assertion A holds for all A, so there is one and only one formal power series solution of (6). To obtain the a–priori bounds on the coefficients $\tilde{\tau}_{km}$, y_{kms} we reexamine the preceding argument. For terms of strength 0, we saw that $\tilde{\tau}_{00} = 0$ and $y_{00s} = \frac{1}{s!}(\pm\sqrt{2})(\frac{d}{dx})^s g(x) \Big|_{x=0}$ with $g^2(x) = \int_0^x W^{1/2}(t)\, dt$.

It follows that $\tilde{\tau}_{00}$, y_{00s} can be bounded a–priori in terms of C_α and c_1 in the statement of the lemma. Also, y_{001} can be bounded below. Assuming we have bounded y_{kms}, $\tilde{\tau}_{km}$ for $k+m < A$, we look at $(9)_{km}$ and observe that the coefficients of the power series f_{km} and g_{km} are bounded a–priori in terms of C_α, c_1, for $k + m = A$. (For f_{km} this follows from the fact that $f_{km} = \sum_s f_{kms} x^s$ with f_{kms} a polynomial in quantities which we have already bounded a–priori. For g_{km} it is immediate from the definitions.)

Therefore (9bis) shows that $\tilde{\tau}_{km}$ is bounded a–priori in terms of C_α, c_1. This in turn shows that the coefficients h_{kms} in $(10)_{km}$ are bounded a–priori. Hence it follows by induction on s that y_{kms} (which we obtain by solving $(10)_{km}$) is bounded a–priori in terms of C_α, c_1.

Assuming a–priori bounds for $\tilde{\tau}_{km}$ and y_{kms} $(k + m < A)$, we have derived a–priori bounds for $\tilde{\tau}_{km}$ and y_{kms} when $k + m = A$. By induction on $A = k + m$, the $\tilde{\tau}_{km}$ and y_{kms} are all bounded a–priori in terms of C_α and c_1. The proof of the lemma is complete. ■

Note that $y_{000} = 0$ and $\tilde{\tau}_{00} = 0$, as we saw in the proof of Lemma 2.

Corollary. *With W, y_{kms} $\tilde{\tau}_{km}$ as in Lemma 2, set*

$$y(x, E, \lambda) = \sum_{k,m,s=0}^{N} y_{kms} \left(\frac{E}{\lambda^2}\right)^k \lambda^{-2m} x^s$$

and

$$\tilde{E}(E, \lambda) = \lambda^2 \sum_{k,m=0}^{N} \tilde{\tau}_{km} \left(\frac{E}{\lambda^2}\right)^k \lambda^{-2m}.$$

Then for $|x|$, $|\frac{E}{\lambda^2}| \leq C_2\lambda^{-\varepsilon}$ *and* $\lambda \geq C_2$ *we have*

$$\Big(\frac{\partial y}{\partial x}\Big)^2(\lambda^2 y^2 - \tilde{E}) + \Big\{-\frac{1}{2}\frac{\partial_x^3 y}{\partial_x y} + \frac{3}{4}\Big(\frac{\partial_x^2 y}{\partial_x y}\Big)^2\Big\} = \lambda^2 W(x) - E + Error(x, E, \lambda)$$

with $|Error(x,E,\lambda)| \leq C_3\lambda^{-\varepsilon N/2}$, $|\frac{\partial}{\partial E}Error(x,E,\lambda)| \leq C_3\lambda^{-\frac{\varepsilon N}{2}-2}$. *The constant* C_3 *depends only on* C_2 *and on* c_1, C_α *in the hypothesis of Lemma 2.*

Proof. Set $f(x,\tau,\xi) = \lambda^{-2}\mathrm{Error}(x,E,\lambda)$ with $\tau = \frac{E}{\lambda^2}$ and $\xi = \lambda^{-2}$. Immediately from the definitions, we see that the C^∞ seminorms of f on $\{|x|, |\tau| \leq c\}$ are bounded a–priori in terms of the C_α, the $|y_{kms}|$, the $|\tilde{\tau}_{km}|$, and a lower bound for y_{001}. The a–priori bounds in the statement of Lemma 2 shows that $|\partial^\alpha_{x,\tau,\xi} f| \leq C'_\alpha$ for $|x|, |\tau|, |\xi| \leq c'$ with C'_α, c' determined by c_1, C_α. On the other hand, the formal power series equation (6) shows that $f(x,\tau,\xi) = 0$ to order at least $N-5$ at $(x,\tau,\xi) = (0,0,0)$. Therefore, Taylor's theorem with remainder gives

$$|f(x,\tau,\xi)| \leq C'\lambda^{-\varepsilon(N-5)} \quad \text{and} \quad \Big|\frac{\partial}{\partial\tau}f(x,\tau,\xi)\Big| \leq C'\lambda^{-\varepsilon(N-6)}$$

for $|x|, |\tau|, |\xi| \leq C_2\lambda^{-\varepsilon}$, with C' determined by C_2, c_1, C_α. This is stronger than the conclusion of the corollary. ■

The previous lemma applies to potentials $\lambda^2 W(x)$. A simple rescaling lets us apply that result to study $V(x)$.

Lemma 3. *Let* $V(x)$, S, B *etc. satisfy (H1*)... (H6*). Then there exist coefficients* $\hat{y}_{ks}$, $\hat{\tau}_k$ *for which the functions*

$$y(x,E) = \sum_{k,s=0}^{N} \hat{y}_{ks}\Big(\frac{E - V(x_0)}{S}\Big)^k\Big(\frac{x - x_0}{B}\Big)^s$$

and

$$\tilde{E}(E) = \lambda^2\sum_{k=0}^{N}\hat{\tau}_k\Big(\frac{E - V(x_0)}{S}\Big)^k$$

satisfy the equation

$$\Big(\frac{\partial y}{\partial x}\Big)^2(\lambda^2 y^2 - \tilde{E}) + \Big\{-\frac{1}{2}\frac{\partial_x^3 y}{\partial_x y} + \frac{3}{4}\Big(\frac{\partial_x^2 y}{\partial_x y}\Big)^2\Big\} = V(x) - E + Error(x,E), \tag{11}$$

with $|Error(x,E)| \leq C_{\#}\lambda^{-\varepsilon N/2}S$, $|\frac{\partial}{\partial E}\ Error(x,E)| \leq C_{\#}\lambda^{-\varepsilon N/2}$ *for* $|x - x_0| < C_{\#}\lambda^{-\varepsilon}B$, $|E - V(x_0)| < C_{\#}\lambda^{-2\varepsilon}S$.

Moreover, we have the a–priori bounds $|\hat{y}_{ks}|$, $|\hat{\tau}_k| \leq C_{\#}$, $|\hat{y}_{00}| \leq C_{\#}\lambda^{-2}$, $|\hat{\tau}_0| \leq C_{\#}\lambda^{-2}$, $\hat{y}_{01} > c_{\#} > 0$.

Proof. Apply Lemma 2 to $W(x) = \frac{V(x_0+Bx)-V(x_0)}{S}$. With y_{kms}, $\tilde{\tau}_{km}$ as in Lemma 2, put $\hat{y}_{ks} = \sum_{m=0}^{N} y_{kms}\lambda^{-2m}$ and $\hat{\tau}_k = \sum_{m=0}^{N} \tilde{\tau}_{km}\lambda^{-2m}$. The estimates asserted for Error(x,E) are then immediate consequences of the corollary to Lemma 2. From Lemma 2 we get $|y_{kms}||\tilde{\tau}_{km}| \leq C_{\#}$, which shows that $|\hat{y}_{ks}|, |\hat{\tau}_k| \leq C_{\#}$. Since $y_{000} = \tilde{\tau}_{00} = 0$, we get also $|\hat{y}_{00}|, |\hat{\tau}_0| \leq C_{\#}\lambda^{-2}$. Since $y_{001} > c_{\#} > 0$ and $|y_{0m1}| \leq C_{\#}$ for $1 \leq m \leq N$ we get $\hat{y}_{01} > c_{\#} > 0$. ∎

Remark. In Lemma 3, we can take $\hat{y}_{ks}$, and $\hat{\tau}_k$ to depend only on N, S, B and the power series of $V(x)$ at $x = x_0$. That's clear from the proof of Lemma 3, since the y_{kms} and $\tilde{\tau}_{km}$ provide the unique solution to a formal power series equation (6) and are consequently determined by the power series of $W(x)$ at $x = 0$.

Corollary. *We have the following a–priori bounds*

$$|\partial_x^{\alpha}\partial_E^{\beta} y(x,E)| \leq C_{\#}^{\alpha\beta} B^{-\alpha} S^{-\beta} \quad \textit{for} \quad |x - x_0| < B,\ |E - V(x_0)| < S; \tag{12}$$

$$\left|\left(\frac{d}{dE}\right)^{\beta} \tilde{E}(E)\right| \leq C_{\#}^{\beta}\lambda^2 S^{-\beta} \quad \textit{for} \quad |E - V(x_0)| < c_{\#}S; \tag{13}$$

$$\frac{\partial y}{\partial x} > c_{\#}B^{-1} \quad \textit{for} \quad |x - x_0| < c_{\#}B,\ |E - V(x_0)| < c_{\#}S; \tag{14}$$

$$\left(\frac{d}{dE}\right)\tilde{E} > c_{\#}\lambda^2 S^{-1} = c_{\#}B^2 \quad \textit{for} \quad |E - V(x_0)| < c_{\#}S. \tag{15}$$

Proof. The upper bounds for $|\partial_x^{\alpha}\partial_E^{\beta} y|$ and $|(\frac{d}{dy})^{\beta}\tilde{E}|$ are immediate from the definitions and $|\hat{y}_{ks}|, |\hat{\tau}_k| \leq C_{\#}$. The lower bound for $\frac{\partial y}{\partial x}$ holds at $x = x_0$, $E = V(x_0)$

since $\hat{y}_{01} > c_\#$. Since $|\frac{\partial^2 y}{\partial x^2}| \le C_\# B^{-2}$ and $|\frac{\partial^2 y}{\partial x \partial E}| \le C_\# B^{-1} S^{-1}$, the lower bound for $\frac{\partial y}{\partial x}$ remains valid throughout $|x - x_0| < c_\# B$, $|E - V(x_0)| < c_\# S$.

To estimate $(\frac{d}{dE})\tilde{E}$, we differentiate (11) with respect to E at the point $x = x_0$, $E = V(x_0)$. Thus

$$\text{(16)} \qquad 2\Big(\frac{\partial y}{\partial x}\Big)\Big(\frac{\partial y}{\partial E \partial x}\Big)(\lambda^2 y^2 - \tilde{E}) + \frac{\partial}{\partial E}\Big\{-\frac{1}{2}\frac{\partial_x^3 y}{\partial_x y} + \frac{3}{4}\Big(\frac{\partial_x^2 y}{\partial_x y}\Big)^2\Big\}$$
$$+\Big(\frac{\partial y}{\partial x}\Big)^2 \cdot 2\lambda^2 y\Big(\frac{\partial y}{\partial E}\Big) - \Big(\frac{\partial y}{\partial x}\Big)^2 \frac{d\tilde{E}}{dE} = -1 + \frac{\partial}{\partial E}\,\text{Error}(x, E).$$

At $x = x_0$, $E = V(x_0)$ we have $y = \hat{y}_{00}$ and $\tilde{E} = \lambda^2 \hat{\tau}_0$, so $|y| \le C_\# \lambda^{-2}$ and $|\lambda^2 y^2 - \tilde{E}| \le C_\#$. From (12) we get

$$\text{(17)} \qquad |2\frac{\partial y}{\partial E \partial x}\Big(\frac{\partial y}{\partial x}\Big)(\lambda^2 y^2 - \tilde{E})| \le (C_\# B^{-1}) \cdot (C_\# B^{-1} S^{-1}) = C_\# \lambda^{-2}$$
$$\text{at } x = x_0, E = V(x_0).$$

Also (12) and (14) show that

$$\text{(18)} \qquad \Big|\frac{\partial}{\partial E}\Big\{-\frac{1}{2}\frac{\partial_x^3 y}{\partial_x y} + \frac{3}{4}\Big(\frac{\partial_x^2 y}{\partial_x y}\Big)^2\Big\}\Big| \le C_\# B^{-2} S^{-1} = C_\# \lambda^{-2}.$$

From (12) and the fact that $|y| \le C_\# \lambda^{-2}$ at $x = x_0$, $E = V(x_0)$ we conclude that

$$\text{(19)} \qquad \Big|\Big(\frac{\partial y}{\partial x}\Big)^2 \cdot 2\lambda^2 y\Big(\frac{\partial y}{\partial E}\Big)\Big| \le (C_\# B^{-1})^2 \cdot C_\# S^{-1} = C_\# \lambda^{-2} \text{ at } x = x_0,$$
$$E = V(x_0).$$

Putting (17), (18), (19), and the bound $|\frac{\partial}{\partial E}\text{Error}(x, E)| \le C_\# \lambda^{-\varepsilon N/2}$ into (16), we see that $|(\frac{\partial y}{\partial x})^2 \frac{d\tilde{E}}{dE} - 1| \le C_\# \lambda^{-2}$ at $x = x_0$, $E = V(x_0)$. So $\frac{d\tilde{E}}{dE} \ge c_\# (\frac{\partial y}{\partial x})^{-2} \ge c_\# B^2$ at $x = x_0$, $E = V(x_0)$, by (12). That is $\frac{d\tilde{E}}{dE} \ge c_\# B^2 = c_\# \lambda^2 S^{-1}$ at $x = x_0$, $E = V(x_0)$. Since $|(\frac{d}{dE})^2 \tilde{E}| \le C_\# \lambda^2 S^{-2}$ by (13), the estimate $\frac{d}{dE}\tilde{E} \ge c_\# \lambda^2 S^{-1}$ remains valid for $|E - V(x_0)| \le c_\# S$. The proof of the corollary is complete. ∎

Lemma 3 says that $x \mapsto y(x, E)$ transforms the equation $[\frac{d^2}{dx^2} + (E - V(x))]F = 0$ to

(20)

$$\Big[\frac{d^2}{dy^2} + (\tilde{E} - \lambda^2 y^2) + \text{Error}\Big]\tilde{F}(y) = 0 \quad \text{with} \quad |\text{Error}| \le C_\# \lambda^{-\varepsilon N/10}$$
$$\text{for } |y| < C_\# \lambda^{-\varepsilon}$$

by taking

$$F(x) = \Big(\frac{\partial y}{\partial x}\Big)^{-1/2} \tilde{F}(y), \quad \text{which defines } \tilde{F}(y) \text{ for } |y| < C_\# \lambda^{-\varepsilon}. \tag{21}$$

Hence we can use our knowledge of the harmonic oscillator to locate the eigenvalues of $-\frac{d^2}{dx^2} + V(x) = H$. This is achieved in the next two lemmas.

Lemma 4. *Suppose E is an eigenvalue of H with $|E - V(x_0)| \le c_\# \lambda^{-2\varepsilon} S$ and $E \le E_\infty$. Then with $\tilde{E}(E)$ as in the previous lemma we have*

$$|\tilde{E}(E) - (2k+1)\lambda| \le C_\# \lambda^{-\varepsilon N/11} \tag{22}$$

for an integer $k \ge 0$. Moreover, no other eigenvalue E' of H with $E' \le E_\infty$ and $|E' - V(x_0)| \le c_\# \lambda^{-2\varepsilon} S$ satisfies $|\tilde{E}(E') - (2k+1)\lambda| \le C_\# \lambda^{-\varepsilon N/11}$ for the same k.

Proof. Let F be the eigenfunction associated to E, and define $\tilde{F}$ by (21). From (20) we get

$$\Big\| \Big[\frac{d^2}{dy^2} + (\tilde{E} - \lambda^2 y^2)\Big] \tilde{F} \Big\|_{L^2(|y| \le C_\# \lambda^{-\varepsilon})} \le C'_\# \lambda^{-\varepsilon N/10} \|\tilde{F}\|_{L^2(|y| \le C_\# \lambda^{-\varepsilon})}. \tag{23}$$

Lemma 1 gives $\int_{|x-x_0| > \lambda^{-\varepsilon} B} |F(x)|^2 \, dx \le C'_\# \lambda^{-N} \int_{|x-x_0| < \lambda^{-\varepsilon} B} |F(x)|^2 \, dx$, hence

$$\|\tilde{F}\|_{L^2(\frac{1}{2} C_\# \lambda^{-\varepsilon} < |y| < C_\# \lambda^{-\varepsilon})} \le C'_\# \lambda^{-N} \|\tilde{F}\|^2_{L^2(|y| < \frac{1}{2} C_\# \lambda^{-\varepsilon})} \tag{24}$$

by virtue of (12), (14). For $|x - x_0| < B$ we have $|\tilde{E} - \lambda^2 y^2| \le C_\# \lambda^2$ by (12), (13), so (24) and (23) imply

$$\Big\| \frac{d^2}{dy^2} \tilde{F} \Big\|_{L^2(\frac{1}{2} C_\# \lambda^{-\varepsilon} < |y| < C_\# \lambda^{-\varepsilon})} \le C'_\# \lambda^{-\varepsilon N/10} \|\tilde{F}\|_{L^2(|y| \le C_\# \lambda^{-\varepsilon})}. \tag{25}$$

From (24), (25) we conclude that

$$\Big\| \frac{d}{dy} \tilde{F} \Big\|_{L^2(\frac{2}{3} C_\# \lambda^{-\varepsilon} < |y| < C_\# \lambda^{-\varepsilon})} \le C'_\# \lambda^{-\varepsilon N/11} \|\tilde{F}\|_{L^2(|y| \le C_\# \lambda^{-\varepsilon})}. \tag{26}$$

Now let $\chi(y)$ be a cutoff function satisfying $|(\frac{d}{dy})^m\chi| \le C_{\#}^m \lambda^{\varepsilon m}$ and $\chi(y) = 1$ for $|y| \le \frac{2}{3}C_{\#}\lambda^{-\varepsilon}$, $\chi(y) = 0$ for $|y| \ge C_{\#}\lambda^{-\varepsilon}$. We can pick $C_{\#}$ large so that $\chi(y(x,E)) = 1$ when $|x - x_0| < \lambda^{-\varepsilon}B$. Estimates (23), (24), (26) yield

$$\Big\| \Big[\frac{d^2}{dy^2} + (\tilde{E} - \lambda^2 y^2)\Big](\chi\tilde{F})\Big\|_{L^2(\mathbb{R})} \le C'_{\#}\, \lambda^{-\varepsilon N/11}\, \|\tilde{F}\|_{L^2(|y|\le C_{\#}\lambda^{-\varepsilon})}. \tag{27}$$

Also, (24) gives $\|\tilde{F}\|_{L^2(|y|\le C_{\#}\lambda^{-\varepsilon})} \le 2\|\chi\tilde{F}\|_{L^2(|y|\le C_{\#}\lambda^{-\varepsilon})}$, so that (27) implies

$$\Big\| \Big[\frac{d^2}{dy^2} + (\tilde{E} - \lambda^2 y^2)\Big](\chi\tilde{F})\Big\|_{L^2(\mathbb{R})} \le C'_{\#}\lambda^{-\varepsilon N/11}\|\chi\tilde{F}\|_{L^2(\mathbb{R})}. \tag{28}$$

For $|x - x_0| < \lambda^{-\varepsilon}B$ we have $\chi(y) = 1$ and $F(x) = (\frac{\partial y}{\partial x})^{-1/2}\tilde{F}(y)$. Hence $\chi\tilde{F}$ cannot vanish identically. From (28) and the known spectrum of the harmonic oscillator, we get

$$|\tilde{E} - (2k+1)\lambda| \le C'_{\#}\lambda^{-\varepsilon N/11} \text{ for an integer } k \ge 0 \tag{29}$$

and also $\chi\tilde{F} = a\tilde{f}_k + \tilde{g}_k$ with $\tilde{f}_k$ the k^{th} eigenfunction of $-\frac{d^2}{dy^2} + \lambda^2 y^2$, a a constant, and $\|\tilde{g}_k\|_{L^2(\mathbb{R})} \le C'_{\#}\lambda^{-\varepsilon N/11}\|\chi\tilde{F}\|_{L^2(\mathbb{R})}$. In $J = \{|x - x_0| < \lambda^{-\varepsilon}B\}$ we have

$$F(x) = \Big(\frac{\partial y}{\partial x}\Big)^{-1/2}\chi(y)\tilde{F}(y) = \Big(\frac{\partial y}{\partial x}\Big)^{-1/2}\cdot a\tilde{f}_k(y) + \Big(\frac{\partial y}{\partial x}\Big)^{-1/2}\tilde{g}_k(y),$$

and therefore

(30)

$$\Big\|F(x) - a\Big(\frac{\partial y}{\partial x}\Big)^{-1/2}\tilde{f}_k(y(x,E))\Big\|_{L^2(J)} \le \max_{x\in J}\Big|\Big(\frac{\partial y}{\partial x}\Big)^{-1}\Big| \|\Big(\frac{\partial y}{\partial x}\Big)^{1/2}\tilde{g}_k(y(x,E))\|_{L^2(J)}$$
$$\le \max_{x\in J}\Big|\Big(\frac{\partial y}{\partial x}\Big)^{-1}\Big|\|\tilde{g}_k\|_{L^2(\mathbb{R})} \le \max_{x\in J}\Big|\Big(\frac{\partial y}{\partial x}\Big)^{-1}\Big|\cdot C'_{\#}\lambda^{-\varepsilon N/11}\|\tilde{F}\|_{L^2(|y|\le C_{\#}\lambda^{-\varepsilon})}.$$

On the other hand,

$$\|F\|_{L^2(I_{\mathrm{BVP}})} \ge \|\Big(\frac{\partial y}{\partial x}\Big)^{-1/2}\tilde{F}(y(x,E))\|_{L^2(|x-x_0|<C''_{\#}\lambda^{-\varepsilon}B)}$$
$$\ge \min_{|x-x_0|<C''_{\#}\lambda^{-\varepsilon}B}\Big|\Big(\frac{\partial y}{\partial x}\Big)^{-1}\Big|\cdot\Big\|\Big(\frac{\partial y}{\partial x}\Big)^{1/2}\tilde{F}(y(x,E))\Big\|_{L^2(|x-x_0|<C''_{\#}\lambda^{-\varepsilon}B)}$$
$$\ge \min_{|x-x_0|<C''_{\#}\lambda^{-\varepsilon}B}\Big|\Big(\frac{\partial y}{\partial x}\Big)^{-1}\Big|\cdot\|\tilde{F}\|_{L^2(|y|<C_{\#}\lambda^{-\varepsilon})}$$

if we take $C''_\#$ large enough so that the image of $\{|x - x_0| < C''_\# \lambda^{-\varepsilon} B\}$ under $x \mapsto y(x,E)$ covers $\{|y| < C_\# \lambda^{-\varepsilon}\}$. Comparing this to (30), and noting that $(\frac{\partial y}{\partial x})$ has a constant order of magnitude by (12), (14), we conclude that

$$\|F(x) - a \cdot \Big(\frac{\partial y}{\partial x}\Big)^{-1/2} \tilde{f}_k(y(x,E))\|_{L^2(J)} \le C'_\# \, \lambda^{-\varepsilon N/11} \, \|F\|_{L^2(I_{\mathrm{BVP}})}.$$

Lemma 1 shows that $\|F\|_{L^2(I_{\mathrm{BVP}})} \le 2\|F\|_{L^2(J)}$, and therefore

$$\|F(x) - a \cdot \Big(\frac{\partial y}{\partial x}\Big)^{-1/2} \tilde{f}_k(y(x,E))\|_{L^2(J)} \le C'_\# \lambda^{-\varepsilon N/11} \|F\|_{L^2(J)}. \tag{31}$$

Thus in $L^2(J)$, F is nearly proportional to $f_E(x) = (\frac{\partial y}{\partial x})^{-1/2} \tilde{f}_k(y(x,E))$. Next we investigate $\frac{\partial}{\partial E} f_E$ for E satisfying (29) and $|E - V(x_0)| < c_\# \lambda^{-2\varepsilon} \cdot S$. We have:

$$\begin{aligned}
\Big\|\frac{\partial}{\partial E} f_E\Big\|_{L^2(J)} &\le \Big\|\Big(\frac{\partial y}{\partial x}\Big)^{-3/2}\Big(\frac{\partial^2 y}{\partial x \partial E}\Big) \tilde{f}_k(y(x,E))\Big\|_{L^2(J)} \\
&\quad + \Big\|\Big(\frac{\partial y}{\partial x}\Big)^{-1/2}\Big(\frac{\partial y}{\partial E}\Big) \tilde{f}'_k(y(x,E))\Big\|_{L^2(J)} \\
&\le \max_{x \in J}\Big|\Big(\frac{\partial y}{\partial x}\Big)^{-1}\Big(\frac{\partial^2 y}{\partial x \partial E}\Big)\Big| \cdot \Big\|\Big(\frac{\partial y}{\partial x}\Big)^{-1/2} \tilde{f}_k(y(x,E))\Big\|_{L^2(J)} \\
&\quad + \max_{x \in J}\Big|\Big(\frac{\partial y}{\partial x}\Big)^{-1}\Big(\frac{\partial y}{\partial E}\Big)\Big| \cdot \Big\|\Big(\frac{\partial y}{\partial x}\Big)^{1/2} \tilde{f}'_k(y(x,E))\Big\|_{L^2(J)} \\
&\le \max_{x \in J}\Big|\Big(\frac{\partial y}{\partial x}\Big)^{-1}\Big(\frac{\partial^2 y}{\partial x \partial E}\Big)\Big| \cdot \Big\|f_E\Big\|_{L^2(J)} + \max_{x \in J}\Big|\Big(\frac{\partial y}{\partial x}\Big)^{-1}\Big(\frac{\partial y}{\partial E}\Big)\Big| \cdot \|\tilde{f}'_k\|_{L^2(\mathbb{R})}
\end{aligned} \tag{32}$$

(by definition of f_E and a change of variable from x to y)

$$\le C'_\# S^{-1} \|f_E\|_{L^2(J)} + C'_\# B S^{-1} \|\tilde{f}'_k\|_{L^2(\mathbb{R})}.$$

We have

$$\begin{aligned}
\|\tilde{f}'_k\|^2_{L^2(\mathbb{R})} \le &< (-\frac{d^2}{dy^2} + \lambda^2 y^2)\tilde{f}_k, \tilde{f}_k >= (2k+1)\lambda \, \|\tilde{f}_k\|^2_{L^2(\mathbb{R})} \\
&\le C_\# \lambda^2 \, \|\tilde{f}_k\|^2_{L^2(\mathbb{R})}
\end{aligned}$$

by (29) and (13). Thus, (32) implies

$$\Big\|\frac{\partial}{\partial E} f_E\Big\|_{L^2(J)} \le C'_\# S^{-1} \|f_E\|_{L^2(J)} + C'_\# B S^{-1} \lambda \|\tilde{f}_k\|_{L^2(\mathbb{R})}. \tag{33}$$

On the other hand,

$$
\begin{aligned}
\|f_E\|_{L^2(J)} &\geq \min_{x\in J}\left|\left(\frac{\partial y}{\partial x}\right)^{-1}\right| \cdot \left\|\left(\frac{\partial y}{\partial x}\right)^{1/2}\tilde{f}_k(y(x,E))\right\|_{L^2(J)} \\
&\geq c'_{\#}B\|\tilde{f}_k\|_{L^2(|y|<c'_{\#}\lambda^{-\varepsilon})} \quad \text{by (12).}
\end{aligned}
\tag{34}
$$

Since we are assuming $|E-V(x_0)| < c_{\#}\lambda^{-2\varepsilon}S$ for a small enough $c_{\#}$, Lemma 3 shows that $|\tilde{E}| \leq c''_{\#}\lambda^{2-2\varepsilon}$ with small $c''_{\#}$. We can make $c''_{\#}$ as small as we please by taking $c_{\#}$ small. By (29) we have $|(2k+1)\lambda| \leq c''_{\#}\lambda^{2-2\varepsilon}$. If we take $c''_{\#}$ small enough depending on $c'_{\#}$, then $|(2k+1)\lambda| \leq c''_{\#}\lambda^{2-2\varepsilon}$ implies that $\tilde{f}_k$ is heavily concentrated in $\{|y| < c'_{\#}\lambda^{-\varepsilon}\}$. Hence (34) implies $\|f_E\|_{L^2(J)} \geq c'_{\#}B\|\tilde{f}_k\|_{L^2(\mathbb{R})}$. Putting this into (33), we get

$$
\left\|\frac{\partial}{\partial E}f_E\right\|_{L^2(J)} \leq C'_{\#}\lambda S^{-1}\|f_E\|_{L^2(J)}. \tag{35}
$$

We proved (35) for any E satisfying

$$
|E-V(x_0)| < c_{\#}\lambda^{-2\varepsilon}S \text{ and } |\tilde{E}(E)-(2k+1)\lambda| \leq C_{\#}\lambda^{-\varepsilon N/11}, \text{and } E \leq E_\infty. \tag{36}
$$

In particular, if E, E' both satisfy (36), then

$$
\|f_E - f_{E'}\|_{L^2(J)} \leq C'_{\#}\lambda S^{-1}|E-E'| \ \ \|f_E\|_{L^2(J)}, \tag{37}
$$

provided

$$
\lambda S^{-1}|E-E'| < c_{\#}. \tag{38}
$$

Since $\frac{d\tilde{E}}{dE} \sim \lambda^2 S^{-1} = B^2$ by (13), (15), condition (36) for E and E' implies $c_{\#}B^2|E-E'| \leq C'_{\#}\lambda^{-\varepsilon N/11}$. This implies (38), and therefore (37) holds. Moreover, from (37) we get

$$
\begin{aligned}
\|f_E - f_{E'}\|_{L^2(J)} &\leq C'_{\#}\lambda S^{-1}\cdot B^{-2}\lambda^{-\varepsilon N/11}\,\|f_E\|_{L^2(J)} \\
&= C_{\#}\lambda^{-1-\varepsilon N/11}\,\|f_E\|_{L^2(J)}.
\end{aligned}
$$

Thus f_E and $f_{E'}$ are nearly proportional. On the other hand, we have seen that the eigenfunctions F, F' corresponding to E, E' are nearly proportional to f_E, $f_{E'}$ respectively, in $L^2(J)$. Therefore F and F' are nearly proportional in $L^2(J)$. Lemma 1 shows that F and F' are almost entirely concentrated in J, hence F and F' are nearly proportional in $L^2(I_{\text{BVP}})$. If $E \neq E'$, then F and F' would have to be orthogonal, and so couldn't be nearly proportional. Hence $E' = E$, proving the uniqueness assertion in Lemma 4. This and estimate (29) complete the proof. ■

Lemma 5. *Let k be an integer satisfying $0 \leq k \leq c_\# \lambda^{1-2\varepsilon}$. Suppose either $E_\infty > V(x_0)+\lambda^{-2\varepsilon}S$, or else $E_\infty \leq V(x_0)+\lambda^{-2\varepsilon}S$ and $(2k+1)\lambda + C_\# \lambda^{-\varepsilon N/11} \leq \tilde{E}(E_\infty)$. Then there is an eigenvalue E' of H satisfying $|E' - V(x_0)| \leq c_\# \lambda^{-2\varepsilon} S$, and $|\tilde{E}(E') - (2k+1)\lambda| < C_\# \lambda^{-\varepsilon N/11}$.*

Proof. We start by finding an E with $|E - V(x_0)| < c_\# \lambda^{-2\varepsilon} S$, $\tilde{E}(E) = (2k+1)\lambda$, and $E + C_\# \lambda^{-\varepsilon N/10} S \leq E_\infty$. In fact, we know that $|\tilde{E}| \leq C_\#$ when $E = V(x_0)$ by the bound for $\hat{\tau}_0$ in Lemma 3. We know also that $\frac{d\tilde{E}}{dE} \geq c_\# \lambda^2 S^{-1} = c_\# B^2$ for $|E - V(x_0)| < c_\# \lambda^{-2\varepsilon} S$.

Therefore, for $0 \leq k < c_\# \lambda^{1-2\varepsilon}$, we can find E satisfying

$$|E - V(x_0)| < c_\# \lambda^{-2\varepsilon} S \quad \text{and} \quad \tilde{E}(E) = (2k+1)\lambda. \tag{39}$$

It remains to check that $E + C_\# \lambda^{-\varepsilon N/10} S < E_\infty$. This is clear from (39) in case $E_\infty \geq V(x_0) + \lambda^{-2\varepsilon} S$. In the alternate case $E_\infty \leq V(x_0) + \lambda^{-2\varepsilon} S$, $(2k+1)\lambda + C_\# \lambda^{-\varepsilon N/11} \leq \tilde{E}(E_\infty)$, we use (39) to get $\tilde{E}(E) + C_\# \lambda^{-\varepsilon N/11} \leq \tilde{E}(E_\infty)$. Then from (13), (15) we conclude that $E_\infty - E \geq c_\# \lambda^{-\varepsilon N/11} (\frac{\lambda^2}{S})^{-1} > C_\# \lambda^{-\varepsilon N/10} S$. Hence in all cases we have $E + C_\# \lambda^{-\varepsilon N/10} S < E_\infty$, as claimed.

Lemma 3 shows that the equation

$$\left[\frac{d^2}{dy^2} + (\tilde{E}(E) - \lambda^2 y^2)\right] \tilde{F}(y) = 0 \tag{40}$$

is transformed into

$$\left[\frac{d^2}{dx^2} + (E - V(x)) + \text{Error}\right] F(x) = 0 \text{ with } |\text{Error}| < C_\# \lambda^{-\varepsilon N/10} S \quad \text{for } |x - x_0| < C_\# \lambda^{-\varepsilon} B, \tag{41}$$

by setting

$$F(x) = \left(\frac{\partial y}{\partial x}\right)^{-1/2} \tilde{F}(y(x,E)) \quad \text{for } |x - x_0| < C_\# \lambda^{-\varepsilon} S. \tag{42}$$

Since $\tilde{E}(E) = (2k+1)\lambda$, equation (40) is satisfied on the whole real line for a suitable Hermite function $\tilde{F}$. Hence, defining F by (42), we get

(43)
$$\left\| \left[\frac{d^2}{dx^2} + (E - V(x))\right] F \right\|_{L^2(|x-x_0|<C_\# \lambda^{-\varepsilon} B)} \leq C_\# \lambda^{-\varepsilon N/10} S \|F\|_{L^2(|x-x_0| \leq C_\# \lambda^{-\varepsilon} B)}$$

by virtue of (41). Moreover, if $0 \leq k \leq c_\# \lambda^{1-2\,\varepsilon}$, then the Hermite function $\tilde{F}$ is concentrated almost entirely in $\{|y| < c'_\# \lambda^{-\varepsilon}\}$. We can make $c'_\#$ small by taking $c_\#$ small.

In particular, we have

$$\|\tilde{F}\|_{L^2(|y|>c'_\# \lambda^{-\varepsilon})} \leq C'_\# \lambda^{-N} \|\tilde{F}\|_{L^2(|y|<c'_\# \lambda^{-\varepsilon})} \quad \text{and}$$
$$\left\| \frac{d}{dy} \tilde{F}(y) \right\|_{L^2(|y|>c'_\# \lambda^{-\varepsilon})} \leq C'_\# \lambda^{-N} \|\tilde{F}\|_{L^2(|y|<c'_\# \lambda^{-\varepsilon})}.$$

From these estimates, (12), and the fact that $c_\# B^{-1} < \frac{\partial y}{\partial x} < C_\# B^{-1}$ for $|x - x_0| < c_\# B$, we see that

$$\|F\|_{L^2(\lambda^{-\varepsilon} B < |x-x_0| < C_\# \lambda^{-\varepsilon} B)} \leq C_\# \lambda^{-N} \|F\|_{L^2(|x-x_0|<\lambda^{-\varepsilon} B)} \tag{44}$$

and

$$\left\| \frac{d}{dx} F \right\|_{L^2(\lambda^{-\varepsilon} B < |x-x_0| < C_\# \lambda^{-\varepsilon} B)} \leq C_\# \lambda^{-N} B^{-1} \|F\|_{L^2(|x-x_0|<\lambda^{-\varepsilon} B)}. \tag{45}$$

Introduce a cutoff function $\chi(x)$ satisfying $\left|\left(\frac{d}{dx}\right)^\alpha \chi\right| \leq C^\alpha_\# (\lambda^{-\varepsilon} B)^{-\alpha}$, $\chi(x) = 1$ for $|x - x_0| \leq \lambda^{-\varepsilon} B$, $\chi(x) = 0$ for $|x - x_0| > 2\lambda^{-\varepsilon} B$. From (43), (44), (45) we get

$$\left\| \left[\frac{d^2}{dx^2} + (E - V(x))\right] (\chi F) \right\|_{L^2(I_{\text{BVP}})} \leq C_\# \lambda^{-\frac{\varepsilon N}{10}} S \|\chi F\|_{L^2(I_{\text{BVP}})}. \tag{46}$$

A glance at the definition (42) with $\tilde{F}$ a Hermite function shows that χF doesn't vanish identically. Hence, (46) implies that the interval $\mathcal{J} = [E - C_{\#}\lambda^{-\varepsilon N/10}S, E + C_{\#}\lambda^{-\varepsilon N/10}S]$ contains some point E' in the spectrum of H. (Note that χF satisfies the boundary conditions for H since supp $\chi \subset\subset I_{\mathrm{BVP}}$.) Since the continuous spectrum of H is contained in $[E_\infty, \infty)$, and since $E_\infty > E + C_{\#}\lambda^{-\frac{\varepsilon N}{10}}S$, it follows that $\mathcal{J}$ contains none of the continuous spectrum of H. Thus E' is an eigenvalue of H. Since $|E - V(x_0)| < c_{\#}\lambda^{-2\varepsilon}S$ and $|E' - E| < C_{\#}\lambda^{-\frac{\varepsilon N}{10}}S$, we get $|E' - V(x_0)| < c_{\#}\lambda^{-2\varepsilon}S$. We have $|E' - E| \leq C_{\#}\lambda^{-\frac{\varepsilon N}{10}}S$ and therefore $|\tilde{E}(E') - (2k+1)\lambda| = |\tilde{E}(E') - \tilde{E}(E)| \leq C_{\#}\lambda^2 S^{-1}|E' - E| \leq C_{\#}\lambda^{2-\varepsilon N/10} \leq C_{\#}\lambda^{-\varepsilon N/11}$. The proof of the lemma is complete. ■

Corollary. *There is a finite sequence $E_0, E_1, \dots, E_{k_{\max}}$ with the following properties.*

(a) *The integers $k = 0, 1, \dots, k_{\max}$ are precisely those for which we can find some E to satisfy*

$$|E - V(x_0)| < c_{\#}\lambda^{-2\varepsilon}S, \;\; E \leq E_\infty, \;\; |\tilde{E}(E) - (2k+1)\lambda| \leq C_{\#}\lambda^{-\frac{\varepsilon N}{11}}.$$

(b) *If $0 \leq k < k_{\max}$ then E_k is an eigenvalue of H.*

(c) *Either $E_{k_{\max}}$ is an eigenvalue of H or else $E_{k_{\max}} = E_\infty$.*

(d) *Every eigenvalue E of H satisfying $|E - V(x_0)| < c_{\#}\lambda^{-2\varepsilon}S$ and $E \leq E_\infty$ is one of the E_k $(0 \leq k \leq k_{\max})$.*

(e) *For $0 \leq k \leq k_{\max}$ we have $|E_k - V(x_0)| \leq 2c_{\#}\lambda^{-2\varepsilon}S$ and $|\tilde{E}(E_k) - (2k+1)\lambda| \leq C_{\#}\lambda^{-\varepsilon N/11}$.*

Sketch of Proof. Define $k_{\max}$ by (a), recalling that $E \mapsto \tilde{E}$ is monotone with $\frac{d\tilde{E}}{dE} \sim \lambda^2 S^{-1}$. If $0 \leq k \leq k_{\max}$ then either the hypotheses of Lemma 5 are satisfied, or else $k = k_{\max}$ and $|E_\infty - V(x_0)| < \lambda^{-2\varepsilon}S$, $|\tilde{E}(E_\infty) - (2k+1)\lambda| \leq C_{\#}\lambda^{-\varepsilon N/11}$. There can be at most one eigenvalue E with $|E - V(x_0)| < \lambda^{-2\varepsilon}S$, $|\tilde{E}(E) - (2k+1)\lambda| \leq$

$C_\# \lambda^{-\varepsilon N/11}$, $E \leq E_\infty$ by virtue of Lemma 4. Also from Lemma 5 there exists such an eigenvalue, except perhaps in the case: $k = k_{\max}$, $|E_\infty - V(x_0)| < \lambda^{-2\varepsilon} S$, $|\tilde{E}(E_\infty) - (2k+1)\lambda| \leq C_\# \lambda^{-\varepsilon N/11}$. Hence we can define $E_k =$ the unique eigenvalue E as above if one exists, $E_k = E_\infty$ if no suitable eigenvalue exists. Properties (a), (b), (c), (e) are thus satisfied, and (d) follows from Lemma 4. We have seen this argument before in establishing the WKB Eigenvalue Theorem. ■

It remains to give an explicit formula for $\tilde{E}(E)$ modulo a small error. We shall see that $\tilde{E}(E) = \frac{2}{\pi}\lambda(\phi(E) + \frac{1}{48}\psi(E)) + (\text{small error})$, where the phases ϕ and ψ are defined as in the WKB Eigenvalue Theorem. Thus the above Corollary extends the range of E under which the WKB Eigenvalue Theorem locates the eigenvalues of H with good precision. Details are as follows.

For $V(x_0) < E < V(x_0) + c_\# S$, the set $\{x \mid |x - x_0| < c'_\# B, V(x) < E\}$ is an interval $(x_{\text{left}}(E), x_{\text{rt}}(E))$ with $V(x_{\text{left}}(E)) = V(x_{\text{rt}}(E)) = E$. We define

$$\phi(E) = \int_{x_{\text{left}}(E)}^{x_{\text{rt}}(E)} (E - V(x))^{1/2}\, dx \quad \text{and} \tag{47}$$

(48)

$$\psi(E) = \lim_{\delta_{\text{left}}, \delta_{\text{rt}} \to 0+} \Big[\int_{x_{\text{left}}(E)+\delta_{\text{left}}}^{x_{\text{rt}}(E)-\delta_{\text{rt}}} V''(x)(E - V(x))^{-3/2}\, dx - q_{\text{left}}(E)\delta_{\text{left}}^{-1/2} - q_{\text{rt}}(E)\delta_{\text{rt}}^{-1/2} \Big]$$

with $q_{\text{left}}(E)$, $q_{\text{rt}}(E)$ uniquely specified by demanding the finitness of the limit. Equivalently,

$$\psi(E) = \lim_{\delta \to 0+} \Big[\int_{\substack{|x - x_0| < c_\# B \\ E - V(x) > \delta}} V''(x)(E - V(x))^{-3/2}\, dx - q(E)\delta^{-1/2} \Big]$$

with $q(E)$ uniquely specified by demanding the finiteness of the limit. Our goal is to compare $\frac{\tilde{E}(E)}{2\lambda}$ with $\phi(E) + \frac{1}{48}\psi(E)$. We begin by studying the smoothness of ϕ

and ψ.

Lemma 6. *The phases $\phi(E)$ and $\psi(E)$ satisfy*

$$\left|\left(\frac{d}{dE}\right)^\beta \phi(E)\right| \le C^\beta_\# \lambda S^{-\beta}, \;\left|\left(\frac{d}{dE}\right)^\beta \psi(E)\right| \le C^\beta_\# \lambda^{-1} S^{-\beta} \quad (\beta \ge 0)$$

for $V(x_0) < E < V(x_0) + c_\# S$.

Proof. Rescaling reduces matters to the following problem. Suppose $|(\frac{d}{dx})^\alpha W(x)| \le C_\alpha$, $W(0) = W'(0) = 0$, $W''(0) > c_1 > 0$. Define $\phi_W(E) = \int_{E-W(x)>0}(E - W(x))^{1/2}\,dx$, $\psi_W(E) = \lim_{\delta\to 0+}\big[\int_{E-W(x)>\delta} W''(x)(E-W(x))^{-3/2}\,dx - q(E)\delta^{-1/2}\big]$ for small, positive E.

Then for $0 < E < c_2$ we have $|(\frac{d}{dE})^\beta \phi_W(E)|, |(\frac{d}{dE})^\beta \psi_W(E)| \le C'_\beta$, with c_2, C'_β depending only on c_1 and the C_α. This implies Lemma 6 by virtue of the scaling relations $\phi(E) = \lambda\phi_W(\frac{E-V(x_0)}{S})$, $\psi(E) = \lambda^{-1}\psi_W(\frac{E-V(x_0)}{S})$ for $W(x) = \frac{V(x_0+Bx)-V(x_0)}{S}$. The rest of our proof studies $\phi_W(E)$, $\psi_W(E)$. We can write $W(x) = (y(x))^2$ for a smooth function $y(x)$ satisfying $y(0) = 0$, $y'(0) > \hat{c}_1 > 0$, $|(\frac{d}{dx})^\alpha y(x)| \le \hat{C}_\alpha$ for $|x| < \hat{c}$. Here, $\hat{c}_1$, $\hat{C}_\alpha$ depend only on c_1 and the C_α. Changing variable from x to $y = y(x)$ in the definitions of ϕ_W, ψ_W gives

$$\phi_W(E) = \int_{E-y^2>0} (E-y^2)^{1/2}\mathcal{T}_1(y)\,dy \tag{49}$$

$$\psi_W(E) = \lim\Big[\int_{E-y^2>\delta} (E-y^2)^{-3/2}\mathcal{T}_2(y)\,dy - q(E)\delta^{-1/2}\Big] \tag{50}$$

with $\mathcal{T}_i(y)$ defined by $dx = \mathcal{T}_1(y)\,dy$, $V''\,dx = \mathcal{T}_2(y)\,dy$. The C^∞–seminorms of $\mathcal{T}_i$ are bounded a–priori in terms of c_1, C_α. In (49), (50), we may replace $\mathcal{T}_i(y)$ by $\frac{1}{2}[\mathcal{T}_i(y)+\mathcal{T}_i(-y)]$ without affecting the answer. Hence we may suppose $\mathcal{T}_i(y)$ is even, and therefore $\mathcal{T}_i(y) = F_i(y^2)$ for smooth functions F_i. The C^∞–seminorms of the F_i are bounded a–priori in terms of c_1, C_α. Changing variable from y to $t = y^2/E$, we get

$$\phi_W(E) = c\int_0^1 E^{1/2}\cdot(1-t)^{1/2}F_1(Et)\cdot E^{1/2}\,\frac{dt}{t^{1/2}} \quad \text{and} \tag{51}$$

$$(52)\quad \psi_W(E) = c \lim_{\delta\to 0+}\Big[\int_0^{1-\delta} E^{-3/2}\cdot(1-t)^{-3/2}F_2(Et)\cdot E^{1/2}\frac{dt}{t^{1/2}} - \tilde{q}(E)\delta^{-1/2}\Big].$$

The desired a–priori estimates for $\phi_W(E)$ are immediate from (51), while $\psi_W(E)$ requires a little more work.

Using the elementary fact that $\int_0^{1-\delta}(1-t)^{-3/2}\frac{dt}{t^{1/2}} = \frac{2t^{1/2}}{(1-t)^{1/2}}\Big]_0^{1-\delta} = \frac{2(1-\delta)^{1/2}}{\delta^{1/2}} = 2\delta^{-1/2}+o(1)$, so that $\lim_{\delta\to 0+}\Big[\int_0^{1-\delta}(1-t)^{-3/2}\frac{dt}{t^{1/2}} - 2\delta^{-1/2}\Big] = 0$, we can rewrite (52) in the form

$$(53)\qquad \psi_W(E) = c\lim_{\delta\to 0}\Big[\int_0^{1-\delta}(1-t)^{-3/2}\frac{F_2(Et)-F_2(E)}{E}\frac{dt}{t^{1/2}} - \hat{q}(E)\delta^{-1/2}\Big]$$

with $\hat{q}(E)$ uniquely specified by the finiteness of the limit. In (53) the integrand has only a weak singularity at $t=1$, so the limit is finite with $\hat{q}(E)=0$. Hence (53) is equivalent to

$$\psi_W(E) = c'\int_0^1 (1-t)^{-1/2}\frac{F_2(Et)-F_2(E)}{Et-E}\frac{dt}{t^{1/2}},$$

which we rewrite as

$$(54)\qquad \psi_W(E) = c'\int_0^1\int_0^1 (1-t)^{-1/2}F_2'(sEt+(1-s)E)\,ds\frac{dt}{t^{1/2}},$$

The derived a–priori estimates for $\psi_W(E)$ are immediate from (54), completing the proof of the Lemma. ■

In comparing $\frac{\pi}{2}\frac{\tilde{E}(E)}{\lambda}$ with $\phi(E)+\frac{1}{48}\psi(E)$, we make an extra assumption on $V(x)$, which we remove later. The extra assumption is as follows.

$$\text{(H7*)}\qquad E_\infty \geq V(x_0)+c_\# S.$$

Lemma 7. *Under hypotheses (H1*)... (H7*), we have*

$$(55)\quad \Big|\frac{\pi}{2\lambda}\tilde{E}(E)-(\phi(E)+\frac{1}{48}\psi(E))\Big| \leq C_\#\lambda^{-2+4\varepsilon}\quad \textit{for } V(x_0)<E<V(x_0)+\lambda^{-2\varepsilon}S.$$

Proof. Let $E_1 < \ldots < E_{j_{\max}}$ be the eigenvalues of H in the interval

$$\mathcal{T} = \{|E - (V(x_0) + \hat{c}_\# \lambda^{-2\varepsilon} S)| < \hat{\hat{c}}_\# \lambda^{-2\varepsilon} S\} \quad \text{with } \hat{\hat{c}}_\# << \hat{c}_\# << 1.$$

We will compute the E_j by using the results of this section, and independently by applying the WKB Eigenvalue Theorem. Our proof is based on comparing the results of these two computations of the E_j.

First of all, Lemma 4 shows that

$$|\frac{\pi}{2\lambda}\tilde{E}(E_j) - \pi(k_j + 1/2)| \leq C_\# \lambda^{-\frac{\varepsilon N}{11} - 1} \quad \text{for integers } k_j. \tag{56}$$

Lemma 4 shows that the k_j are all distinct. Since $\tilde{E}(E)$ is increasing, we must have $k_{j+1} \geq k_j$. Thus $k_{j+1} \geq k_j + 1$ since the k_j are all distinct. We could not have $k_{j+1} \geq k_j + 2$. Otherwise, Lemma 5 would give us an eigenvalue E' with $|\frac{\pi}{2\lambda}\tilde{E}(E') - \pi(k_j + 1)| \leq C_\# \lambda^{-\frac{\varepsilon N}{11} - 1}$, and monotonicity of $\tilde{E}(\cdot)$ would imply $E_j < E' < E_{j+1}$, contradicting the fact that E_{j+1} is the next eigenvalue after E_j.

Thus $k_{j+1} = k_j + 1$, so that $k_j = j + m$, with m an integer independent of j. Equation (56) becomes

$$\left|\frac{\pi}{2\lambda}\tilde{E}(E_j) - \pi(j + m + \frac{1}{2})\right| \leq C_\# \lambda^{-\frac{\varepsilon N}{11} - 1} \quad (1 \leq j \leq j_{\max}). \tag{57}$$

On the other hand, we can analyze $E_1 \ldots E_{j_{\max}}$ using the WKB Eigenvalue Theorem. In fact, set $\hat{I}_{\text{BVP}} = I_{\text{BVP}}$, $\hat{I} = \{|x - x_0| < \lambda^{-\varepsilon} B\}$, $\hat{V}(x) = V(x) - V(x_0)$, $\hat{S}(x) = \lambda^{-2\varepsilon} S$ and $\hat{B}(x) = \lambda^{-\varepsilon} B$ for $x \in \hat{I}$, $\hat{E}_\infty = E_\infty - V(x_0)$, $\hat{E}_0 = \hat{c}_\# \lambda^{-2\varepsilon} S$. These satisfy the hypotheses (Hyp0)... (Hyp10) of the WKB Eigenvalue Theorem, and the constants c_1, C_α etc. in (Hyp0)...(Hyp5) depend only on the constants c_1, C_α, in our present hypotheses (H1*)...(H7*). In particular, we find that $S_{\min} = \lambda^{-2\varepsilon} S$, $\lambda(x) = \lambda^{1-2\varepsilon}$ for $x \in \hat{I}$, and $\Lambda = \lambda^{1-2\varepsilon}$. The WKB Eigenvalue Theorem therefore implies that the eigenvalues of H which lie in $\mathcal{T}$ may be written as $E^+_{p_{\min}} < \ldots < E^+_{p_{\max}}$ with

$$\left|\phi(E^+_k) + \frac{1}{48}\psi(E^+_k) - \pi(k + \frac{1}{2})\right| \leq C_\# \Lambda^{-2} = C_\# \lambda^{-2+4\varepsilon} (p_{\min} \leq k \leq p_{\max}). \tag{58}$$

The eigenvalues in $\mathcal{T}$ are given as $E_1 < \ldots < E_{j_{\max}}$ and as $E^+_{p_{\min}} < \ldots < E^+_{p_{\max}}$. Thus $E^+_k = E_{j-m'}$ with $m' = p_{\min} - 1$, so (58) becomes

$$\Big|\phi(E_j) + \frac{1}{48}\psi(E_j) - \pi(j + m' + \frac{1}{2})\Big| \leq C_\# \lambda^{-2+4\varepsilon} \quad (1 \leq j \leq j_{\max}). \tag{59}$$

Now set $f(E) = \frac{\pi}{2\lambda}\tilde{E}(E) - (\phi(E) + \frac{1}{48}\psi(E)) + \pi(m' - m)$. Estimates (57) and (59) imply

$$|f(E_j)| \leq C_\# \lambda^{-2+4\varepsilon} \quad (1 \leq j \leq j_{\max}). \tag{60}$$

Lemma 6 and estimate (13) show that

(61)

$$\Big|\Big(\frac{d}{dE}\Big)^\beta \Big\{\frac{\pi}{2\lambda}\tilde{E}(E) - (\phi(E) + \frac{1}{48}\psi(E))\Big\}\Big| \leq C^\beta_\# \lambda S^{-\beta} \text{ for } V(x_0) < E < V(x_0) + c_\# S.$$

Take $\beta = 0$, $E = E_j$, and compare this estimate with (60), to conclude that $|m - m'| \leq C_\# \lambda$. This and (61) show that

$$\Big|\Big(\frac{d}{dE}\Big)^\beta f(E)\Big| \leq C^\beta_\# \lambda S^{-\beta} \quad \text{for } V(x_0) < E < V(x_0) + c_\# S. \tag{62}$$

By (62) and Taylor's theorem with remainder, we can write

$$f(E) = P(E) + \text{error}(E), \tag{63}$$

with $P(E)$ a polynomial of degree N, and

(64)

$$|\text{error}(E)| \leq C_\# \lambda \Big(\frac{E - V(x_0)}{S}\Big)^N \leq C_\# \lambda^{-2+4\varepsilon} \text{ for } V(x_0) < E < V(x_0) + \lambda^{-2\varepsilon} S.$$

From (60), (63), (64) and $E_j \in \mathcal{T}$, we get

$$|P(E_j)| \leq C_\# \lambda^{-2+4\varepsilon}. \tag{65}$$

Any E in $\mathcal{T}$ lies within $C_\# \frac{S}{\lambda}$ of an E_j, by (57) and the fact that $\frac{d}{dE}(\frac{\pi}{2\lambda}\tilde{E}(E)) \sim \lambda S^{-1}$ in $\mathcal{T}$ by (13). Hence we can find $E_{j_0}, E_{j_1}, \ldots, E_{j_N} \in \mathcal{T}$ with $|E_{j_{\nu+1}} - E_{j_\nu}| \sim \frac{\text{diam } \mathcal{T}}{N}$. Using (65) for the E_{j_ν} and applying the Lagrange interpolation formula, we obtain

$$|P(E)| \leq C_\# \lambda^{-2+4\varepsilon} \text{ for } |E - V(x_0)| < \lambda^{-2\varepsilon} S. \tag{66}$$

Estimates (63), (64), (66) together show that

$$|f(E)| \leq C_{\#}\lambda^{-2+4\varepsilon} \text{ for } V(x_0) < E < V(x_0) + \lambda^{-2\varepsilon}S. \tag{67}$$

Let us specialize (67) to $E \to V(x_0)+$. Lemma 3 shows that $|\tilde{E}(V(x_0))| = |\lambda^2\hat{\tau}_0| \leq C_{\#}$, and Lemma 6 gives $|\psi(E)| \leq C_{\#}\lambda^{-1}$. Immediately from the definition we get $\phi(E) \to 0$ as $E \to V(x_0)+$. Hence for E very slightly greater than $V(x_0)$, the definition of $f(E)$ gives $|f(E) + \pi(m - m')| \leq C_{\#}\lambda^{-1}$. Comparing this with (67) and recalling that m, m' are integers, we conclude that $m - m' = 0$. Therefore, (67) says that

$$\left|\frac{\pi}{2\lambda}\tilde{E}(E) - (\phi(E) + \frac{1}{48}\psi(E))\right| \leq C_{\#}\lambda^{-2+4\varepsilon} \text{ for } V(x_0) < E < V(x_0) + \lambda^{-2\varepsilon}S,$$

which is the conclusion of the Lemma. ■

Corollary. *The conclusion of Lemma 7 holds also without the extra assumption (H7*).*

Proof. Suppose we modify the potential $V(x)$ in $\{|x-x_0| > c_{\#}B\}$, without changing it in $\{|x - x_0| \leq c_{\#}B\}$. The remark following the proof of Lemma 3 shows that $\tilde{E}(E)$ will not change. Also $\phi(E)$, $\psi(E)$ will not change, provided $V(x_0) < E < V(x_0)+c'_{\#}S$. Therefore the conclusion of Lemma 7 is unaffected by changing $V(x)$ in $\{|x-x_0| > c_{\#}B\}$. We can make the change in V so that $V(x) \geq V(x_0)+\frac{c_{\#}S}{B^2}(x-x_0)^2$ globally, and we then change E_∞ to the value $V(x_0) + c'_{\#}S$ for a small $c'_{\#} > 0$. Hypotheses (H1*)...(H7*) hold for the modified $V(\cdot)$, E_∞, and therefore the conclusion of Lemma 7 holds for the original V. ■

We summarize the results of this section in the next section.

The WKB Theorem on Low Eigenvalues

Let $\varepsilon, K, N > 0$ be given, with $\varepsilon N \geq 100$. Let $V(x)$ be a potential defined on a (possibly unbounded) interval I_{BVP}. Let S, B be positive numbers, and let $x_0 \in I_{\mathrm{BVP}}$ be given. Define $\lambda = S^{1/2}B$. Let E_∞ be a given energy, with $E_\infty > V(x_0)$. We make the following assumptions.

(H0*) $I = \{|x - x_0| < cB\} \subset I_{\mathrm{BVP}}$

(H1*) $|(\frac{d}{dx})^\alpha V(x)| \le C_\alpha S B^{-\alpha}$ in I

(H2*) $\frac{d^2}{dx^2} V \ge c' S B^{-2}$ in I

(H3*) $V'(x_0) = 0$

(H4*) For $x \in I_{\mathrm{BVP}} \backslash I$ we have $V(x) \ge \min\{E_\infty, V(x_0) + c''\lambda^{-2\varepsilon} S\}$.

(H5*) For $x \in I_{\mathrm{BVP}}$ with $|x - x_0| > \frac{1}{2}\lambda^K B$, we have $V(x) \ge E_\infty + \frac{1000}{|x-x_0|^2}$.

(H6*) λ is bounded below by a positive constant depending only on c, c', c'', C_α in (H0*)…(H1*), and on ε, K, N.

Let $H = -\frac{d^2}{dx^2} + V(x)$ on $L^2(I_{\mathrm{BVP}})$ with Dirichlet or Neumann conditions at the endpoints.

For $V(x_0) < E < V(x_0) + \lambda^{-2\varepsilon} S$, define $x_{\mathrm{left}}(E) < x_{\mathrm{rt}}(E)$ to be the two values of $x \in I$ at which $V(x) = E$. Then define

$$\phi(E) = \int_{x_{\mathrm{left}}(E)}^{x_{\mathrm{rt}}(E)} (E - V(x))^{1/2}\, dx$$

$$\begin{aligned}\psi(E) &= \lim_{\delta \to 0+} \Big[\int\limits_{\substack{x \in I \\ E - V(x) > \delta}} V''(x)(E - V(x))^{-3/2}\, dx - q(E)\delta^{-1/2} \Big] \\ &= \lim_{\delta_{\mathrm{left}}, \delta_{\mathrm{rt}} \to 0+} \Big[\int_{x_{\mathrm{left}}(E)+\delta_{\mathrm{left}}}^{x_{\mathrm{rt}}(E)-\delta_{\mathrm{rt}}} V''(x)(E - V(x))^{-3/2}\, dx - q_{\mathrm{left}}(E)\delta_{\mathrm{left}}^{-1/2} \\ &\qquad - q_{\mathrm{rt}}(E)\delta_{\mathrm{rt}}^{-1/2} \Big]\end{aligned}$$

with $q(E)$, $q_{\mathrm{left}}(E)$, $q_{\mathrm{rt}}(E)$ uniquely specified by demanding the finiteness of the limits.

Lemma 1. *The phases $\phi(E)$, $\psi(E)$ satisfy the estimates*

$$\Big|\Big(\frac{d}{dE}\Big)^\beta \phi(E)\Big| \le C_\#^\beta \lambda S^{-\beta}$$

$$\left|\left(\frac{d}{dE}\right)^{\beta}\psi(E)\right| \leq C_{\#}^{\beta}\lambda^{-1}S^{-\beta}$$
$$\frac{d}{dE}\phi(E) \geq c_{\#}\lambda S^{-1}$$

for $V(x_0) < E < V(x_0) + \lambda^{-2\varepsilon}S$.

The constants $c_{\#}$, $C_{\#}^{\beta}$ *depend only on* c, c',c'', C_α, ε, K, N *in hypotheses (H0*)... (H6*).*

WKB Theorem on Low Eigenvalues. *Assume (H0*)...(H6*). Then there is a finite sequence* $E_0, E_1, \dots, E_{k_{\max}}$ *with the following properties.*

(a) *Let* $w = \phi(E_*) + \frac{1}{48}\psi(E_*)$ *with* $E_* = min\{E_\infty, V(x_0) + c_{\#}\lambda^{-2\varepsilon}S\}$, *and let* $\overline{n}$ *be the largest integer with* $\pi(\overline{n}+1/2) \leq w$. *If* $\min_{k\in\mathbb{Z}}|w - \pi(k+1/2)| > C_{\#}\lambda^{-2+4\varepsilon}$, *then* $k_{\max} = \overline{n}$. *In any case,* $|k_{\max} - \overline{n}| \leq 1$.

(b) *If* $0 \leq k < k_{\max}$, *then* E_k *is an eigenvalue of* H.

(c) *Either* $E_{k_{\max}} = E_\infty$ *or else* $E_{k_{\max}}$ *is an eigenvalue of* H.

(d) *Every eigenvalue* E *of* H *satisfying* $E \leq E_\infty$, $|E - V(x_0)| < c_{\#}\lambda^{-2\varepsilon}S$ *is one of the* E_k.

(e) *For* $0 \leq k \leq k_{\max}$ *we have* $V(x_0) < E_k < V(x_0) + 2c_{\#}\lambda^{-2\varepsilon}S$ *and* $|\phi(E_k) + \frac{1}{48}\psi(E_k) - \pi(k+1/2)| \leq C_{\#}\lambda^{-2+4\varepsilon}$.

The constants $c_{\#}$, $C_{\#}$ *depend only on* ε, K, N, c, c', c'', C_α *in hypotheses (H0*)...(H6*).* ∎

Lemma 2. *Assume (H0*)...(H6*). Let* $F(x)$ *be an eigenfunction of* H *whose eigenvalue* E *satisfies* $V(x_0) < E < V(x_0) + c_{\#}\lambda^{-2\varepsilon}S$ *and* $E \leq E_\infty$. *Then*

$$\int_{I_{\mathrm{BVP}}\setminus\{|x-x_0|<\lambda^{-\varepsilon}B\}} |F(x)|^2\,dx \leq C_{\#}\lambda^{-N}\int_{\{|x-x_0|<\lambda^{-\varepsilon}B\}} |F(x)|^2\,dx.$$

The constants $c_{\#}$, $C_{\#}$ *depend only on* ε, K, N, c, c', c'', C_α *in the hypotheses (H0*)...(H6*).*

Proofs of the results. Lemma 6 of the preceding section contains the present Lemma 1, except for the trivial estimate $\frac{d\phi}{dE} > c_{\#}\lambda S^{-1}$, which we leave to the reader. The

WKB Theorem on low eigenvalues follows at once from the corollaries to Lemmas 5 and 7 in the preceding section. Finally, our present Lemma 2 just restates Lemma 1 from the preceding section. ■

WKB Theory with Weak Turning Points

In this section, we develop a crude form of WKB Theory that requires only very weak hypotheses on the potential near the turning points.

Set–up. We are given an energy E_0 and a potential $V(x)$ defined on a (possibly unbounded) interval I_{BVP}. The interval I_{BVP} is partitioned into subintervals $I_{\text{far left}}$, I_{left}, I_{center}, I_{rt}, $I_{\text{far rt}}$ with $I_{\text{far left}}$ to the left of I_{left}, I_{left} to the left of I_{center}, etc. Here, $I_{\text{far left}}$ and $I_{\text{far right}}$ may be empty. On I_{center} we are given positive weight functions $S(x)$, $B(x)$. Set $\lambda(x) = S^{1/2}(x)B(x)$ and $\Lambda = (\int_{I_{\text{center}}} \frac{dx}{\lambda(x)B(x)})^{-1}$. We make the following assumptions.

Hypotheses

(H$\hat{0}$) I_{center} is non–empty, and for $x, y \in I_{\text{center}}$ with $|x-y| < cB(x)$ we have $c < B(y)/B(x) < C$ and $c < S(y)/S(x) < C$, and $|I_{\text{center}}| > cB(x)$.

(H$\hat{1}$) For $x \in I_{\text{center}}$ we have $|(\frac{d}{dx})^\alpha V(x)| \le C_\alpha S(x) B^{-\alpha}(x)$ and $cS(x) < E_0 - V(x) < CS(x)$.

(H$\hat{2}$) Λ is bounded below by a large number depending only on c, C, C_α in (H$\hat{0}$), (H$\hat{1}$).

(H$\hat{3}$) I_{left}, I_{rt} are non–empty. If $I_{\text{center}} = [x_{\text{left}}, x_{\text{rt}}]$, then we have $|I_{\text{left}}| \le \underline{C}B(x_{\text{left}})$, $|I_{\text{rt}}| \le \underline{C}B(x_{\text{rt}})$, $\lambda(x_{\text{left}}) \le \underline{C}$, $\lambda(x_{\text{rt}}) \le \underline{C}$.

(H$\hat{4}$) If $I_{\text{left}} = [x_{\text{far left}}, x_{\text{left}}]$, then we have $|V(x) - E_0| \le \underline{C}|I_{\text{left}}|^{-1}(x - \hat{x}_{\text{far left}})^{-1}$ in I_{left}. Here $\hat{x}_{\text{far left}} \le x_{\text{far left}}$ with strict inequality unless $I_{\text{far left}} = \emptyset$.

(H$\hat{5}$) For $x \in I_{\text{rt}}$ we have $|V(x) - E_0| \le \underline{C}|I_{\text{rt}}|^{-2}$.

(H$\hat{6}$) For $x \in I_{\text{far left}}$ we have $V(x) - E_0 \ge \underline{c}|I_{\text{left}}|^{-2}$. $V(x)$ is C^∞ in the interior of $I_{\text{far left}}$.

(H$\hat{7}$) For $x \in I_{\text{far rt}}$ we have $V(x) - E_0 \ge \frac{-10^{-9}}{(x - x_{\text{rt}})^2}$.

Our goal is to understand the eigenfunctions and eigenvalues of $H = -\frac{d^2}{dx^2} + V(x)$ on $L^2(I_{\mathrm{BVP}})$ with Dirichlet boundary conditions.

Denote by $C_\#$ a constant depending on c, C, C_α in $(\hat{\mathrm{H0}})$, $(\hat{\mathrm{H1}})$. Denote by C_*, c_* etc. constants depending only on c, C, C_α, $\underline{c}$, $\underline{C}$ in $(\hat{\mathrm{H0}})\ldots(\hat{\mathrm{H7}})$.

Note that I_{left} and I_{rt} don't play completely analogous rôles in our hypotheses.

We start by studying I_{center}.

Lemma 1. *Let $I_\nu = \{|x - x_\nu| < c_\# B(x_\nu)\} \subset I_{\mathrm{center}}$. Then the equation $[\frac{d^2}{dx^2} + E_0 - V(x)]u = f$ has a solution on I_ν, with $\|u\|_{L^\infty(I_\nu)} \leq C_\# \frac{B^2(x_\nu)}{\lambda(x_\nu)} \|f\|_{L^\infty(I_\nu)}$ and $\|\frac{du}{dx}\|_{L^\infty(I_\nu)} \leq C_\# B(x_\nu) \|f\|_{L^\infty(I_\nu)}$.*

Proof. Put $F_c(x) = \frac{\exp(i\int_{x_{\mathrm{left}}}^x (E_0 - V(t))^{1/2}\,dt)}{(E_0 - V(x))^{1/4}}$ on I_ν. Then

$$\left|\left[\frac{d^2}{dx^2} + (E_0 - V(x))\right]F_c(x)\right| = \left|\frac{5}{16}\frac{(V')^2}{(E_0 - V(x))^{9/4}} + \frac{1}{4}\frac{V''}{(E_0 - V(x))^{5/4}}\right| \leq C_\# B^{-2}(x_\nu) S^{-1/4}(x_\nu),$$

while $|F_c(x)| \leq C_\# S^{-1/4}(x_\nu)$. Using F_c we construct the approximate Green's function

$$G(x,y) = F_c(x)\overline{F_c(y)} \text{ for } x \leq y, \quad G(x,y) = \overline{F_c(x)}F_c(y) \text{ for } x > y.$$

Thus

$$|G(x,y)| \leq C_\# S^{-1/2}(x_\nu),$$

and as distributions we have

$$\left[\frac{\partial^2}{\partial x^2} + (E_0 - V(x))\right]G(x,y) = H(y)\delta(x-y) + K(x,y)$$

with $c\,H(y) = \mathrm{Im}(F_c'(y)\overline{F_c(y)})$ and $|K(x,y)| \leq C_\# B^{-2}(x_\nu) S^{-1/2}(x_\nu)$. We have $F_c'\overline{F_c}(y) = i + \frac{(\mathrm{const.})\ V'(y)}{(E_0 - V(y))^{3/2}} = i + \mathcal{E}(y)$, with $|\mathcal{E}(y)| \leq C_\# S^{-1/2}(x_\nu) B^{-1}(x_\nu) = C_\#/\lambda(x_\nu)$. This implies $|H(y)| > c_\#$ on I_ν.

Now set $u(x) = \int_{I_\nu} G(x,y)h(y)\,dy$ with h to be determined. Thus

$$\Big[\frac{d^2}{dx^2} + E_0 - V(x)\Big]u(x) = H(x)h(x) + \int_{I_\nu} K(x,y)h(y)\,dy,$$

so u solves our ODE provided we take h to solve

$$H(x)h(x) + \int_{I_\nu} K(x,y)h(y)\,dy = f(x).$$

As an operator on $L^\infty(I_\nu)$, $Th(x) = \int_{I_\nu} K(x,y)h(y)\,dy$ has norm ess $\sup_{x\in I_\nu}[\int_{I_\nu} |K(x,y)|\,dy] \le C_\# B^{-1}(x_\nu)S^{-1/2}(x_\nu) = C_\#/\lambda(x_\nu)$. Since $|H(x)| \ge c_\#$, we can solve the integral equation by a Neumann series, obtaining also the bound $\|h\|_{L^\infty(I_\nu)} \le C_\# \|f\|_{L^\infty(I_\nu)}$. To estimate u, we note that

$$\begin{aligned}|u(x)| &\le \int_{I_\nu} |G(x,y)|\,dy \cdot \|h\|_{L^\infty(I_\nu)} \le C_\# B(x_\nu)S^{-1/2}(x_\nu)\|h\|_{L^\infty(I_\nu)}\\ &\le C_\# B(x_\nu)S^{-1/2}(x_\nu)\|f\|_{L^\infty(I_\nu)} = \frac{C_\# B^2(x_\nu)}{\lambda(x_\nu)}\|f\|_{L^\infty(I_\nu)}.\end{aligned}$$

Similarly,

$$\Big|\frac{du(x)}{dx}\Big| \le \int_{I_\nu} \Big|\frac{\partial}{\partial x}G(x,y)\Big|\,dy \cdot \|h\|_{L^\infty(I_\nu)} \le C_\# S^{-1/4}(x_\nu)\int_{I_\nu}\Big|\frac{dF_c(x)}{dx}\Big|\,dx \cdot \|f\|_{L^\infty(I_\nu)}.$$

We saw that $|F_c'(y)\overline{F_c(y)}| = |i + \mathcal{E}(y)| < 2$, so $|F_c'(y)| \le 2(E_0 - V(y))^{1/4} \le C_\# S^{+1/4}(x_\nu)$. Therefore the previous estimate becomes $|\frac{du(x)}{dx}| \le C_\# B(x_\nu)\|f\|_{L^\infty(I_\nu)}$, which completes the proof. ■

Lemma 2. *Let* $I_\nu = \{|x - x_\nu| < c_\# B(x_\nu)\} \subset I_{\text{center}}$. *Then we can find a complex-valued solution of* $[\frac{d^2}{dx^2} + E_0 - V(x)]F_\nu(x) = 0$ *on* I_ν, *of the form*

$$F_\nu(x) = (E_0 - V(x))^{-1/4}\exp(i\int_{x_{\text{left}}}^{x}(E_0 - V(t))^{1/2}\,dt) - Error_\nu(x),$$

with $|Error_\nu| \le \frac{C_\# S^{-1/4}(x_\nu)}{\lambda(x_\nu)}$ *and* $|\frac{d}{dx}Error_\nu(x)| \le \frac{C_\# S^{+1/4}(x_\nu)}{\lambda(x_\nu)}$.

Proof. Let $f = [\frac{d^2}{dx^2} + (E_0 - V(x))]\{(E_0 - V(x))^{-1/4}\exp(i\int_{x_{\text{left}}}^{x}(E_0 - V(t))^{1/2}\,dt)\}$, and let $\text{Error}_\nu(x)$ be the function $u(x)$ given by the previous lemma. At the start of the proof of Lemma 1, we saw that $\|f\|_{L^\infty(I_\nu)} \le C_\# B^{-2}(x_\nu) S^{-1/4}(x_\nu)$, hence by lemma 1,

$$\|\text{Error}_\nu\|_{L^\infty(I_\nu)} \le C_\# \frac{B^2(x_\nu)}{\lambda(x_\nu)} \cdot B^{-2}(x_\nu) S^{-1/4}(x_\nu) = \frac{C_\# S^{-1/4}(x_\nu)}{\lambda(x_\nu)}$$

and

$$\begin{aligned}\|\frac{d}{dx}\ \text{Error}_\nu\|_{L^\infty(I_\nu)} &\le C_\# B(x_\nu) \cdot C_\# B^{-2}(x_\nu) S^{-1/4}(x_\nu) \\ &= \frac{C_\# S^{+1/4}(x_\nu)}{S^{1/2}(x_\nu) B(x_\nu)} = \frac{C_\# S^{+1/4}(x_\nu)}{\lambda(x_\nu)}.\end{aligned}$$

Also $(E_0 - V(x))^{-1/4} \exp(i \int_{x_{\text{left}}}^x (E_0 - V(t))^{1/2}\, dt) - \text{Error}_\nu(x)$ solves our ODE. ■

Lemma 3. *Let* I_ν *be as in the previous lemma. Suppose* $\theta_\nu \in C_0^\infty(I_\nu)$ *with* $|(\frac{d}{dx})^\alpha \theta_\nu(x)| \le C_\#^\alpha B^{-\alpha}(x_\nu)$. *Then*

(a) $\left| \int_{I_\nu} \theta_\nu \exp(\pm 2i \int_{x_{\text{left}}}^x (E_0 - V(t))^{1/2}\, dt)\, dx \right| \le C_\# \frac{B(x_\nu)}{\lambda(x_\nu)}$.

(b) *Let* $F_\nu(x)$ *be the ODE solution given by the previous lemma. For* A *a (complex) constant, we have*

$$\begin{aligned}\Big| \int_{I_\nu} \theta_\nu(x) |Re(AF_\nu(x))|^2\, dx - \frac{1}{2}|A|^2 \int_{I_\nu} \frac{\theta_\nu(x)\, dx}{(E_0 - V(x))^{1/2}} \Big| \\ \le C_\# \frac{|A|^2 S^{-1/2}(x_\nu) B(x_\nu)}{\lambda(x_\nu)}.\end{aligned}$$

(c) F_ν *and* $\overline{F}_\nu$ *are linearly independent functions.*

Proof. $| \int_{I_\nu} \theta_\nu(x) \exp(\pm 2i \int_{x_{\text{left}}}^x (E_0 - V(t))^{1/2}\, dt)\, dx| =$
$(\text{const.}) \Big| \int_{I_\nu} \frac{\theta_\nu(x)}{(E_0 - V(x))^{1/2}} \frac{d}{dx} \Big\{ \exp(\pm 2i \int_{x_{\text{left}}}^x (E_0 - V(t))^{1/2}\, dt) \Big\}\, dx \Big|$
$= (\text{const.}) \Big| \int_{I_\nu} \exp(\pm 2i \int_{x_{\text{left}}}^x (E_0 - V(t))^{1/2}\, dt) \frac{d}{dx} \Big\{ \frac{\theta_\nu(x)}{(E_0 - V(x))^{1/2}} \Big\}\, dx \Big|$
$\le C_\# \int_{I_\nu} \Big\{ \frac{|\theta_\nu'|}{(E_0 - V)^{1/2}} + \frac{|\theta_\nu||V'|}{(E_0 - V)^{3/2}} \Big\}\, dx \le C_\# \int_{I_\nu} S^{-1/2}(x_\nu) B^{-1}(x_\nu)\, dx$
$= \frac{C_\# B(x_\nu)}{\lambda(x_\nu)}$, proving (a).

To prove (b), we write $F_\nu(x) = (1 + \text{Error}_\nu(x)) \cdot (E_0 - V(x))^{-1/4} \cdot \exp(i \int_{x_{\text{left}}}^x (E_0 -$

$V(t))^{1/2}\,dt)$ with $|\mathrm{Error}_\nu(x)| \le \frac{C_\#}{\lambda(x_\nu)}$ by virtue of Lemma 2. Hence

$$[\mathrm{Re}(AF_\nu)]^2 = \frac{1}{2}|A|^2|F_\nu|^2 + \frac{1}{4}A^2F_\nu^2 + \frac{1}{4}\overline{A}\,\overline{F}_\nu^2 =$$

$$\frac{1}{2}|A|^2\cdot(E_0-V(x))^{-1/2}\cdot|1+\mathrm{Error}_\nu(x)|^2 + \frac{1}{4}A^2\cdot(E_0-V(x))^{-1/2}\cdot(1+\mathrm{Error}_\nu(x))^2$$
$$\cdot\exp(2i\int_{x_{\mathrm{left}}}^{x}(E_0-V(t))^{1/2}\,dt)$$
$$+\frac{1}{4}\overline{A}^2\cdot(E_0-V(x))^{-1/2}\cdot(1+\overline{\mathrm{Error}_\nu(x)})^2\cdot\exp(-2i\int_{x_{\mathrm{left}}}^{x}(E_0-V(t))^{1/2}\,dt).$$

This implies

$$\Big|\int_{I_\nu}\theta_\nu\Big[\mathrm{Re}(AF_\nu)\Big]^2\,dx - \frac{1}{2}\int_{I_\nu}\theta_\nu|A|^2(E_0-V(x))^{-1/2}\,dx\Big| \le$$
$$\sum_{\pm}C_\#|A|^2\Big|\int_{I_\nu}\frac{\theta_\nu(x)}{(E_0-V(x))^{1/2}}\,e^{\pm 2i\int_{x_{\mathrm{left}}}^{x}(E_0-V(t))^{1/2}\,dt}\,dx\Big|$$
$$+\,C_\#|A|^2\int_{I_\nu}\frac{|\theta_\nu(x)|}{(E_0-V(x))^{1/2}}|\mathrm{Error}_\nu(x)|\,dx,$$

since $\Big||1+\mathrm{Error}_\nu|^2-1\Big|, |(1+\mathrm{Error}_\nu)^2-1|, |(1+\overline{\mathrm{Error}_\nu})^2-1| \le C_\#|\mathrm{Error}_\nu|$.

The sum over $\pm$ may be estimated by part (a), and the last term on the right is estimated by using $|\mathrm{Error}_\nu(x)| \le \frac{C_\#}{\lambda(x_\nu)}$.

The result is

$$\Big|\int_{I_\nu}\theta_\nu\Big[\mathrm{Re}(AF_\nu)\Big]^2\,dx - \frac{1}{2}|A|^2\int_{I_\nu}\frac{\theta_\nu\,dx}{(E_0-V)^{1/2}}\Big| \le C_\#\frac{|A|^2S^{-1/2}(x_\nu)B(x_\nu)}{\lambda(x_\nu)},$$

which is (b).

Finally (c) follows from (b) since F_ν, $\overline{F}_\nu$ linearly dependent implies $\mathrm{Re}(AF_\nu)=0$ for a complex number A with $|A|=1$. From (b) we would then learn that $|\int_{I_\nu}\frac{\theta_\nu\,dx}{(E_0-V)^{1/2}}| \le C_\#\frac{S^{-1/2}(x_\nu)B(x_\nu)}{\lambda(x_\nu)}$, which is plainly false. ■

Lemma 4. *Let $I_\nu = \{|x-x_\nu| < c_\# B(x_\nu)\}$ and $I_{\nu+1} = \{|x-x_{\nu+1}| < c_\# B(x_{\nu+1})\}$ be subintervals of I_{center}, with $|I_\nu \cap I_{\nu+1}| > c_\#|I_\nu|$. Let $F_\nu(x)$, $F_{\nu+1}(x)$ be the solutions of $[\frac{d^2}{dx^2}+(E_0-V(x))]F=0$ given by Lemma 2 on I_ν, $I_{\nu+1}$ respectively. Then on $I_\nu\cap I_{\nu+1}$ we have*

(1) $$F_{\nu+1}(x) = A_\nu^1F_\nu(x) + A_\nu^2\overline{F_\nu(x)} \quad \textit{with } |A_\nu^1-1|,\ |A_\nu^2| \le \frac{C_\#}{\lambda(x_\nu)}.$$

Proof. Since F_ν, $\overline{F}_\nu$ are two independent solutions of the ODE for $F_{\nu+1}$, we can represent $F_{\nu+1}$ by (1), and the only problem is to bound the A^i_ν. We will integrate (1) against $g^\pm_\nu(x) = (E_0 - V(x))^{+1/4}\theta_\nu(x)\exp(\pm i\int_{x_{\text{left}}}^x (E_0 - V(t))^{1/2}\,dt)$. Here we take $\theta_\nu \in C_0^\infty(I_\nu \cap I_{\nu+1})$ with $\theta_\nu \geq 0$, $\max_{x\in I_\nu}\theta_\nu(x) = 1$, and $|(\frac{d}{dx})^\alpha\theta_\nu| \leq C^\alpha_\# B^{-\alpha}(x_\nu)$. Lemma 2 gives

$$\int_{I_\nu} g^-_\nu F_\nu\,dx = \int_{I_\nu}\theta_\nu(x)\,dx + 0(\int_{I_\nu}|g^-_\nu(x)||\text{Error}_\nu(x)|\,dx),$$

with the last integral dominated by $C_\#\frac{B(x_\nu)}{\lambda(x_\nu)}$. Thus, $|\int_{I_\nu} g^-_\nu F_\nu\,dx - \int_{I_\nu}\theta_\nu(x)\,dx| \leq C_\# B(x_\nu)/\lambda(x_\nu)$. Similarly, $|\int_{I_\nu} g^-_\nu F_{\nu+1}\,dx - \int_{I_\nu}\theta_\nu(x)\,dx| \leq C_\# B(x_\nu)/\lambda(x_\nu)$ and $|\int_{I_\nu} g^+_\nu \overline{F}_\nu\,dx - \int_{I_\nu}\theta_\nu(x)\,dx| \leq C_\# B(x_\nu)/\lambda(x_\nu)$. Also

$$\Big|\int_{I_\nu} g^+_\nu F_\nu\,dx\Big| \leq \Big|\int_{I_\nu}\theta_\nu(x)\exp(+2i\int_{x_{\text{left}}}^x (E_0 - V(t))^{1/2}\,dt)\,dx\Big| + \int_{I_\nu}|g^+_\nu(x)||\text{Error}_\nu(x)|\,dx.$$

Lemmas (2) and (3a) show that both terms on the right are dominated by $C_\#\frac{B(x_\nu)}{\lambda(x_\nu)}$, so $|\int_{I_\nu} g^+_\nu F_\nu\,dx| \leq C_\# B(x_\nu)/\lambda(x_\nu)$. Similarly, $|\int_{I_\nu} g^+_\nu F_{\nu+1}\,dx| \leq C_\# B(x_\nu)/\lambda(x_\nu)$ and $|\int_{I_\nu} g^-_\nu \overline{F}_\nu\,dx| \leq C_\# B(x_\nu)/\lambda(x_\nu)$.

Using these estimates, we can integrate (1) against $g^\pm_\nu(x)\,dx$ on I_ν, to obtain the following equations

$$\text{(2)}\qquad \begin{cases} X_\nu = A^1_\nu P^\nu_{11} + A^2_\nu P^\nu_{12} \\ Y_\nu = A^1_\nu P^\nu_{21} + A^2_\nu P^\nu_{22} \end{cases}$$

with

$$|X_\nu - \int_{I_\nu}\theta_\nu\,dx| \leq C_\# B(x_\nu)/\lambda(x_\nu)$$

$$|Y_\nu| \leq C_\# B(x_\nu)/\lambda(x_\nu)$$

$$|P^\nu_{11} - \int_{I_\nu}\theta_\nu\,dx|,\ |P^\nu_{22} - \int_{I_\nu}\theta_\nu\,dx| \leq C_\# B(x_\nu)/\lambda(x_\nu)$$

$$|P^\nu_{12}|, |P^\nu_{21}| \leq C_\# B(x_\nu)/\lambda(x_\nu).$$

Dividing (2) by $\int_{I_\nu} \theta_\nu \, dx \sim B(x_\nu)$, we obtain equations (2) with $|X_\nu - 1|$, $|Y_\nu|$, $|P^\nu_{11} - 1|$, $|P^\nu_{22} - 1|$, $|P^\nu_{12}|$, $|P^\nu_{21}| \leq \frac{C_\#}{\lambda(x_\nu)}$. Hence $|A^1_\nu - 1|$, $|A^2_\nu| \leq \frac{C_\#}{\lambda(x_\nu)}$ as asserted. ■

Lemma 5. *Let u be a real–valued solution of $[-\frac{d^2}{dx^2} + (E_0 - V(x))]u = 0$ on I_{center}. For a (complex) constant Q we can express u as*

$$(3) \quad u(x) = Re\Big[Q(E_0 - V(x))^{-1/4}\Big\{e^{i\int_{x_{\text{left}}}^x (E_0 - V(t))^{1/2}\, dt} + Error(x)\Big\}\Big] \quad with$$

$$|Error(x)| \leq C_\#/\Lambda \quad and \; |\frac{d}{dx} Error(x)| \leq C_\# S^{1/2}(x)/\Lambda.$$

Equivalently,

(4)

$$u(x) = Re\Big[Q(E_0 - V(x))^{-1/4}(1 + Error(x))\exp(i\int_{x_{\text{left}}}^x (E_0 - V(t))^{1/2}\, dt)\Big] \quad with$$

$$|Error(x)| \leq C_\#/\Lambda \quad and \; |\frac{d}{dx} Error(x)| \leq C_\# S^{1/2}(x)/\Lambda.$$

Proof. Cover I_{center} by $I_\nu = \{|x - x_\nu| < c_\# B(x_\nu)\}$ so that $|I_\nu \cap I_{\nu+1}| \geq c_\# |I_\nu|$. For each I_ν, let F_ν be the ODE solution given by lemma 2 on I_ν. Since F_ν, $\overline{F}_\nu$ are independent solutions of the ODE for u on I_ν, we can express u on I_ν as $u = \text{Re}[Q_\nu F_\nu]$. The complex constant Q_ν is uniquely determined. On $I_{\nu+1} \cap I_\nu$ we have $F_{\nu+1} = A^1_\nu F_\nu + A^2_\nu \overline{F}_\nu$ as in lemma 4, hence on $I_{\nu+1} \cap I_\nu$ $u = \text{Re}[Q_{\nu+1} F_{\nu+1}] =$ $\text{Re}[Q_{\nu+1}(A^1_\nu F_\nu + A^2_\nu \overline{F}_\nu)] = \text{Re}[Q_{\nu+1} A^1_\nu F_\nu + Q_{\nu+1} A^2_\nu \overline{F}_\nu] =$ $\text{Re}[(Q_{\nu+1} A^1_\nu + \overline{Q}_{\nu+1} \overline{A}^2_\nu) F_\nu]$. Since Q_ν is uniquely determined, this implies $Q_\nu = Q_{\nu+1} A^1_\nu + \overline{Q}_{\nu+1} \overline{A}^2_\nu$. Lemma 4 gives $|A^1_\nu - 1|$, $|A^2_\nu| \leq \frac{C_\#}{\lambda(x_\nu)}$, hence $|Q_\nu - Q_{\nu+1}| \leq \frac{C_\# |Q_{\nu+1}|}{\lambda(x_\nu)}$. Equivalently, $Q_{\nu+1} = Q_\nu \exp(b_\nu)$ with b_ν a complex number and $|b_\nu| \leq \frac{C_\#}{\lambda(x_\nu)}$. Since $\sum_\nu \frac{C_\#}{\lambda(x_\nu)} \leq \frac{C'_\#}{\Lambda}$, it follows that $Q_\nu = Q \exp(b'_\nu)$ for a complex number Q independent of ν, and with $|b'_\nu| \leq C_\#/\Lambda$. Equivalently, $|Q_\nu - Q| \leq \frac{C_\# |Q|}{\Lambda}$.

On I_ν we have

$$u(x) = \mathrm{Re}[Q_\nu F_\nu(x)] = \mathrm{Re}[Q_\nu (E_0 - V(x))^{-1/4} \exp(i \int_{x_{\mathrm{left}}}^{x} (E_0 - V(t))^{1/2}\, dt) + Q_\nu w_\nu(x)]$$

with $|w_\nu(x)| \le \frac{C_\# S^{-1/4}(x_\nu)}{\lambda(x_\nu)}$, $|\frac{dw_\nu(x)}{dx}| \le \frac{C_\# S^{+1/4}(x_\nu)}{\lambda(x_\nu)}$, by Lemma 2. Hence

$$(5) \quad u = \mathrm{Re}\Big[Q(E_0 - V(x))^{-1/4} \exp(i \int_{x_{\mathrm{left}}}^{x} (E_0 - V(t))^{1/2}\, dt) + \mathcal{E}_\nu(x)\Big] \text{ on } I_\nu,$$

with $\mathcal{E}_\nu(x) = (Q_\nu - Q)(E_0 - V(x))^{-1/4} \exp(i \int_{x_{\mathrm{left}}}^{x} (E_0 - V(t))^{1/2}\, dt) + Q_\nu w_\nu(x)$.

Our bounds on $|Q_\nu - Q|$ and on $|w_\nu(x)|$, $|\frac{d}{dx} w_\nu(x)|$ gives us the estimates

$$(6) \qquad |\mathcal{E}_\nu(x)| \le C_\# S^{-1/4}(x_\nu)|Q|/\Lambda,\ |\frac{d}{dx}\mathcal{E}_\nu(x)| \le C_\# S^{+1/4}(x_\nu)|Q|/\Lambda.$$

Replacing $\mathcal{E}_\nu(x)$ by its real part, we preserve (5) and (6). Now $\mathcal{E}_\nu(x)$ is uniquely specified on I_ν by (5) and Im $\mathcal{E}_\nu \equiv 0$, so $\mathcal{E}_{\nu+1}(x) = \mathcal{E}_\nu(x)$ on $I_\nu \cap I_{\nu+1}$. Therefore, (5) and (6) become

$$(7) \quad u(x) = \mathrm{Re}\Big[Q(E_0 - V(x))^{-1/4} \exp(i \int_{x_{\mathrm{left}}}^{x} (E_0 - V(t))^{1/2}\, dt) + \mathcal{E}(x)\Big] \text{ on } I_{\mathrm{center}}$$

with

$$(8) \qquad |\mathcal{E}(x)| \le \frac{C_\# S^{-1/4}(x)|Q|}{\Lambda},\ |\frac{d}{dx}\mathcal{E}(x)| \le \frac{C_\# S^{1/4}(x)|Q|}{\Lambda}.$$

Writing $\mathrm{Error}(x) = \frac{\mathcal{E}(x)}{Q} \cdot (E_0 - V(x))^{+1/4}$, we get (3) with $|\mathrm{Error}(x)| \le \frac{C_\#}{\Lambda}$, $|\frac{d}{dx}\,\mathrm{Error}(x)| \le \frac{C_\# S^{+1/4}(x)|Q|}{\Lambda} \cdot \frac{S^{+1/4}(x)}{|Q|} + \frac{C_\# S^{-1/4}(x)|Q|}{\Lambda} \cdot \frac{S^{+1/4}(x)B^{-1}(x)}{|Q|} \le \frac{C_\#}{\Lambda}\{S^{1/2}(x) + B^{-1}(x)\} = \frac{C_\# S^{1/2}(x)}{\Lambda}\{1 + \lambda^{-1}(x)\} \le \frac{C'_\# S^{1/2}(x)}{\Lambda}$. These are the desired estimates for (3).

We can also put $\mathrm{Error}(x) = \frac{\mathcal{E}(x)}{Q}(E_0 - V(x))^{+1/4} \exp(-i \int_{x_{\mathrm{left}}}^{x} (E_0 - V(t))^{1/2}\, dt)$, so that (4) holds, together with the estimate $|\mathrm{Error}(x)| \le C_\#/\Lambda$ and

$$\Big|\frac{d}{dx}\,\mathrm{Error}(x)\Big| \le \frac{C_\# S^{+1/4}(x)|Q|}{\Lambda} \cdot \frac{S^{+1/4}(x)}{|Q|} + \frac{C_\# S^{-1/4}(x)|Q|}{\Lambda} \cdot \frac{S^{+1/4}(x)B^{-1}(x)}{|Q|} + \frac{C_\# S^{-1/4}(x)|Q|}{\Lambda} \cdot \frac{S^{3/4}(x)}{|Q|} \le \frac{C_\# S^{1/2}(x)}{\Lambda}\{1 + \lambda^{-1}(x)\} \le \frac{C_\# S^{1/2}(x)}{\Lambda}$$

as asserted. The proof of Lemma 5 is complete. ∎

Corollary 1. *In the setting of Lemma 5, the number of zeroes of* $u(x)$ *on* I_{center} *differs by at most* $C_\#$ *from* $\frac{1}{\pi}\int_{I_{\text{center}}}(E_0 - V(t))^{1/2}\,dt$.

Proof. Equation (4) shows that the zeros of $u(x)$ on I_{center} are the points of I_{center} where

$$f(x) = (\arg Q) + \int_{x_{\text{left}}}^{x} (E_0 - V(t))^{1/2}\,dt + \text{Im}\;\ell n(1 + \text{Error}(x)) \equiv 0 \bmod \pi,$$

with $|\text{Error}(x)| \le \frac{C_\#}{\Lambda}, |\frac{d}{dx}\text{ Error}(x)| \le \frac{C_\# S^{1/2}(x)}{\Lambda}$, and with a branch of $\ell n(1+\zeta)$ defined on $\{|\zeta| < \frac{1}{2}\}$ and yielding $\ell n 1 = 0$. The function $g(x) = \text{ Im }\ell n(1 + \text{Error}(x))$ satisfies $|g(x)| \le \frac{C_\#}{\Lambda}$, $|\frac{d}{dx}g(x)| \le C_\# \frac{S^{1/2}(x)}{\Lambda}$, while $\frac{d}{dx}(f - g) = (E_0 - V(x))^{1/2} \ge c_\# S^{1/2}(x)$. If $x_k \in I_{\text{center}}$ satisfies $(f-g)(x_k) = \pi k$, then $|f(x_k) - \pi k| \le \frac{C_\#}{\Lambda}$, and $f'(x) \ge c_\# S^{1/2}(x_k)$ for $|x - x_k| < c_\# B(x_k)$. Hence there is an $x'_k \in I_{\text{center}}$ with $f(x'_k) = \pi k$ and $|x_k - x'_k| < \frac{(C_\#/\Lambda)}{c_\# S^{1/2}(x_k)} = \frac{C_\# B(x_k)}{\lambda(x_k)\Lambda}$, unless x_k lies within $C_\# B(x_k)/(\lambda(x_k)\Lambda)$ of an endpoint of I_{center}, i.e. unless $|x_k - x_{\text{left, rt}}| \le \frac{C_\# B(x_{\text{left. rt}})}{\lambda(x_{\text{left }, rt})\Lambda}$. So to every x_k not too near an endpoint of I_{center} there corresponds an x'_k. Different k give rise to different $f(x'_k)$, hence to distinct x'_k. Since $(f-g)' > 0$ there is at most one x_k for each k. Hence the number of solutions of $f(x) \equiv 0 \bmod \pi$ in I_{center} is at least as great as the number of solutions of $f(x) - g(x) \equiv 0 \bmod \pi$ in $I_{\text{center}} \setminus \bigcup_{\text{left, rt}}\{|x - x_{\text{left, rt}}| \le \frac{C_\# B(x_{\text{left rt}})}{\lambda(x_{\text{left, rt}})\Lambda}\} \equiv I_{\text{center}} \setminus \bigcup_{\text{left, rt}} J_{\text{left, rt}}$.

The roles of f and $f - g$ can be reversed in the above argument, so the number of solutions of $f(x) \equiv 0 \bmod \pi$ in I_{center} differs from the corresponding number of solutions for $f - g$ by at most the number of solutions of f or $f - g \equiv 0 \bmod \pi$ in $J_{\text{left}} \cup J_{\text{rt}}$. In J_{left} we have $0 < f'$, $(f-g)' < C_\# S^{1/2}(x_{\text{left}})$, and $|J_{\text{left}}| \le \frac{C_\# B(x_{\text{left}})}{\lambda(x_{\text{left}})\Lambda}$. Hence on J_{left}, f and $f - g$ are strictly increasing and vary by at most $\frac{C_\#}{\Lambda}$. So there is at most one solution of $f(x) \equiv 0 \bmod \pi$ in J_{left}. Similarly, there is at most one solution of $f - g \equiv 0 \bmod \pi$ in J_{left}, and the same argument applies to J_{rt}. Thus, the number of solutions of $f \equiv 0 \mod \pi$ and of $f - g \equiv 0 \mod \pi$ in I_{center} differ by at most 4. (We could have done better.) The number of solutions of $f - g \equiv 0 \mod \pi$ differs from $\frac{1}{\pi}\int_{I_{\text{center}}}(E_0 - V(x))^{1/2}\,dx$ by at most 20 while we have seen that the number of solutions of $f \equiv 0 \mod \pi$ is equal to the number of zeroes of u. Hence $|(\text{Number of zeroes of } u \text{ in } I_{\text{center}}) - \frac{1}{\pi}\int_{I_{\text{center}}}(E_0 - V)^{1/2}| \leq 24$. ■

Corollary 2. *In the setting of Lemma 5 we have*

$$\int_{I_{\text{center}}} u^2\,dx \geq c_{\#}|Q|^2 \int_{I_{\text{center}}} \frac{dx}{(E_0 - V(x))^{1/2}}.$$

Proof. For $\theta_\nu \geq 0$ as in lemma 3 we have

$$\int_{I_\nu} \theta_\nu u^2\,dx = \int_{I_\nu} \theta_\nu(x)\Big(\text{Re}\Big[Q(E_0 - V(x))^{-1/4}\exp(i\int_{x_{\text{left}}}^{x}(E_0 - V(t))^{1/2}\,dt)$$
$$\cdot(1 + \text{Error}(x))\Big]\Big)^2\,dx$$

with $|\text{Error}(x)| \leq \frac{C_{\#}}{\Lambda}$.

The proof of Lemma 3(b) shows that the right–hand side differs from $\frac{1}{2}|Q|^2\int_{I_\nu}\theta_\nu(x)(E_0 - V(x))^{-1/2}\,dx$ by at most $\frac{C_{\#}|Q|^2}{\Lambda}S^{-1/2}(x_\nu)B(x_\nu)$. Covering I_{center} by I_ν, we see that

$$\int_{I_{\text{center}}}(\sum_\nu \theta_\nu)u^2\,dx \geq \frac{1}{2}|Q|^2\int_{I_{\text{center}}}(\sum_\nu \theta_\nu)(E_0 - V(x))^{-1/2}\,dx$$
$$-\frac{C_{\#}|Q|^2}{\Lambda}\sum_\nu S^{-1/2}(x_\nu)B(x_\nu).$$

We can arrange the θ_ν, I_ν so that each $x \in I_{\text{center}}$ lies in at most $C_{\#}$ of the I_ν, and so that $\sum\limits_\nu \theta_\nu = 1$ on $\overset{\circ}{I} = [x_{\text{left}} + c_{\#}B(x_{\text{left}}), x_{\text{rt}} - c_{\#}B(x_{\text{rt}})] \subset I_{\text{center}}$. Therefore the preceding inequality gives

$$\int_{I_{\text{center}}} u^2\,dx \geq \frac{1}{2}\int_{\overset{\circ}{I}}(E_0 - V)^{-1/2}\,dx \cdot |Q|^2 - \frac{C_{\#}}{\Lambda}\int_{I_{\text{center}}}(E_0 - V)^{-1/2}\,dx \cdot |Q|^2.$$

Since $c_{\#}\int_{I_{\text{center}}}(E_0 - V)^{-1/2}\,dx \leq \int_{\overset{\circ}{I}}(E_0 - V)^{-1/2}\,dx$, the last inequality implies $\int_{I_{\text{center}}} u^2\,dx \geq |Q|^2 \cdot (\frac{1}{2}c_{\#} - \frac{C_{\#}}{\Lambda})\int_{I_{\text{center}}}(E_0 - V(x))^{-1/2}\,dx$, which yields the desired corollary. ■

Now we study $-\frac{d^2}{dx^2} + V(x)$ on I_{left}, I_{rt}, $I_{\text{far left}}$, $I_{\text{far rt}}$.

Lemma 6.

(a) *Let* $H_{\text{left}} = -\frac{d^2}{dx^2} + V(x)$ *on* I_{left} *with the following boundary conditions. If* $I_{\text{far left}}$ *is non–empty, then we use Neumann conditions.*

If $I_{\text{far left}}$ *is empty, then we impose a Dirichlet condition at the left endpoint and a Neumann condition at the right endpoint.*

(b) *Let* $H_{\text{rt}} = -\frac{d^2}{dx^2} + V(x)$ *on* I_{rt} *with Neumann conditions.*

(c) *Let* $H_{\text{far left}} = -\frac{d^2}{dx^2} + V(x)$ *on* $I_{\text{far left}}$ *with Dirichlet boundary condition on the left and Neumann condition on the right endpoint. (We consider* $H_{\text{far left}}$ *only if* $I_{\text{far left}}$ *is non–empty).*

(d) *Let* $H_{\text{far right}} = -\frac{d^2}{dx^2} + V(x)$ *on* $I_{\text{far right}}$ *with Dirichlet boundary condition on the right and Neumann boundary condition at the left endpoint. (We consider* $H_{\text{far right}}$ *only if* $I_{\text{far rt}}$ *is non–empty.)*

Then all of the above operators have at most C_* *negative eigenvalues.*

Proof. For H_{left}, note first of all that H_{left} is bounded below. After passing to a subspace of codimension 0 or 1 in the domain of H_{left}, we may assume a Dirichlet boundary condition at the left endpoint. Then after rescaling, we have only to prove that $-\frac{d^2}{dx^2} - \underline{C}x^{-1}$ on $[0,1]$ with Dirichlet condition at 0 and Neumann condition at 1 has at most C_* negative eigenvalues. Here C_* depends only on $\underline{C}$. This bound is trivial.

After rescaling H_{right}, we are left with the operator $-\frac{d^2}{dx^2} - \underline{C}$ on $[0,1]$ with Neumann boundary conditions. We must prove that this operator has at most C_* negative eigenvaleus, which is trivial.

$H_{\text{far left}}$ has no eigenvalues $< E_0$, since $V(x) - E_0 \geq 0$ on $I_{\text{far left}}$.

For $H_{\text{far right}}$, we again impose a Dirichlet boundary condition at the left endpoint after passing to a subspace of codimension at most 1.

Now we see that $H_{\text{far right}}$ has no eigenvalues $< E_0$ by virtue of the elementary inequality $\int_0^\infty |\frac{1}{y}\int_0^y f(t)\,dt|^2\,dy \le 10^5 \int_0^\infty |f(y)|^2\,dy$. ■

Lemma 7. *The number of eigenvalues $< E_0$ for $-\frac{d^2}{dx^2}+V(x)$ on I_{BVP} with Dirichlet boundary conditions differs by at most C_* from $\frac{1}{\pi}\int_{I_{\text{center}}}(E_0 - V(x))^{1/2}\,dx$.*

Proof. We compare our eigenvalue problem with separate problems on $I_{\text{far left}}$, I_{left}, I_{center}, I_{right}, $I_{\text{far right}}$. To get a lower bound on the number of eigenvalues $< E_0$, we impose Dirichlet conditions at all the endpoints.

To get an upper bound on the number of eigenvalues $< E_0$, we impose Dirichlet conditions at the endpoints of I_{BVP} and Neumann conditions at all the other endpoints. Lemma 6 shows that all the intervals except I_{center} contribute at most C_* eigenvalues to the upper and lower bounds. Hence it is enough to study $H_{\text{center}} = -\frac{d^2}{dx^2} + V(x)$ on I_{center}, with Dirichlet or Neumann boundary conditions. Sturm–Liouville theory shows that the number of eigenvalues $< E_0$ for H_{center} differs by at most C_* from the number of zeroes of u on I_{center}. Here u is the real–valued solution of $[-\frac{d^2}{dx^2} + E_0 - V(x)]u = 0$ with a Dirichlet or Neumann boundary condition at one endpoint of I_{center}. Hence Lemma 7 follows from Corollary 1 to Lemma 5. ■

We turn to the study of an eigenfunction on I_{left}, I_{right}.

Lemma 8. *Suppose $|u''(x)| \le C_1 x^{-1}|u(x)|$ on $(0,1)$, and let $\hat{I} \subset (0,1)$ be an interval with $|\hat{I}| > c_1$. Then $\max_{x\in(0,1)}|u(x)| \le \hat{C}\max_{x\in\hat{I}}|u(x)|$ with $\hat{C}$ depending only on c_1, C_1.*

Proof. Let x_0, x_1 be respectively the midpoint and right endpoint of $\hat{I}$. Thus x_0, $x_1 - x_0 > \frac{1}{2}c_1$. By the mean–value theorem we can find an $\overline{x}$ between x_0 and x_1 with $|u'(\overline{x})| = |\frac{u(x_1)-x(x_0)}{x_1-x_0}| \le 2\max_{\hat{I}}|u|/(\frac{1}{2}c_1)$. Also $|u(\overline{x})| \le \max_{\hat{I}}|u|$ since $\overline{x}\in\hat{I}$.

Now $\frac{d}{dx}\begin{pmatrix} u'(x) \\ u(x)\end{pmatrix} = \begin{pmatrix} 0 & W(x) \\ 1 & 0\end{pmatrix}\begin{pmatrix} u'(x) \\ u(x)\end{pmatrix}$ with

$W(x) = \begin{cases} \frac{u''}{u(x)} & \text{if } u(x) \neq 0 \\ 0 & \text{if } u(x) = 0 \end{cases}$. Hypothesis implies $|W(x)| \leq C_1 x^{-1}$, so $\left|\frac{d}{dx}\begin{pmatrix} u'(x) \\ u(x) \end{pmatrix}\right| \leq (1 + C_1 x^{-1})\left|\begin{pmatrix} u'(x) \\ u(x) \end{pmatrix}\right|$, so $\frac{d}{dx}\left[x^{(1+C_1)}\left|\begin{pmatrix} u'(x) \\ u(x) \end{pmatrix}\right|\right] \geq 0 \geq \frac{d}{dx}\left[x^{-(1+C_1)}\left|\begin{pmatrix} u'(x) \\ u(x) \end{pmatrix}\right|\right]$. Therefore

$$\left|\begin{pmatrix} u'(x) \\ u(x) \end{pmatrix}\right| \leq \left|\begin{pmatrix} u'(\overline{x}) \\ u(\overline{x}) \end{pmatrix}\right| \cdot (x/\overline{x})^{(1+C_1)} \quad \text{for } \overline{x} \leq x < 1, \text{ and}$$

$$\left|\begin{pmatrix} u'(x) \\ u(x) \end{pmatrix}\right| \leq \left|\begin{pmatrix} u'(\overline{x}) \\ u(\overline{x}) \end{pmatrix}\right| \cdot (\overline{x}/x)^{(1+C_1)} \quad \text{for } 0 < x \leq \overline{x}.$$

Substituting our estimates for $|u(\overline{x})|$, $|u'(\overline{x})|$ into the last two estimates, we see that

$$|u'(x)|, |u(x)| \leq \hat{C}_1 \max_{\hat{I}} |u| \quad \text{for } x_* \leq x < 1, \tag{9}$$

where $\hat{C}_1$ depends only on c_1, C_1 provided $x_* > 0$ depends only on c_1, C_1. We pick x_* so that

$$x_* C_1 \int_0^1 (\ell n \frac{1}{t})\, dt = \frac{1}{10}.$$

For a constant $\hat{C}_2 >> \hat{C}_1$ depending only on c_1, C_1, we claim that

$$|u(x)| \leq \hat{C}_2 \max_{\hat{I}} |u| \quad \text{for } 0 < x \leq x_*. \tag{10}$$

To see (10), let $x_{\min}$ be the smallest number $y \geq 0$ for which $\max_{[y,x_*]} |u| \leq \hat{C}_2 \max_{\hat{I}} |u|$. Either (10) holds or else $0 < x_{\min} < x_*$, $\max_{[x_{\min},x_*]} |u| \leq \hat{C}_2 \max_{\hat{I}} |u|$ and $|u(x_{\min})| = \hat{C}_2 \max_{\hat{I}} |u|$. We assume the latter case and derive a contradiction. Since $|u'(x_*)| \leq \hat{C}_1 \max_{\hat{I}} |u|$ by (9), we have for $x \in [x_{\min}, x_*]$ the estimate

$$\begin{aligned} |u'(x)| &\leq |u'(x_*)| + \int_x^{x_*} |u''(t)|\, dt \leq |u'(x_*)| + \int_x^{x_*} C_1 t^{-1} |u(t)|\, dt \\ &\leq \hat{C}_1 \max_{\hat{I}} |u| + (\hat{C}_2 \max_{\hat{I}} |u|) \int_x^{x_*} C_1 t^{-1}\, dt \\ &= (\hat{C}_1 + C_1 \hat{C}_2\ \ell n(x_*/x)) \cdot \max_{\hat{I}} |u|. \end{aligned}$$

Since also $|u(x_*)| \leq \hat{C}_1 \max_{\hat{I}} |u|$ by (9), it follows that

$$
\begin{aligned}
|u(x_{\min})| &\leq |u(x_*)| + \int_{x_{\min}}^{x_*} |u'(x)|\, dx \\
&\leq \hat{C}_1 \max_{\hat{I}} |u| + (\max_{\hat{I}} |u|) \int_{x_{\min}}^{x_*} (\hat{C}_1 + C_1\hat{C}_2 \ \ell n(x_*/x))\, dx \\
&\leq (2\hat{C}_1 + x_* C_1 \hat{C}_2 \int_0^1 \ell n(1/t)\, dt) \cdot \max_{\hat{I}} |u|.
\end{aligned}
$$

We have $2\hat{C}_1 < \frac{1}{10}\hat{C}_2$ since we picked $\hat{C}_2 >> \hat{C}_1$, and $x_* C_1 \hat{C}_2 \int_0^1 \ell n(1/t)\, dt = \frac{1}{10}\hat{C}_2$ by our choice of x_*. Hence the previous estimate implies $|u(x_{\min})| \leq \frac{1}{5}\hat{C}_2 \max_{\hat{I}} |u|$, which contradicts $|u(x_{\min})| = \hat{C}_2 \max_{\hat{I}} |u|$. This contradiction proves (10). The conclusion of lemma 8 is contained in (9), (10). ■

Corollary. *If* $[\frac{d^2}{dx^2} + (E_0 - V(x))] = 0$ *on* I_{BVP} *and* u *is expressed as in lemma 5 on* I_{center}*, then we have*

$$
\max_{I_{\text{left}}} |u| \leq C_* |Q| (E_0 - V(x_{\text{ left}}))^{-1/4} \quad \textit{and}
$$
$$
\max_{I_{\text{rt}}} |u| \leq C_* |Q| \cdot (E_0 - V(x_{\text{right}}))^{-1/4}.
$$

Proof. Take $J = I_{\text{left}} \cup \hat{J}$ with $\hat{J} = [x_{\text{left}}, x_{\text{left}} + c_\# B(x_{\text{left}})]$. Then $\hat{J} \subset J$, $|\hat{J}| > c_* |J|$, and $|V(x) - E_0| \leq \frac{C_* |J|^{-1}}{|x - \min\ J|}$ on J. Rescaling this situation brings us to the setting of lemma 8, which implies the estimate $\max_J |u| \leq C_* \max_{\hat{J}} |u|$. Lemma 5 shows that $\max_{\hat{J}} |u| \leq C_\# |Q| \cdot (E_0 - V(x_{\text{left}}))^{-1/4}$, so we obtain the desired estimate for $\max_{I_{\text{left}}} |u|$. The estimate for $\max_{I_{\text{right}}} |u|$ is analogous and easier, since on I_{right} we make a stronger assumption on $|V - E_0|$. ■

We next study how u behaves on $I_{\text{far left}}$. (We will not need to understand $I_{\text{far right}}$.)

Lemma 9. *Let* $u(x)$ *be a real–valued solution of* $[\frac{d^2}{dx^2} + E_0 - V(x)]u = 0$ *on* I_{BVP} *with Dirichlet conditions, with* u *described by Lemma 5 on* I_{center}*. Then for* $x \in I_{\text{far left}}$

we have

$$|u(x)| \leq \frac{C_*|Q|}{(E_0 - V(x_{\text{left}}))^{1/4}} \exp(-c_*(x_{\text{left}} - x)/B(x_{\text{left}})). \tag{11}$$

Proof. The corollary of Lemma 8 shows that (11) holds for $x \in I_{\text{left}}$, in particular for $x = \min J_{\text{left}} = x_{\text{far left}}$. Now set $w = \pm u$ and

$$f(x) = \frac{C_*|Q|}{(E_0 - V(x_{\text{left}}))^{1/4}} \exp(-\tilde{c}_*(x_{\text{left}} - x)/B(x_{\text{left}})) - w(x)$$

with $\tilde{c}_* > 0$ to be picked in a moment. We know that $f(x_{\text{far left}}) \geq 0$, and also that $\lim_{x \to \min I_{\text{BVP}}} f(x) \geq 0$ since u satisfies Dirichlet boundary conditions.

Also, $f(x)$ is twice differentiable in the interior of $J_{\text{far left}}$, so either $f(x) \geq 0$ throughout $J_{\text{far left}}$ or else $f(x)$ has a strictly negative interior minimum at some point $x_* \in I_{\text{far left}}$. We assume the latter case and derive a contradiction. Thus we assume $f(x_*) < 0$, $f'(x_*) = 0$, $f''(x_*) \geq 0$. In particular $w(x_*) \geq \frac{C_*|Q|}{(E_0 - V(x_{\text{left}}))^{1/4}} \exp(-\tilde{c}_*(x_{\text{left}} - x_*)/B(x_{\text{left}})) > 0$ since $f(x_*) < 0$. Also $V(x) - E_0 \geq c_* B^{-2}(x_{\text{left}})$ on $I_{\text{far left}}$, by our basic assumptions on $V(x)$. Hence

$$w''(x_*) = (V(x_*) - E_0)w(x_*) \geq \frac{c_* B^{-2}(x_{\text{left}})|Q|}{(E_0 - V(x_{\text{left}}))^{1/4}} \exp(-\tilde{c}_*(x_{\text{left}} - x_*)/B(x_{\text{left}}))$$

whereas

$$\Big(\frac{C_*|Q|}{(E_0 - V(x_{\text{left}}))^{1/4}} \exp(-\tilde{c}_*(x_{\text{left}} - x)/B(x_{\text{left}}))\Big)'' = \frac{C_*|\tilde{c}_*|^2 B^{-2}(x_{\text{left}})|Q|}{(E_0 - V(x_{\text{left}}))^{1/4}} \cdot$$
$$\cdot \exp\Big(\frac{-\tilde{c}_*(x_{\text{left}} - x_*)}{B(x_{\text{left}})}\Big) \quad \text{at } x = x_*.$$

Taking $\tilde{c}_*$ small enough and subtracting, we find that $f''(x_*) < 0$, which contradicts our previous assumption $f''(x_*) \geq 0$.

This contradiction shows that $f(x) \geq 0$ in $I_{\text{far left}}$ for either choice of sign in the definitions of f, w. That is $|u(x)| \leq \frac{C_*|Q|}{(E_0 - V(x_{\text{left}}))^{1/4}} \exp(-\tilde{c}_*(x_{\text{left}} - x_*)/B(x_{\text{left}}))$, as asserted. ■

We summarize our knowledge of the eigenfunctions and eigenvalues in the following result.

Theorem 1. *Under the assumptions* $(\hat{H0})\dots(\hat{H7})$ *we have*

$$|(\textit{Number of eigenvalues of } H < E_0) - \frac{1}{\pi}\int_{I_{\mathrm{center}}} (E_0 - V(t))^{1/2}\,dt| \le C_*.$$

Suppose E_0 *is an eigenvalue of* H, *and suppose* u *is the corresponding eigenfunction, with* u *real–valued and having* L^2*–norm* 1 *on* I_{BVP}. *Then*

$$|u(x)|^2 \le C_*\Big(\int_{I_{\mathrm{center}}} (E_0 - V(t))^{-1/2}\,dt\Big)^{-1} (E_0 - V(x))^{-1/2} \textit{ for } x \in I_{\mathrm{center}}$$

$$|u(x)|^2 \le C_*\Big(\int_{I_{\mathrm{center}}} (E_0 - V(t))^{-1/2}\,dt\Big)^{-1} (E_0 - V(x_{\mathrm{left}}))^{-1/2} \exp\Big(\frac{-c_*(x_{\mathrm{left}} - x)}{B(x_{\mathrm{left}})}\Big)$$

for $x \in I_{\mathrm{BVP}}, x \le x_{\mathrm{left}} = \min I_{\mathrm{center}}$.

$$|u(x)|^2 \le C_*\Big(\int_{I_{\mathrm{center}}} (E_0 - V(t))^{-1/2}\,dt\Big)^{-1} (E_0 - V(x_{\mathrm{right}}))^{-1/2} \quad \textit{for } x \in I_{\mathrm{right}}.$$

As an application of this theorem, we study the following situation.

Set–up. $V(x)$ is a potential defined on a (possibly unbounded) interval I_{BVP}. We are given a subinterval $I \subset I_{\mathrm{BVP}}$ and weight functions $S(x)$, $B(x) > 0$ defined on I. Set $\lambda(x) = S^{1/2}(x)B(x)$. We are given an energy E_0.

We make the following assumptions.

Hypotheses

($\bar{H0}$) For $x, y \in I$ with $|x - y| < cB(x)$ we have $c < \frac{B(y)}{B(x)} < C$ and $c < \frac{S(y)}{S(x)} < C$, and $|I| > cB(x)$.

($\bar{H1}$) For $x \in I$ we have $|(\frac{d}{dx})^\alpha V(x)| \le C_\alpha S(x) B^{-\alpha}(x)$.

($\bar{H2}$) $\{x \in I_{\mathrm{BVP}} \mid V(x) < E_0\} = (x_{\mathrm{left}}, x_{\mathrm{right}}) \subset I$ with $\mathrm{dist}(x_{\mathrm{left}}, \partial I) > cB(x_{\mathrm{left}})$, $\mathrm{dist}(x_{\mathrm{right}}, \partial I) > cB(x_{\mathrm{right}})$.

($\bar{H3}$) In $[x_{\mathrm{left}}, x_{\mathrm{left}} + c_1 B(x_{\mathrm{left}})]$ we have $-V'(x) \ge cS(x_{\mathrm{left}})/B(x_{\mathrm{left}})$, and in $[x_{\mathrm{right}} - c_1 B(x_{\mathrm{right}}), x_{\mathrm{right}}]$ we have $+V'(x) \ge cS(x_{\mathrm{right}})/B(x_{\mathrm{right}})$.

($\mathrm{H\bar{4}}$) In $[x_{\text{left}} + c_1 B(x_{\text{left}}), x_{\text{right}} - c_1 B(x_{\text{right}})]$ we have $cS(x) < E_0 - V(x) < CS(x)$.

($\mathrm{H\bar{5}}$) In $I_{\text{BVP}}^{\text{interior}} \cap (-\infty, x_{\text{left}})$, $V(x)$ is decreasing and C^∞.

($\mathrm{H\bar{6}}$) $\Lambda = (\int_{x_{\text{left}}}^{x_{\text{right}}} \frac{dx}{\lambda(x)B(x)})^{-1}$ is bigger than a large positive number depending only on c, C, c_1, C_α in ($\mathrm{H\bar{0}}$)... ($\mathrm{H\bar{4}}$).

Let $H = -\frac{d^2}{dx^2} + V(x)$ on $L^2(I_{\text{BVP}})$ with Dirichlet boundary conditions.

Theorem 2. *Under assumptions ($H\bar{0}$)... ($H\bar{6}$) above, we have*

$$|(\textit{Number of eigenvalues of } H < E_0) - \frac{1}{\pi}\int_{I_{\text{BVP}}} (E_0 - V(t))_+^{1/2}\, dt| \leq C_*.$$

If E_0 is an eigenvalue of H, and if $u(x)$ (real–valued, with norm 1 in $L^2(I_{\text{BVP}})$) is the corresponding eigenfunction, then

$$|u(x)|^2 \leq C_* \Big(\int_{I_{\text{BVP}}} (E_0 - V(t))_+^{-1/2}\, dt \Big)^{-1} (E_0 - V(x))^{-1/2} \quad \textit{for } x_{\text{left}} < x < x_{\text{right}}$$

$$|u(x)|^2 \leq C_* \Big(\int_{I_{\text{BVP}}} (E_0 - V(t))_+^{-1/2}\, dt \Big)^{-1} \Big[S(x_{\text{left}}) \lambda^{-2/3}(x_{\text{left}}) \Big]^{-1/2} \cdot$$
$$\cdot \exp\Big(-c_* \lambda^{2/3}(x_{\text{left}}) \frac{(x_{\text{left}} - x)}{B(x_{\text{left}})} \Big) \quad \text{for } x \in I_{\text{BVP}} \cap (-\infty, x_{\text{left}}].$$

The constants c_*, C_* depend only on c, C, c_1, C_α, in ($H\bar{0}$)... ($H\bar{4}$).

Proof. We define intervals $\check{I}_{\text{far left}}$, $\check{I}_{\text{left}}$, $\check{I}_{\text{center}}$, $\check{I}_{\text{right}}$, $\check{I}_{\text{far right}}$ and weight functions $\check{S}(x)$, $\check{B}(x)$ so that theorem 1 applies. With a large constant $\check{C}$ to be picked later, we define the intervals and weights as follows.

$$\check{I}_{\text{far left}} = I_{\text{BVP}} \cap (-\infty, x_{\text{left}} - \frac{\check{C} B(x_{\text{left}})}{\lambda^{2/3}(x_{\text{left}})}]$$

$$\check{I}_{\text{left}} = \{ |x - x_{\text{left}}| < \frac{\check{C} B(x_{\text{left}})}{\lambda^{2/3}(x_{\text{left}})} \}$$

$$\check{I}_{\text{center}} = [x_{\text{left}} + \frac{\check{C} B(x_{\text{left}})}{\lambda^{2/3}(x_{\text{left}})}, x_{\text{right}} - \frac{\check{C} B(x_{\text{right}})}{\lambda^{2/3}(x_{\text{right}})}]$$

$$\check{I}_{\text{right}} = \{ |x - x_{\text{right}}| \leq \frac{\check{C} B(x_{\text{right}})}{\lambda^{2/3}(x_{\text{right}})} \}$$

$$\check{I}_{\text{far right}} = I_{\text{BVP}} \cap [x_{\text{right}} + \check{C} \frac{B(x_{\text{right}})}{\lambda^{2/3}(x_{\text{right}})}, \infty).$$

For $x \in \check{I}_{\text{center}}$, we take $\check{B}(x) = \min\{|x - x_{\text{left}}|, |x - x_{\text{right}}|, B(x)\}$ and set $\check{S}(x) = \frac{S(x_{\text{left}})}{B(x_{\text{left}})}|x - x_{\text{left}}|$ if $|x - x_{\text{left}}| < cB(x_{\text{left}})$, $\check{S}(x) = \frac{S(x_{\text{right}})}{B(x_{\text{right}})}|x - x_{\text{right}}|$ if $|x - x_{\text{right}}| < cB(x_{\text{right}})$, $\check{S}(x) = S(x)$ otherwise.

We check that hypotheses (H$\hat{0}$)...(H$\hat{7}$) hold. (H$\hat{0}$) and (H$\hat{1}$) are obvious. To check (H$\hat{2}$) we note that

$$\int_{\check{I}_{\text{center}}} \frac{dx}{\check{\lambda}(x)\check{B}(x)} \leq \int_{x_{\text{left}} + \frac{\check{C}B(x_{\text{left}})}{\lambda^{2/3}(x_{\text{left}})}}^{x_{\text{left}} + cB(x_{\text{left}})} \frac{dx}{\frac{S^{1/2}(x_{\text{left}})}{B^{1/2}(x_{\text{left}})}|x - x_{\text{left}}|^{5/2}} + \int_{x_{\text{left}} + cB(x_{\text{left}})}^{x_{\text{right}} - cB(x_{\text{right}})} \frac{dx}{\lambda(x)B(x)} + \int_{x_{\text{right}} - cB(x_{\text{right}})}^{x_{\text{right}} - \frac{\check{C}B(x_{\text{right}})}{\lambda^{2/3}(x_{\text{right}})}} \frac{dx}{\frac{S^{1/2}(x_{\text{right}})}{B^{1/2}(x_{\text{right}})}|x - x_{\text{right}}|^{5/2}}.$$

The first integral on the right is $\sim \frac{1}{\frac{S^{1/2}(x_{\text{left}})}{B^{1/2}(x_{\text{left}})}\left(\frac{\check{C}B(x_{\text{left}})}{\lambda^{2/3}(x_{\text{left}})}\right)^{3/2}} = \frac{1}{(\check{C})^{3/2}}$. The last integral on the right is $\sim \frac{1}{(\check{C})^{3/2}}$ as well, by a similar calculation. The second integral on the right is at most $\int_{x_{\text{left}}}^{x_{\text{right}}} \frac{dx}{\lambda(x)B(x)} = \Lambda^{-1}$. Hence $\int_{\check{I}_{\text{center}}} \frac{dx}{\check{\lambda}(x)\check{B}(x)} \leq \frac{C_*}{(\check{C})^{3/2}} + \Lambda^{-1}$, with C_* depending only on c, C, c_1, C_α in (H$\overline{0}$)...(H$\overline{7}$). Hence (H$\hat{2}$) holds if we take $\check{C}$ large enough, which we now do.

To check (H$\hat{3}$) we note that $\check{I}_{\text{center}} = [\check{x}_{\text{left}}, \check{x}_{\text{right}}]$ with $\check{x}_{\text{left}} = x_{\text{left}} + \frac{\check{C}B(x_{\text{left}})}{\lambda^{2/3}(x_{\text{left}})}$, $\check{S}(\check{x}_{\text{left}}) = \frac{S(x_{\text{left}})\cdot\check{C}}{\lambda^{2/3}(x_{\text{left}})}$, $\check{B}(\check{x}_{\text{left}}) = \frac{\check{C}B(x_{\text{left}})}{\lambda^{2/3}(x_{\text{left}})}$ so that $\check{\lambda}(\check{x}_{\text{left}}) = \check{S}^{1/2}(\check{x}_{\text{left}})\check{B}(\check{x}_{\text{left}}) = (\check{C})^{3/2}$. This verifies the desired a–priori bound on $\check{\lambda}(x_{\text{left}})$ and shows that $|\check{I}_{\text{left}}| = 2\check{B}(\check{x}_{\text{left}})$.

Analogous arguments work at x_{right}, completing the proof of (H$\hat{3}$). To check (H$\hat{4}$), (H$\hat{5}$), note that $|V(x) - E_0| \leq C_* \frac{S(x_{\text{left}})}{B(x_{\text{left}})}|x - x_{\text{left}}| \leq C_*\check{S}(\check{x}_{\text{left}})$ in $\check{I}_{\text{left}}$, and $\check{S}(\check{x}_{\text{left}}) = (\check{\lambda}(\check{x}_{\text{left}}))^2(\check{B}(\check{x}_{\text{left}}))^{-2} \leq C_*(\check{B}(\check{x}_{\text{left}}))^{-2} \leq C_*|\check{I}_{\text{left}}|^{-2}$. Here we used $\check{\lambda}(\check{x}_{\text{left}}) \leq C_*$, which we verified before. Thus $\max_{\check{I}_{\text{left}}} |V(x) - E_0| \leq C_*|\check{I}_{\text{left}}|^{-2}$. Similarly for $\check{I}_{\text{right}}$, completing the proofs of (H$\hat{4}$), (H$\hat{5}$).

To check (H$\hat{6}$), we use the fact that $V(x)$ is decreasing for $x < x_{\text{left}}$. Hence for $x \in \check{I}_{\text{far left}}$ we have $V(x) - E_0 \geq V(x_{\text{left}} - \frac{\check{C}B(x_{\text{left}})}{\lambda^{2/3}(x_{\text{left}})}) - E_0 \geq \frac{c_*S(x_{\text{left}})}{B(x_{\text{left}})}$ ·

$\frac{\check{C}B(x_{\text{left}})}{\lambda^{2/3}(x_{\text{left}})} = c_*\check{S}(\check{x}_{\text{left}}) = c_*(\check{\lambda}(\check{x}_{\text{left}}))^2(\check{B}(\check{x}_{\text{left}}))^{-2} \geq c_*|\check{I}_{\text{left}}|^{-2}$ since $\check{\lambda}(\check{x}_{\text{left}}) = (\check{C})^{3/2}$ and $|\check{I}_{\text{left}}| \sim \check{B}(\check{x}_{\text{left}})$. Also $V(x)$ is C^∞ on $\check{I}_{\text{far left}}$ which completes the proof of (H$\hat{6}$).

To check (H$\hat{7}$) we just note that $\{V < E_0\} = (x_{\text{left}}, x_{\text{right}})$, so $V(x) - E_0 \geq 0$ in $I_{\text{far right}}$. Thus (H$\hat{0}$)...(H$\hat{7}$) hold, so we can apply Theorem 1.

The number of eigenvalues of $H < E_0$ therefore differs by at most C_* from $\int_{\check{I}_{\text{center}}}(E_0 - V(t))^{1/2}\,dt$. Also $\int_{\check{I}_{\text{left}}}|E_0 - V(t)|^{1/2}\,dt \leq \int_{\check{I}_{\text{left}}}(C_*|\check{I}_{\text{left}}|^{-2})^{1/2}\,dt \leq C_*$, and similarly for $\check{I}_{\text{right}}$. Thus $\int_{\check{I}_{\text{center}}}(E_0 - V(t))^{1/2}\,dt$ and $\int_{I_{\text{BVP}}}(E_0 - V(t))_+^{1/2}\,dt$ differ by at most C_*. This proves the part of Theorem 2 on eigenvalues. Regarding eigenfunctions, the conclusions of Theorem 2 are obvious consequences of those of Theorem 1, once we note that $\int_{\check{I}_{\text{center}}}(E_0 - V(t))^{-1/2}\,dt$ has the same order of magnitude as $\int_{I_{\text{BVP}}}(E_0 - V(t))_+^{-1/2}\,dt$, and that $\check{B}(\check{x}_{\text{left}}) = \frac{\check{C}B(x_{\text{left}})}{\lambda^{2/3}(x_{\text{left}})}$. ■

When we apply our ODE results to three–dimensional potentials arising from an atom of nuclear charge Z, Theorem 2 will be used for angular momenta in the range (Large Const.) $< \ell < Z^\varepsilon$, and Theorem 1 will be used for angular momenta $\ell \leq$ (Large Const.). For angular momenta $\ell \geq Z^\varepsilon$, we use the much more refined WKB theory of the previous sections.

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