New Families of Distributions fitting L-moments for Modelling Financial Data

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Abstract

The classical problem in statistics of estimating an unknown distribution from a given a series of observations is approached from the point of view of interpolating primary features of the shape of any distribution. Unlike traditional approaches that aim at matching descriptive measures based on algebraic moments, we choose to match more robust statistics: the L-moments. Our main contribution is to present a new system of parametric families that are capable of interpolating an arbitrary finite set of L-moments. Our methodology is based on the representation of certain subsets of quantile functions by means of positive measures and the concept of entropy for making up for missing information. This approach also allows to incorporate additional constraints of a more qualitative nature such as unimodality. The calibration of these families is accomplished by simply matching sample L-moments. We justify the feasibility of this method almost surely, and discuss a numerically tractable algorithm for its implementation.

1 Introduction

A challenging problem in the area of risk management is finding distributions that resemble the complex random behavior exhibited by most financial data. It is a widely known that the gaussian model despite its simplicity and ease of calibration, can not account for empirical features observed in historical series such as asymmetries and fat tails.

It is common practice to increase the flexibility of location-scale families like the gaussian, by adding additional shape parameters. This is particularly so, when the divergent features of the sample can be described in terms of familiar concepts such as skewness or kurtosis, conveniently summarized by means of a given descriptive measure. The problem that arises then, is to build models with the flexibility of matching a wide range of these patterns, while keeping the variability of the estimates under acceptable levels.

The approach towards model building based on descriptive measures goes back to Karl Pearson and its discovery of the Pearsonian curves. It is well known that distributions in these families can be described and its parameters estimated, by means of the first fourth algebraic moments (see, for example, Pearson (1902), Johnson et al (1994)).

In Finance, a popular class of parametric models closely related to descriptive measures, have arisen from functional expansions. We start with a basic model such as the normal distribution, an add correction terms based on the algebraic moments of higher order. The Edgeworth and Gram-Charlier expansions, are probably the most widely known examples in this class. Applications of these families in Finance can be found in Jondeau and Rockinger (1998), Corrado and Su (1997).

As it was shown in previous work, descriptive measures based on algebraic moments fail to provide a reliable description of the shape of a distribution. Based on these results, one reasonably expects that models designed to have good explanatory power in the directions described by L-moments and related statistics, shall improve the accuracy and stability of standard models originally designed to describe features conveyed by conventional moments.

Therefore, we shall be interested in matching general descriptive measures that can be expressed as ratios $\frac{\tau_1}{\tau_2}$, where the functionals τ_i have the general form

$$\tau_i(F) = \int_0^1 F^{-1}(u) d\lambda(u)$$

and λ is a convenient real measure. The variety of descriptive measures that fall in the above class, increases our possibilities of modelling effectively data from different sources and patterns. We refer the readers to MacGillivray (1986) for a list, by no means exhaustive, of such measures.

We also shall be interested in constraints of a more qualitative nature, such as unimodality, the later being a stylized feature of most financial data. None of the examples mentioned above is consistent with this restriction.

2 The L-moments Inverse Problem.

The question of finding models with the capability of matching a finite number of descriptive measures, can be restated as a linear interpolation problem for quantile functions. We wish to estimate an unknown cdf F on the basis of a finite number of observed L-statistics,

$$\int_0^1 F^{-1}(u) J_i(u) du = c_i \quad i = 1, \ldots, k. \tag{1}$$

and the additional qualitative constraint: $F^{-1} \in \Psi$.

The constraint imposed by the set Ψ will typically depend on the specific context. Qualitative constraints usually act as an aid for reducing the inherent bias in parametric models. In a general setting, one wishes to obtain estimates $\hat{F}(x)$ that are smooth at the points of a given set. In certain instances it may be plausible to impose restrictions on the estimate such as unimodality, tail behavior, etc.

Traditional approaches for solving linear systems in infinite dimensional spaces fail to be satisfactory in this setting. Well known regularization procedures such as The Maximum Entropy Method, go as far as providing a positive function satisfying a finite of linear equations. They are not tailored to deal with the type of qualitative constraints imposed by the set Ψ. See Kay and Marple (1981) for a review of such methods.

Our approach is based on applying a simple linear transformation on the set Ψ so that it is mapped into a more tractable set, for which the above methods directly apply. This is described in the next section.

2.1 General Methodology

Let Ψ be a set of quantile functions satisfying the following representation constraint

(**RC**) There exists a parametric subclass $\Gamma_a = \{\chi_a(u),\ a \in A\}$, where A is a measurable subset of $\mathbb{R}^n$, such that every $Q \in \Psi$ admits the following representation

$$Q(u) = \int_A \chi_a(u) d\lambda(a) \tag{2}$$

for some positive Borel measure λ related to $Q(u)$.

The above condition resembles Choquet's Representation theorem for compact and convex subsets of a vector space (see, for example, Rudin (1985)). The interesting fact is that the property above holds for large non-compact sets Ψ of quantile functions, as it will be shown in the next sections.

Let M be the subset of positive measures associated to elements in Ψ through the above representation. Using the above representation, and assuming that the orders of integration can be changed (in fact, this assumption can be easily justified in all the cases that shall be considered), the constraints in (1) are transformed in

$$\int_A J_i^*(a) d\lambda(a) = c_i \quad i = 1, \ldots, k,$$
$$\lambda \in M\,,$$

where $J_i^*(a) = \int_0^1 J_i(u)\,\chi_a(u) du$.

The above equations establish an equivalence between the problem of finding an element of Ψ satisfying the constraints in (1) and the more familiar one of finding a positive measure $\lambda \in M$, satisfying a finite number of linear equations. The later is a problem which commonly arises in diverse areas of physics, engineering, and statistics (see, for example, Navaza (1986) and, Johnson and Shore (1984)). A popular technique is to choose the maximum entropy estimate, i.e., the solution of the optimization problem

$$\begin{aligned} &\text{minimize} \quad \int_A \phi(x(a)) d\upsilon(a) \\ &\text{subject to} \quad \int_A J_i^*(a)\, x(a) d\upsilon(a) = c_i \quad i = 1, \ldots, k, \\ &\quad 0 \le x(a) \in L_1(\upsilon) \end{aligned}$$

where $\phi : R \to (-\infty, +\infty]$ is a given convex function and υ is a reference finite measure on the set A.

Computationally feasible dual methods are readily available for solving this problem. These are described in Appendix A.

In the following sections we study the above inverse problem for well known sets of distributions under L-moments constraints. We show that the representation property holds in such cases and offer an explicit solution for the transformed problem. Our solution takes the form of a parametric family of quantile functions. The feasibility of the method of empirical L-moments for calibrating the parameters of such models is justified by means of a probabilistic argument. A simple and numerically tractable algorithm is presented.

3 L-moments Interpolation for smooth distributions.

In this section we address the L-moment inverse problem when the qualitative constraint is determined by the set

$\Psi^* = \{$All absolutely continuous distributions having finite expectation and having a continuous density $F' = \rho(x)$, that is strictly positive for all $x \in R.\}$

We review some elementary properties and definitions. Let $F(x)$ be the distribution function of a random variable X and $Q(u)$ be the associated quantile function defined by

$$Q(u) = F^{-1}(u) = \inf \{x : F(x) \geq u\}$$

$Q(u)$ is a right continuous and increasing function that completely determines $F(x)$. The opposite is also true: every increasing and right continuous function defined on (0,1) is the quantile function associated to a distribution function $F(x)$. When $F(x)$ is strictly increasing in its domain then $Q(u)$ is just the inverse function of $F(x)$.

Let Ψ denote the set of quantile functions of distributions in Ψ^*. The restrictions we wish to interpolate can be restated in a "quantile" setting as follows

1. $Q(u)$ has a continuous and strictly positive derivative.
2. $\lim_{u\to 0} Q(u) = -\infty$ and $\lim_{u\to 1} Q(u) = +\infty$
3. L-moments Interpolation,

$$l_r = \int_0^1 Q(u)P_{r-1}(u)du \quad r = 1, \ldots, k,$$

where $(l_1, \ldots, l_k)$ are L-moments of an unknown $\chi(u) \in \Psi$.

We shall denote by Ψ_0 the subset of Ψ satisfying the normalizing condition: $Q(1/2) = 0$. It follows that every $Q \in \Psi$ can be written as: $Q = Q_0 + \mu$, where $Q_0 \in \Psi_0$ and $\mu \in R$. From the orthogonality of the polynomials $P_{r-1}(u)$, we obtain: $L_r(Q) = L_r(Q_0)$ for all $r \geq 2$. The usefulness of this decomposition comes from the fact that the elements of the set Ψ_0 can be "represented" by means of positive measures, which makes it possible to solve the L-moment interpolation problem for the set Ψ_0. For these reasons we can safely restrict our attention to the set Ψ_0 and consider L-moment functionals of order greater than 1.

We shall require the following definition.

Definition 1. *A function $b(a) : (0,\, 1) \rightarrow R$ shall be called a growth function if*

i) $b(a)$ *is strictly increasing and continuous for all* $a \in (0,\, 1), \;\; a \neq 1/2$.

ii) $\lim_{a\to 1/2^+} b(a) = -\lim_{a\to 1/2^-} b(a) \neq 0$

iii) $b(a) < 0, \; a < \frac{1}{2}$ *and* $b(a) > 0, \; a > \frac{1}{2}$

iv) $\lim_{a\to 0} b(a) = -\infty$ *and* $\lim_{a\to 1} b(a) = +\infty$

The "extremal" elements in the set Ψ can be conveniently parametrized by means of a reference growth function $b(a)$. For each $a \in (0, 1)$, let $\chi_a(u)$ be the step function given by

$$\chi_a(u) = \begin{cases} b(a) \cdot 1_{[a,1]}(u) & \text{if } a \geq \frac{1}{2} \\ b(a) \cdot 1_{[0,a]}(u) & \text{if } a < \frac{1}{2} \end{cases}$$

The representation condition for the set Ψ_0 follows from the following more general result.

Theorem 1. *Let $Q(u)$ be a quantile function continuous at $\frac{1}{2}$ and satisfying $Q(\frac{1}{2}) = 0$. Then, there exists a unique positive measure λ on (0,1) such that the following representation holds:*

$$Q(u) \;=\; \int_0^1 \chi_a(u) d\lambda(a) \tag{3}$$

Further, $\lambda(A) \;<\; \infty$ for every compact set $A \subset$ (0,1).

Proof. Since $Q(u)$ is increasing and right-continuous on $(0,1)$, there exists a unique Borel positive measure μ such that the following relation holds:

$$Q(u_2) - Q(u_1) = \mu(]u_1, u_2]) = \int_{u_1}^{u_2} d\mu(a)$$

for all $0 < u_1 \leq u_2 < 1$. In particular, since $\mu(\,\{\frac{1}{2}\}\,) = 0$, by the continuity of Q we obtain

$$Q(u) = \int_{1/2}^{u} d\mu(a) \quad \text{for all } \; u \geq u_0$$

and

$$Q(u) = \int_{u}^{1/2} d\mu(a) \quad \text{for all } \; u < u_0$$

Now, taking $d\lambda(a) = \frac{1}{|b(a)|}\, d\mu(a)$, relation (3) follows.

Uniqueness of λ follows directly from uniqueness of μ and the fact that $|b(a)| > 0$ for all $a \in (0,1)$.

Finally,

$$\lambda(A) = \int_A \frac{1}{|b(a)|} d\mu(a) \leq \; \sup_{a \in A} b(a) \cdot \mu(A) \; < \infty.$$

□

From the above representation it follows immediately that when Q is absolutely continuous and strictly increasing, the associated spectral measure λ becomes absolutely continuous as well. Further, its density $\frac{d\lambda}{da}$ is related to Q' by means of the formula,

$$Q'(a) = \begin{cases} -b(a)\,\frac{d\lambda(a)}{da} & u < \frac{1}{2} \\ b(a)\,\frac{d\lambda(a)}{da} & u \geq \frac{1}{2} \end{cases}$$

almost everywhere in (0,1). The function $\frac{d\lambda(a)}{da}$ shall be termed the spectral density of $Q(u)$. From the above representation it follows immediately that the set Ψ_0 can be mapped into the set $\mathcal{M}_0$ of spectral densities given by

$$\mathcal{M}_0 = \{x : (0,1) \to R,\ \ x(a) > 0,\ \ x(a) \text{ continuous for all } a \in (0,1)\}.$$

As for the L-moment constraints we have the following result.

Corollary 1. *Suppose $Q(u) \in \Psi_0$ and $\int_0^1 Q(u)du < \infty$. Let λ be the spectral measure associated to $Q(u)$. Then the following equations hold:*

$$\int_0^1 P_{r-1}(u)Q(u)du \;=\; \int_0^1 P^*_{r-1}(a)\; d\lambda(a) \quad r \;=\; 1,\ldots,k,$$

where

$$P^*_{r-1}(a) \;=\; \begin{cases} \sum_{i=0}^{r-1} p_{i,r-1} \frac{(1-a^{i+1})b(a)}{(i+1)} & a > u_0 \\ \\ \sum_{i=0}^{r-1} p_{i,r-1} \frac{a^{i+1}b(a)}{(i+1)} & a < u_0 \end{cases}$$

Proof. The above equations follow from changing orders of integration, which is justified next. Using the representation equation we have

$$
\begin{aligned}
\int_0^1 P_{r-1}(u)Q(u)du &= \int_0^1 \left(\int_0^1 P_{r-1}(u)1_{[0,u]}b(a)\,d\lambda(a)\right) du \qquad (4)\\
&\leq \int_0^1 \left(\int_0^1 |P_{r-1}(u)\,1_{[0,u]}\,b(a)|\,d\lambda(a)\right) du\\
&= \int_0^1 |P_{r-1}(u)\,Q(u)|\,du\\
&\leq \int_0^1 |Q(u)|du \;<\; +\infty.
\end{aligned}
$$

Thus, we can apply Fubini's Theorem to change orders of integration in (4), which gives the result.

□

In view of the previous corollary the transformed interpolation problem becomes

$$
\mathcal{P}^* \qquad \begin{cases} \int_0^1 P^*_{r-1}(a)\,x(a)da \;=\; l_r \quad r \;=\; 2,\ldots,k \\ x \in \mathcal{M}_0, \end{cases} \tag{5}
$$

Under the feasibility constraint $(F.C)$,

$$
l_r \;=\; \int_0^1 P^*_{r-1}(a)\,x_0(a)da.
$$

for some $x_0(a) \in \mathcal{M}_0$.

Following the general methodology mentioned in section 2.1, we would solve the above problem and map the solution back to the Ψ_0.

Remark 1. *¿From elementary properties of the polynomials $P_r(a)$, it easily follows that the functions $P^*_r(a)$ are continuous in (0,1) for all $r \geq 0$. Furthermore, $P^*_r(a)$ will remain bounded on $[0,\,1]$ if and only if $\lim_{a\to 0^+} b(a)a < \infty$ and $\lim_{a\to 1^-} b(a)(1-a) < \infty$. In general, boundedness of the functions $P^*_r(a)$ becomes a constant requirement in almost any tractable interpolation procedure. From this point on, it shall be assumed that the growth function $b(a)$ satisfies the limiting conditions above.*

It becomes evident that $\mathcal{P}^*$, is still an ill-posed problem. In general, there will be an infinite number of positive functions that are consistent with a finite number of linear constraints, even if we impose stronger smoothness assumptions. To deal with this uncertainty, we employ a well known regularization technique: the maximum entropy method.

Definition 2. *An entropy function* $\phi(x) : R \to R \cup \{+\infty\}$ *is a proper lower semi-continuous function.*

More details about this concept can be found in Appendix A.

In analogy with the Maximum Entropy Principle, we choose a solution $x(a)$ of $\mathcal{P}^*$ which maximizes a given measure of entropy, $H_\phi(x(a))$, or equivalently, minimizes the corresponding information measure, $I_\phi(x(a)) = -H_\phi(x(a))$, which is defined as

$$I_\phi(x(a)) = \int_0^1 \phi(x(a))da.$$

Information measures most frequently encountered in practice include: the Boltzman-Shanon, Burg, and Fermi-Dirac information measures, defined respectively by

$$\text{Boltzman-Shannon} \begin{cases} x\ln(x) & ,x \geq 0 \\ +\infty & ,x < 0 \end{cases} \qquad \text{Burg} \begin{cases} \ln(x), & ,x > 0 \\ +\infty & ,x \leq 0 \end{cases}$$

$$\text{Fermi-Dirac} \begin{cases} (x-a)\ln(x-a) + (b-x)\ln(b-x) & ,a < x < b \quad a,\, b > 0 \\ (b-a)\ln(b-a) & ,x = a \text{ or } b \\ +\infty & \text{otherwise} \end{cases}$$

We refer to Borwein and Lewis (1991) for an extensive list of such measures.

In the present context, the above optimization problem can be formulated as follows

$$(\mathcal{P}') \qquad \begin{cases} \text{minimize } \int_0^1 \phi(x(a))da \\ \text{subject to} \int_0^1 P^*_{r-1}(a)x(a)da = l_r, \quad r = 2, \ldots, k, \\ x(a) \in \mathcal{M}_0. \end{cases}$$

3.1 Existence of solutions

Solvability conditions for the problem $\mathcal{P}'$ can be derived from a well known result in Partial Linear Programming Theory due to Borwein and Lewis (1991), which we briefly summarize next. We refer to Appendix A for a more comprehensive exposition of this result.

Consider the following primal problem

$$\begin{cases} \text{minimize } \int_T \phi(x(a)da \\ \text{subject to} \int_T x(a)\theta_i(a)da \;=\; l_i, \quad i = 1, \ldots k, \\ 0 \leq x(a) \;\in\; L_p[\alpha, \beta], \; p \geq 1. \end{cases}$$

and its associated dual,

$$\begin{cases} \text{maximize } \sum_{i=1}^{k} \lambda_i l_i \;-\; \int_T (\phi)^* \{ \sum_{i=1}^{k} \lambda_i \theta_i(a) \} da \\ \lambda \;\in\; \mathbb{R}^k \end{cases}$$

Assume that the following conditions hold:

A. The functions $\theta_i(a) \in L_q[\alpha, \beta]$ are locally Lipchitz (or in particular continuously differentiable).

B. There exists a feasible $e(a) \in \text{dom } I_\phi$ with $e(a) \geq 0$ almost everywhere. In particular, for this assumption to be met it is sufficient that ess sup $x(a) < +\infty$ and ess inf $x(a) > 0$.

C. There exists $\mu \in \mathbb{R}^n$ with

$$\sum_{i=1}^{n} \mu_i \theta_i(a) \;\leq\; d \qquad \forall\, a \in\; [\alpha, \beta],$$

where $d = \lim_{x \to \infty} (\phi(x)/x)$, and if $d < \infty$ define $c = \lim_{y \to d} (d - y)((\phi)^*)'(y)$. Suppose that either $d = \infty$ or $d < \infty$ and $c \geq 0$.

D. The entropy function ϕ satisfies an additional number of regularity conditions (reviewed in Appendix A).

Under the above assumptions, Borwein and Lewis showed the existence of a unique primal optimal solution x^*(a) given by

$$x^*(a) = (\phi^*)'\{\sum_{i=1}^{k} \lambda_i \theta_i(a)\}, \quad (6)$$

where $\lambda = (\lambda_1, \dots, \lambda_k)$ is a dual optimal solution and ϕ^* denotes the Fenchel conjugate of ϕ.

In order to clarify some points let us assume that the assumptions above hold in our setting.

Remark 2. *First, let us focuss on the L-moments constraint of problem $\mathcal{P}^*$. Although condition* $\boldsymbol{B}$ *might seem a little stronger than the feasibility condition imposed in $\mathcal{P}^*$, it turns out that both conditions are equivalent. This fact will be proved in the next section.*

Remark 3. *If we choose an entropy functional ϕ for which $(\phi^*)'$ is continuous and strictly positive, then from remark 1 of section 3 we can guarantee the continuity of the estimated spectral density x^*. Now, letting x^* play the role of the spectral measure in equation (3), we shall recover a quantile function $Q^* \in \Psi_0$, satisfying the L-moments constraints.*

3.2 Solvability Conditions

In this section we prove the existence of the solvability conditions needed in Borwein and Lewis theorem. We begin by proving a statement already mentioned without proof, in remark 2 of the previous section, which implies the feasibility of condition **B**.

We shall denote by $\mathcal{C}^+[0,1]$ the set of continuous and strictly positive functions on $[0,1]$. The symbol $\mathcal{M}_b$ will represent the subset of $\mathcal{M}_0$, of bounded measures. We shall require some other definitions and lemmas.

Definition 3. *A set of measurable functions $\theta_i : T \to R, \ \ i = 1, \dots, k$ on a measure space T are called pseudo-Haar if they are linearly independent on every nonnull subset of T.*

Lemma 1. *The functions $P_r^*, \ \ r = 0, \dots, k$, are pseudo-Haar in the interval $[0,1]$ with Lebesgue measure.*

Proof. We will show that the functions are pseudo-Haar on the interval $[\frac{1}{2}, 1]$, the proof being similar for the case $[0, \frac{1}{2}]$. The result then will follow immediately.

Suppose that there exists a set A, $\nu(A) > 0$ and a nonnull vector $\alpha = (\alpha_k, \ldots, \alpha_k)$ such that: $\sum_{r=1}^{k} \alpha_r \cdot P_r^*(a) = 0$ for all $a \in A$. Since $b(a)$ is strictly positive the last equation becomes:

$$\sum_{r=1}^{k} \alpha_r \cdot (\sum_{i=0}^{r-1} p_{i,r-1} \frac{(1 - a^{i+1})}{i+1}) = 0, \tag{7}$$

for all $a \in A$. Since the function above is a polynomial of degree k, it can have at most k roots. This fact contradicts (7).

□

Definition 4. *Given a set of measurable functions $\{h_1, \ldots, h_k\}$ and a subset $\mathcal{M}^*$ of positive measures, we shall denote by $\mathcal{R}(\mathcal{M}^*)$ the subset of $\mathbb{R}^k$*

$$\mathcal{R}(\mathcal{M}^*) = \{(l_1, \ldots, l_k) \in \mathbb{R}^k : \ l_r = \int_0^1 h_r(a) d\lambda(a) : \ \lambda \in \mathcal{M}^*\}.$$

Lemma 2. *Let $P_1^*, \ldots, P_k^*$ be the constraint functions for the transformed L-moment problem. Then*

(1) $\mathcal{R}(\mathcal{M}_b)$ is dense in $\mathcal{R}(\mathcal{M}_0)$.

(2) $\mathcal{R}(\mathcal{C}^+[0, 1])$ is dense in $\mathcal{R}(\mathcal{M}_b)$.

Proof. Let $x \in \mathcal{M}_0$ and define the sequence $x_m \in \mathcal{M}_b$,

$$x_m(a) = \begin{cases} m & x(a) > m \\ x(a) & x(a) \leq m. \end{cases}$$

The sequence x_m converges almost surely to the function x. Further, $x_m \leq x$. Applying Fubini's Theorem (see also proof of corollary 1) gives,

$$\begin{aligned}\int_0^1 |P_r^*(a)|x_m(a)da &\leq \int_0^1 |P_r^*(a)|x(a)da \\ &= \int_0^1 |P_r(u)\,Q(u)|du \\ &\leq \int_0^1 |Q(u)|du \\ &< +\infty,\end{aligned}$$

where $Q(u)$ stands for the quantile function having spectral density x. By the dominated convergence theorem, $\int_0^1 x_m(a)P_r^*(a)da \to \int_0^1 x(a)P_r^*(a)da$ as $m \to \infty$, for each $r = 1, \ldots, k$. This proves (1).

Statement (2) follows from the fact that $\mathcal{C}^+[0,1]$ is a dense subset of the set of positive and bounded measures, in the weak topology. See, for example, Parthasaraty (1980).

□

Lemma 3. *Let A be a convex set in $\mathbb{R}^n$ with $\operatorname{int} A \neq \emptyset$. Let C be a convex set that is dense in $\operatorname{int} A$. Then, $\operatorname{int} A \subseteq C$.*

Proof. Let $x \in \operatorname{int} A$ and $V_\delta(x)$ be a ball centered at x with radius $\delta > 0$, contained in $\operatorname{int} A$. It follows that the set $C' = V_\delta(x) \cap C$ is a convex and dense subset of $V_\delta(x)$. We will show that $x \in C'$.

Without loss of generality, let us put $x = 0$. Define V_δ^{n-1} to be the set given by

$$V_\delta^{n-1} = \{(x_1, \ldots, x_n) \in V_\delta(0) : \ x_1 = 0\}.$$

We claim that $C' \cap V_\delta^{n-1}$ is dense in V_δ^{n-1}. For any point $\mathbf{x} \in V_\delta^{n-1}$, we can find two points $\mathbf{y}, \mathbf{z} \in V_\delta(0)$ with $y_1 = -z_1 > 0$ such that: $\mathbf{x} = \frac{1}{2}\mathbf{y} + \frac{1}{2}\mathbf{z}$. Now taking sequences $\{\mathbf{y_m}\}$ and $\{\mathbf{z_m}\}$ in C' such that: $\mathbf{y_m} \to \mathbf{y}$ and $\mathbf{z_m} \to \mathbf{z}$, it follows that for n sufficiently large, the sequence

$$\mathbf{x_m} = \frac{-z_{1m}}{y_{1m} - z_{1m}}\mathbf{y_m} + \frac{y_{1m}}{y_{1m} - z_{1m}}\mathbf{z_m}$$

is contained in $C' \cap V_\delta^{n-1}$. Clearly, $\mathbf{x_m} \to \mathbf{x}$.

Similarly it can easily argued that the intersection of C' with the convex sets

$$V_\delta^{n-k} = \{(x_1, \ldots, x_n) \in V_\delta(0) :\ x_1 = 0,\, x_2 = 0, \ldots, x_k = 0\}$$

are dense in V_δ^{n-k} for each $1 \leq k \leq n$, respectively. This gives the result.

□

Proposition 1. *Let $x(a) \in \mathcal{M}_0, x \neq 0$ on a non-null subset T. Define the vector $\boldsymbol{c} = (c_1, \ldots, c_k)$ where, $c_r = \int_0^1 P_r^*(a)x(a)da$. Then, $\boldsymbol{c} \in int\,\mathcal{R}(\mathcal{M})$*

Proof. Since $x(a) \neq 0$ on a non-null subset T, there exists $\delta > 0$ and non-null subset T_1 such that $x(a) \geq \delta$ for all $a \in T_1$. We claim

$$\left\{(l_1, \ldots, l_k) \in \mathbb{R}^k :\ l_r = \int_0^1 P_r^*(a)y(a)da,\ \ y(a) \in L_\infty(T_1)\right\} = \mathbb{R}^k.$$

Suppose not. Since the above subset is clearly a subspace, it follows that for all $y \in L_\infty(T_1)$ there exists a non-zero vector $(\alpha_1, \ldots, \alpha_k)$ such that

$$\int_{T_1} (\sum_{r=1}^{k} \alpha_r P_r^*(a)\,)y(a)da = 0.$$

Clearly this implies that $\sum_{r=1}^k \alpha'_r P_r^*(a) = 0$ almost everywhere in T_1 for some non-null vector $(\alpha'_1, \ldots, \alpha'_k)$. This contradicts the fact that the functions P_r^* are pseudo-Haar.

Let V_δ and B be the sets

$$V_\delta = \{y(a) \in L_\infty(T_1) :\ ||y||_\infty \leq \tfrac{\delta}{2}\}$$

$$B = \{(b_1, \ldots, b_k) \in \mathbb{R}^k :\ b_r = \textstyle\int_0^1 P_r^*(a)y(a)da,\ \ y \in V_\delta\}$$

respectively. Since $P\,B = \mathbb{R}^k$, it follows immediately that 0 is contained in int B.

For $y \in V_\delta$ define its extension to $L_\infty[0,1]$ as

$$y^*(a) = \begin{cases} y(a) & a \in T_1 \\ 0 & \text{otherwise} \end{cases}$$

We shall keep the same notation V_δ for the extended set. Note the equality: $B = \mathcal{R}(V_\delta)$.

Now, define the sequence $x_m \in L_\infty[0,1]$ as

$$x_m(a) \;=\; \begin{cases} \frac{1}{m} & x(a) < \frac{1}{m} \\ x(a) & \frac{1}{m} \le x(a) \le m \\ m & x(a) > m. \end{cases}$$

The sequence of functions $x_m(a)$ converges almost surely in $[0,1]$ to the function $x(a)$. Further, $x_m(a) \le \min\{x(a),1\}$. Let $K \ge |P_r^*(a)|$ for all $r = 1,\ldots,k$. Now, Fubini's Theorem gives (see also proof of corollary 1)

$$\begin{aligned}
\int_0^1 |P_r^*(a)| \min\{x(a),1\} \;&\le\; \int_{\{a:x(a)<1\}} |P_r^*(a)|da \;+\; \int_0^1 x(a)|P_r^*(a)|da \\
&\le\; K + \int_0^1 |P_r(u)\,Q(u)|du \\
&<\; \infty\,,
\end{aligned}$$

where Q stands for the quantile function having spectral density x. By the dominated convergence theorem, $\int_0^1 x_m(a)P_r^*(a)da \to c_r$ as $m \to \infty$, for each $r = 1,\ldots,k$. For simplicity let us put: $\mathbf{c}_m = (\,\int_0^1 x_m(a)P_1^*(a)da\,,\ldots,\,\int_0^1 x_m(a)P_k^*(a)da\,)$.

Since $0 \in \operatorname{int}\mathcal{R}(V_\delta)$, there exists $m \ge 1/\delta$, such that $\mathbf{c} - \mathbf{c}_m \in \operatorname{int}\mathcal{R}(V_\delta)$. Thus, it follows that $\mathbf{c} \in \operatorname{int}\{\mathcal{R}(V_\delta) + \mathbf{c}_m\}$. Notice that the last set equals $\mathcal{R}(V_\delta + x_m)$. We claim that $\{V_\delta + x_m\} \subseteq \mathcal{M}$, from which the result will follow.

To see this, note that if $y \in V_\delta$ we have: $y(a) + x_m(a) \ge \delta/2$, for all $a \in T_1$; on the other hand, $x_m(a) + y(a) = x_m(a) \ge 1/m$ for $a \in T_1^c$.

The proof is complete. □

The above proposition combined with lemmas 2 and 3 guarantees the existence of a continuous and strictly positive density that is consistent with the constraints of problem $\mathcal{P}^*$. Therefore, assumption **B** is satisfied.

We turn our attention to the other assumptions. Condition **A** follows straightforward by definition of $P_r^*(a)$. Conditions **C** and **D** can be easily checked for many common choices of entropy functions. In particular they hold for the examples mentioned in section 3.

One last remark. Although the results obtained in this section were stated for the set $\mathcal{M}_0$, the proofs actually apply for a wider range of other subsets of $\mathcal{M}$. In particular, they apply to any convex subset contained in the set of non-null spectral densities associated to quantile functions with finite expectation.

3.3 Parametric Models

In this section we discuss some properties of the parametric model recovered from the family of maximum entropy spectral measures, letting the vector λ vary freely in $\mathbb{R}^{k-1}$.

For a general set of parameters $(\lambda_1, \dots, \lambda_{k-1}) \in \mathbb{R}^{k-1}$, the resulting parametric family $\Gamma(\lambda_1, \dots, \lambda_{k-1}) \in \Psi_0$ will take the general form

$$Q(u; \lambda_1, \dots, \lambda_{k-1}) \;=\; \begin{cases} \int_{1/2}^{u} b(a)\,(\phi^*)'\{\,\sum_{r=1}^{k-1} \lambda_r P_r^*(a)\}\,da & u \geq \frac{1}{2} \\ \\ \int_{u}^{1/2} b(a)\,(\phi^*)'\{\,\sum_{r=1}^{k-1} \lambda_r P_r^*(a)\}\,da & u < \frac{1}{2} \end{cases} \tag{8}$$

As it was mentioned in section 3, the original interpolation problem in the set Ψ is solved by considering the extension

$$\Gamma(\mu, \lambda_1, \dots, \lambda_{k-1}) = \{\,\Gamma(\lambda_1, \dots, \lambda_{k-1}) + \mu\,\} \tag{9}$$

and simply solving the L-moments system for λ_r first, and then choosing the parameter μ that matches the constraint given by the mean, which in fact, could be replaced by any other measure of location.

In order to summarize these results more conveniently, we introduce the following definition.

Definition 5. *Given a set of functionals $T = \{T_1, \dots, T_k\}$, we shall denote by $\mathcal{R}(\Lambda)$, the range of the vector $(T_1(Q), \dots, T_k(Q))$ over a set Λ of quantile functions, i.e.,*

$$\mathcal{R}(\Lambda) \;=\; \{\,(T_1(Q), \dots, T_k(Q)) \in \mathbb{R}^k : \; Q \in \Lambda\}$$

Theorem 2. *Let $L = \{L_2, \ldots, L_k\}$ be L-moments functionals. For the family $\Gamma(\lambda_1, \ldots, \lambda_{k-1})$, we have*

1) $\Gamma(\lambda_1, \ldots, \lambda_{k-1}) \subseteq \Psi_0$.

2) $\mathcal{R}(\,\Gamma(\lambda_1, \ldots, \lambda_{k-1})\,) \;=\; \mathcal{R}(\Psi_0)$.

3) There exists a unique correspondence between the vectors $(\lambda_1, \ldots, \lambda_{k-1}) \in \mathbb{R}^{k-1}$ *and* $(l_2, \ldots, l_k) \in \mathcal{R}(\Psi_0)$.

If we add any functional of location $\mu(Q)$ to the system L, then the above assertions also hold for the family $\Gamma(\mu, \lambda_1, \ldots, \lambda_{k-1})$ replacing Ψ_0 by Ψ.

Some examples of entropy functions satisfying the continuity requirement include

1. Boltzman-Shannon entropy

   $\phi^*(y) = \exp(y-1)$

   $x^*(a) = \exp\{\sum_{i=1}^{k} \lambda_n \theta_i(a)$

2. Fermi-Dirac entropy

   $\phi^*(y) = cy + (d-c)\ln 1 + \exp^y - (d-c)\ln d - c$

   $x^*(a) = (c + d\,\exp\{\sum_{i=1}^{k} \lambda_i \theta_i(a)\})\,/\,(1 + \exp\{\sum_{i=1}^{k} \lambda_i \theta_i^*(a)\}\,)$

   $c,\, d > 0.$

Unfortunately the Burg measure, despite its theoretical appeal, is not suitable in this sense since $(\phi^*(y))' = \frac{1}{y}$ has a singularity at the origin.

Remark 4. *The choice of the entropy functional, though being a relevant issue, is out of the scope of this paper. The debate over the relative merits of the various entropies has been as intense as controversial (see, for example, Johnson and Shore(1984)). The issues in this debate can be grouped into two rather distinct areas. The first might be termed a priory reasons for* *selecting a particular entropy, generally involving a probabilistic, statistical, or information-theoretic discussion for the particular phenomenon we seek to model (see, for example, Nityananda and Narayan (1982), and Navaza (1986)). The second area of debate is empirical: the performance of the entropy functional is judged by its ability to reconstruct the unknown object on the basis of the partial information that is known (see, for example, Skilling and Gull (1984)). We believe that the second approach is closer in spirit to one of the highest goals in statistical data modelling: to find a distribution that provides a satisfactory fit of the observed data.*

Remark 5. *The above theorem also reveals the numerical feasibility of the entropy regularization approach. First, for the most common choices of the entropy function ϕ, the Fenchel conjugate ϕ^* can be computed explicitly, consequently, the estimated densities can be obtained in a closed form. Second, the unknown values of the parameters $(\lambda_1, \ldots, \lambda_k)$ arises as the solution of a concave optimization problem in a finite dimensional space without restrictions, for which a range of well known numerically feasible procedures could be used.*

¿From this point on, we shall adopt the Shanon entropy for ease of exposition, but clearly any other feasible entropy satisfying the regularity conditions stated in Borwein and Lewis's Theorem could be used instead.

Asymptotic Decay. By definition of the family $\Gamma(\lambda_1, \ldots, \lambda_{k-1})$ it follows

$$\frac{\partial\, \Gamma(\lambda_1, \ldots, \lambda_{k-1})}{\partial u} \;=\; b(u)\, \exp\{\sum_{r=1}^{k-1} \lambda_i P_r^*(u)\}.$$

Thus the asymptotic behavior of the family is proportional to that of any quantile function having derivative $b(a)$, independently of the vector of parameters $(\lambda_1, \ldots, \lambda_{k-1})$.

3.4 An alternative scale invariant family

An undesirable feature of the models considered in the previous section, lies in the fact that they are not scale invariant, even though they can match any value of scale statistics such as the second l-moment. The reason for this apparent anomaly, comes from the fact that the scale parameter is considered as a non-linear parameter in the model. Based on this observation, one would like to modify this class of models slightly, to add the scale-invariance property without loosing, of course, the interpolation capability of the model. To deal with this issue we introduce a location-scale invariant version of the family $\Gamma(\mu, \lambda_1, \ldots, \lambda_{k-1})$.

Let Ψ_0' be the set

$$\Psi_0' = \{\, Q \in \Psi_0 : \; L_2(Q) = 1 \,\}.$$

Definition 6. *We shall denote by $\Gamma(\mu, l_2, \lambda_1, \ldots, \lambda_{k-1})$ the location-scale invariant family given by*

$$\Gamma(\mu, l_2, \lambda_1, \ldots, \lambda_{k-1}) = \{ l_2 \cdot Q(u; \lambda_1, \ldots, \lambda_{k-1}) \, + \, \mu \} \, ,$$

where $Q(u; \lambda_1, \ldots, \lambda_k) \subset \Psi_0'$ is the parametric subfamily given by

$$Q(u; \lambda_1, \ldots, \lambda_{k-1}) = \begin{cases} \int_{1/2}^{u} b(a) \exp\{\sum_{r=1}^{k-1} \lambda_r P_r^*(a) \} da & u \geq 1/2 \\ \\ \int_{u}^{1/2} b(a) \exp\{\sum_{r=1}^{k-1} \lambda_r P_r^*(a)\} da & u \, < \, 1/2 \end{cases}$$

where the parameters $\lambda_1, \ldots, \lambda_{k-1}$ are constrained to satisfy the "normalization" condition:

$$L_2(Q(u; \lambda_1, \ldots, \lambda_{k-1}) \,) \; = \; 1.$$

Remark 6. *The parameters μ and l_2 are parameters of location and scale respectively. Therefore, their range of values is the usual one for this type of parameters. Indeed, l_2 and μ agree with the second L-moment and the median of the distribution, respectively. Other features of the shape, namely those summarized by the functionals $L_3(Q), \ldots, L_k(Q)$, are described by the vector, $\lambda = (\lambda_1, \ldots, \lambda_{k-1})$. The normalization condition allows us to separate higher order features, such as skewness and fat tails, from any particular value of the parameter of scale l_2. The range of feasible values for the vector $(\lambda_1, \ldots, \lambda_{k-1})$ is constrained only by the normalization equation.*

A descriptive justification for the above model is given next.

Theorem 3. *For the family $\Gamma(\mu, l_2, \lambda_1, \ldots, \lambda_{k-1}$ the following statements hold* *(a)* $\mathcal{R}(\Gamma(\mu, l_2, \lambda_1, \ldots, \lambda_{k-1}) \;=\; \mathcal{R}(\Psi)$.

(b) The L-moments equations

$$L_r(\Gamma(\mu, l_2, \lambda_1, \ldots, \lambda_{k-1})) \;=\; l_r, \quad r = 1, \ldots, k$$

establish a unique correspondence between the L-moments vectors $(l_1, \ldots, l_k) \in \mathcal{R}(\Psi_0)$ *and the parameters of the model.*

Proof. Let $Q \in \Psi$ and let $(l_1, \ldots, l_k)$ be the vector of L-moments of Q. Define $Q_0(u) = Q(u) - u_0/l_2$. Then to prove both (a) and (b), it is sufficient to show that the system

$$\begin{aligned} L_2(Q(u; \lambda_1, \ldots, \lambda_{k-1})) \;&=\; 1 \\ L_r(Q(u; \lambda_1, \ldots, \lambda_{k-1})) \;&=\; l_r, \quad r = 3, \ldots, k \end{aligned} \tag{10}$$

has a unique solution $(\lambda_1^*, \ldots, \lambda_{k-1}^*)$. By corollary 1, this last assertion becomes equivalent to showing that the transformed equations

$$\int_0^1 P_1^*(a) \exp\{\textstyle\sum_{r=1}^{k-1} \lambda_r P_r^*(a)\}\, da = 1$$

$$\int_0^1 P_{r-1}^*(a) \exp\{\textstyle\sum_{r=1}^{k-1} \lambda_r P_r^*(a)\}\, da = l_r \quad r = 3, \ldots, k.$$

have the unique solution $(\lambda_1^*, \ldots, \lambda_{k-1}^*)$. This fact follows as a direct consequence of Borwein and Lewis theorem. The proof is complete.

□

The new family preserves the good interpolating properties of the family $\Gamma(\mu, \lambda_1, \ldots, \lambda_{k-1})$, in the sense that the statements of theorem 2 continue to hold. An attractive feature of the above model comes from its flexibility to extend arbitrary location-scale families in the directions summarized by the higher order L-moments. This is possible due to the flexibility in choosing the growth function $b(a)$. To see this assertion, consider a general location-scale family $\Im \;=\; \{l_2 \cdot Q_0 \;+\; \mu\}$ where Q_0 satisfies: $Q_0(\frac{1}{2}) = 0$ and $L_2(Q_0) = 1$. The family $\Im$ is recovered by choosing $b(a)$ as,

$$b(a) \;=\; \begin{cases} \frac{Q_0'(a)}{e} & a \geq \frac{1}{2} \\ -\frac{Q_0'(a)}{e} & a < \frac{1}{2} \end{cases}$$

and setting $\lambda_r = 0 \;\; r \;=\; 1, \ldots, k-1$.

4 Parameter Estimation

4.1 Feasibility with empirical L-moments

In practice the L-moments values are not given, and must be estimated possibly from a series of observations generated by an unknown distribution F. The fact that sample L-moments are computed from the empirical quantile function $Q_n(u)$, which obviously is not a member of Ψ, invalidates the feasibility constraint imposed in the deterministic setting. In this section we address the question of interpolating L-moments with high probability.

Let $X_1, \ldots, X_n$ be an iid sample from a random variable X with unknown distribution function F. We shall denote the higher order sample L-moments by $\hat{l}_r$. By definition

$$\hat{l}_r \;=\; L_r(Q_n) = \sum_{i=1}^{n} p_{i:n} X_{i:n} \tag{11}$$

with weights $p_{i:n} = \frac{1}{n}\int_{(i-1)/n}^{i/n} P_{r-1}(u)du$.

Let $b(a)$ be a growth function that shall be kept fixed through the following discussion. We shall denote by $\mathcal{C}^+[0,1]$ the set of continuous and strictly positive functions $x(a)$ on $[0,1]$.

The feasibility of the L-moments estimation method to calibrate the parameters of the families considered in previous sections, is justified in our main result, which is based on the following proposition.

Proposition 2. *Let $X_1, \ldots, X_n$ be an iid sample from a distribution $F \in \Psi_0$. Let $\Lambda \subset \Psi_0$ be the set of distributions having a spectral density $x(a)$ in $\mathcal{C}^+[0,1]$. Then,*

$$(\hat{l}_2, \ldots, \hat{l}_k) \in \mathcal{R}(\Lambda) \text{ almost surely for all } n \geq k-1.$$

For the proof we shall require the following lemmas.

Lemma 4. *Let Λ' be the set quantile functions continuous at $u_0 = \frac{1}{2}$, and having a bounded spectral measure. Then, $\mathcal{R}(\Lambda)$ is dense in $\mathcal{R}(\Lambda')$.*

Proof. Let $Q(u) \in \Lambda'$ with L-moments $(l_1, \ldots, l_k)$. Note that for our purposes, we can safely exclude the functional of location L_1 from the analysis and assume that $Q(1/2) = 0$. Let λ be its associated spectral measure. From corollary 1,

$$l_r \; = \; \int_0^1 P^*_{r-1}(a)\, d\lambda(a) \quad r \; = \; 2, \ldots, k$$

By standard results, we know that $\mathcal{C}^+[0,1]$ is a dense subset of the set of positive and bounded measures in the weak topology. From this, the result follows.

□

Proof of proposition 2.

Without loss of generality we can exclude the location functional L_1 from the analysis. We define the set $\mathcal{X}$ and the linear application $L : \mathcal{X} \to \mathbb{R}^{k-1}$

$$\mathcal{X} = \{ (x_1, \ldots, x_n) \in \mathbb{R}^n : \; x_1 \leq x_2 \leq \ldots x_n) \}$$

and

$$L(x_1, \ldots, x_n) \; = \; (L_2(X), \ldots, L_k(X)\,), \;\; \text{where} \;\; L_r(X) = \sum_{i=1}^{n} p_{i:n} x_i$$

respectively, where $p_{i:n}$ are given in 11. The set $\mathcal{X}$ is convex and satisfies, $\text{int}(\mathcal{X}) \neq \emptyset$.

Since by assumption $F' > 0$, it follows by elementary properties of order statistics (see, for example, Davis (1981)) that the joint distribution of $X_{1:n}, \ldots, X_{n:n}$, will also be absolutely continuous with a strictly positive density. Consequently,

$$(X_{1:n}, \ldots, X_{n:n}) \; \in \; \text{int}\, \mathcal{X} \; \text{ almost surely.} \tag{12}$$

We claim that $L(\mathcal{X})$, the image of the set $\mathcal{X}$ by the application L, has a non-empty interior in $\mathbb{R}^{k-1}$. It can be easily argued that this is equivalent to showing that $\text{rank}(L) = k-1$. Denote the rows of this matrix by $m_r, \; 1 \leq r \leq k-1$. Using the fact that there exists functions of the form $u^r + c_r, \; 1 \leq r \leq k-1$ which lie in *span* $\{ P_1, \ldots, P_{k-1} \}$, it follows that *span* $\{ m_1, \ldots, m_{k-1} \}$ will contain the rows v_r,

$$v_r = (\, 1^r + \frac{c_r}{n},\; 2^r - 1^r + \frac{c_r}{n},\; \ldots,\; n^r - (n-1)^r + \frac{c_r}{n}\,)$$

Finally, one can easily see that the last matrix can be transformed, by basic addition between columns, to the matrix

$$\begin{pmatrix} 1^2 & 2^2 & \ldots & n^2 \\ \vdots & & & \\ 1^k & 2^k & \ldots & n^k \end{pmatrix}$$

which trivially has rank $k-1$.

Since L is a linear transformation, we have: $L(\mathrm{int}\,\mathcal{R}(\Lambda)\,) = \mathrm{int}\, L(\mathcal{X})$. Thus, from (12) it follows that

$$(\hat{l}_2, \ldots, \hat{l}_k) \;\in \mathrm{int}\, L(\mathcal{X}) \text{ almost surely.} \tag{13}$$

To complete the proof note that in view of lemma 3 and (13), it is sufficient to show that the set $\mathcal{R}(\Lambda)$ is dense in $L(\mathcal{X})$.

To see this fact, note that if $(l_2, \ldots, l_k) \in L(\mathcal{X})$ then, $l_r = \int_0^1 P_{r-1}(u)Q(u)du$, where $Q(u)$ is the step quantile function

$$Q(u) = X_i \quad \frac{i}{n} < u \leq \frac{i+1}{n}, \;\; i = 0, \ldots, k-1.$$

Since $Q(u)$ can be approximated in L_p norm by a sequence of quantile functions $Q_m(u)$ continuous at $u_0 = \frac{1}{2}$, the result follows from lemma 4.

Next, we state the main result of the section.

Theorem 4. *Let $X_1, \ldots, X_n$ be an iid sample from a distribution $F \in \Psi$. If $n \geq k-1$, then*

(a) The L-moments system,

$$L_r(Q) \;=\; \hat{l}_r, \qquad r = 1, \ldots, k$$

$$Q \in \Gamma(\mu, \lambda_1, \ldots, \lambda_{k-1}),$$

is solvable and has a unique solution almost surely.

(b) The L-moments system,

$$L_r(Q) = \hat{l}_r, \quad r = 1, \ldots, k$$

$$Q(u) \in \Gamma(\mu, l_2, \lambda_1, \ldots, \lambda_{k-1}),$$

is solvable and has a unique solution almost surely.

Proof. Both assertions follow directly from proposition 2 and theorem 2.

□

4.2 Calibration Algorithm

We devote this section to the exposition of a rather general algorithm to calibrate the proposed models. For ease of exposition we have chosen the scale-location family $\Gamma(\mu, l_2, \lambda_1, \ldots, \lambda_{k-1})$ to describe the procedure. As it should be expected, the methodology proposed, shall be centered around the idea of matching the empirical L-moments. Since the estimated parameters $\hat{\lambda}_1, \ldots, \hat{\lambda}_{k-1}$ are smooth functions of the sample L-moments, it is possible to provide explicit formulas for the asymptotic variances of these estimators.

Let $X_1, \ldots, X_n$ denote an iid sample from a random variable X with unknown distribution function F. We shall use the notation m and $\hat{m}$ for the theoretical and sample median respectively. Likewise, we shall use the notation σ and $\hat{\sigma}$ for the theoretical and empirical second L-moments respectively. Denote the higher order sample L-moments by $\hat{l}_r$.

Steps of Calibration

1. Select a growth function $b(a)$.

2. Compute the empirical median and the empirical L-moments.

3. Normalize the sample to obtain: $Q_n(1/2) = 0$ and $\hat{l}_2 = 1$.

4. From the normalized sample, compute the higher order sample L-moments $\hat{l}_3, \ldots, \hat{l}_k$.

5. Find the unique vector $\hat{\lambda} = (\hat{\lambda}_1, \ldots, \hat{\lambda}_{k-1})$ satisfying the equations:

$$\int_0^1 P_1^*(a)\exp\{\sum_{r=1}^{k-1}\lambda_r P_{r-1}^*(a)\}\,da \;=\; 1$$

$$\int_0^1 P_{r-1}^*(a)\exp\{\sum_{r=1}^{k-1}\lambda_r P_r^*(a)\}\,da \;= \hat{l}_r, \quad r = 3,\ldots,k.$$

By theorem 4 we can guarantee that this estimator exists almost surely.

6. Estimate the parameter l_2 as the empirical second L-moment.

7. Estimate the parameter μ solving the equation

$$L_1(Q) = \hat{l}_1.$$

For the estimation of the parameter μ, we could choose a general functional of location $\mu(Q)$ such that its empirical version, $\mu(Q_n)$ is a consistent estimate of $\mu(Q)$, and find the unique value of the parameter μ solving the equation

$$\mu(Q) = \mu(Q_n).$$

5 Unimodality Constraints

There is instances where unimodality arises as a natural restriction. For example, in the field of Empirical Finance, there exists a general consensus that most historical series have been drawn from unimodal distributions (see, for example, Boothe and Glassman, (1987)). In this section we shall deal with L-moment interpolation under this additional shape constraint.

We review some properties and definitions. A continuous distribution F is called unimodal with mode m_0 if $F(x)$ is convex for all $x \le m_0$ and concave for all $x \ge m_0$. Similarly, a quantile function $Q(u)$ is called unimodal with mode u_0 if $Q(u)$ is concave for all $u \le u_0$ and $Q(u)$ is convex for all $u \ge u_0$ It can be seen that the condition of unimodality of $F(x)$ is equivalent to the unimodality of $F^{-1}(u)$, in particular we have: $u_0 = F(m_0)$.

Denote by Ψ_U the set of unimodal quantile functions that are twice continuously differentiable, and satisfy the additional conditions: i) $Q''(u) \neq 0$ for all $u \neq u_0$, ii) $\lim_{u\to 0} Q(u) = -\infty$ and $\lim_{u\to 1} Q(u) = +\infty$. In a parametric setting, the L-moment problem $(\mathcal{P})$ in Ψ_U can be reformulated as follows.

Find a parametric family of quantile functions Γ_λ satisfying:

$$\begin{cases} \Gamma_\lambda \subseteq \Psi_U \\ \mathcal{R}(\Gamma_\lambda) = \mathcal{R}(\Psi_U) \end{cases}$$

Our approach for dealing with problem $(\mathcal{P})$, is based on slicing the set Ψ_U into the classes Ψ_{u_0} given by

$$\Psi_{u_0} = \{Q(u) \in \Psi_U \text{ with mode } u_0\}$$

for $u_0 \in [0, 1]$. The usefulness of such decomposition arises from the fact that the sets Ψ_{u_0}, unlike Ψ_U, are essentially representable by positive measures in the sense explained in the previous sections. Therefore, the techniques already used should continue to apply. In the discussion that follows u_0 shall always be kept fixed.

Denote by $\Psi^*_{u_0}$ the subset of Ψ_{u_0} given by

$$\Psi^*_{u_0} = \{Q(u) \in \Psi_{u_0} : \; Q(u_0) = 0, \; Q'(u_0) = 0\}.$$

From the above definition it follows that every $Q \in \Psi_{u_0}$ can be written as: $Q(u) = Q_0(u) + \alpha(u - u_0) + \mu$, where $Q_0(u) \in \Psi^*_{u_0}$, μ is any real number and α is a positive constant satisfying: $\alpha = Q^{'}(u_0)$. In this discussion α will not be considered as a descriptive parameter of interest but rather as a "nuisance" parameter that allows to reduce the problem of estimating a general quantile function with a strictly positive derivative to that of estimating a quantile function $Q(u)$ in the set $\Psi^*_{u_0}$. The usefulness of this approach comes from the fact that the elements of the set $\Psi^*_{u_0}$ can be directly represented by means of positive measures, which, as we already know from previous sections, makes it possible to solve the L-moment interpolation problem for this set. As we shall see later, this technique automatically provides us with a solution to the original L-moment problem in Ψ_U. In the discussion that follows α will be assumed to be known, either because it is given as a function of the parameters of interest, or because it is replaced by a consistent estimator based on the sample. A more detailed discussion on the estimation or parametrization of α is given in section 5.3.

5.1 Representation of Unimodal Quantiles

Our results in this section parallels those from section 3. We obtain an analogous of theorem 1 for the set $\Psi^*_{u_0}$.

Definition 7. *A function $b(a) : (0,\, 1) \rightarrow R$ shall be called a unimodal growth function if*

i) $b(a)$ is continuous for all $a \in (0,\, 1)$.

ii) $b(u_0) \; = \; 0, \quad b(a) \; > \; 0, \;\; a \; > \; u_0$ and $b(a) \; < 0, \;\; a \; < \; u_0$.

iii) $\lim_{a\rightarrow 0} b(a) = -\infty$ and $\lim_{a\rightarrow 1} b(a) = +\infty$.

The "extremal" elements in the set $\Psi^*_{u_0}$ can be conveniently parametrized by means of a reference unimodal growth function $b(a)$. For each $a \in (0,1)$, let $\chi_a(u)$ be the step function given by

$$\chi_a(u) = \frac{b(a)}{1-a}(x-a)1_{[a,1]} \;\; u_0 \; \geq \; a$$

$$\chi_a(u) = b(a)(\frac{-x}{a} \; + \; 1)1_{[0,a]} \;\; a \; < \; u_0.$$

The representation condition for the set $\Psi^*_{u_0}$ is justified in the following more general result.

Theorem 5. *Let $Q(u)$ be a unimodal quantile function with mode $u_0 \in [0,1]$. Assume that Q' exists at u_0 and is continuous at that point. Then,*

there exists a unique positive measure λ on $(0,1)$ such that the following representation holds:

$$Q(u) \;\; = \;\; \int_0^1 \chi_a(u)d\lambda(a) \tag{14}$$

Further, $\lambda(A) \; < \; \infty$ for every compact set $A \subset (0,1)$ that does not contain u_0.

Proof. Let us prove the assertion for the case $u \; > \; u_0$. Since $Q(u)$ is an absolutely continuous function satisfying $Q'(u_0) \; = \; 0$, it can be identified with $Q'(u)$. Thus, equation (14) will hold if and only if:

$$Q'(u) \;=\; \int_{u_0}^{u} \frac{b(a)}{1-a} d\lambda(a) \tag{15}$$

Since $Q'(u)$ is a right-continuous and increasing function on $(u_0,\, 1)$, there exists a unique Borel positive measure μ such that the following relation holds:

$$Q'(u_2) - Q'(u_1) = \mu(\,]u_1, u_2]\,) = \int_{u_1}^{u_2} d\mu(a)$$

for all $u_0 < u_1 \leq u_2 < 1$. In particular, since Q' is continuous at u_0, we have: $Q'(u) = \int_{u_0}^{u} d\mu(a)$, where the integral is taken over the set $]u_0, u]$.

Now, taking $d\lambda_1(a) \;=\; \frac{1-a}{|b(a)|} d\mu(a)$, equation (15) follows.

Uniqueness of λ_1 follows directly from uniqueness of μ and the facts: $|b(a)| > 0$ for all $a > u_0$ and $\mu\{u_0\} = 0$.

Finally, we have

$$\lambda_1(A) = \int_A \frac{1-a}{|b(a)|} d\mu(a) \leq \sup_{a \in A} \frac{1-a}{|b(a)|} \cdot \mu(A) \; < \infty.$$

Similarly, we can prove the existence of a unique measure λ_2 defined on $(0, u_0)$, satisfying

$$Q'(u) \;=\; \int_{u_0}^{u} \frac{b(a)}{a} \lambda_2(a)$$

for all $u < u_0$, where the above integral is taken over the set (u, u_0). Using the same argument as before, it follows that λ_2 is bounded on compact sets contained in $(0, u_0)$. Defining λ as,

$$\lambda(A) = \lambda_1(\,A \cap (0, u_0\,)\,) \,+\, \lambda_2(A \cap (u_0, 1)\,).$$

for a general Borel set $A \subset (0,1)$, the result follows. □

From the above representation it follows that when $Q \in \Psi^*_{u_0}$, the associated spectral measure λ becomes absolutely continuous. Further, its density $\frac{d\lambda}{da}$ is related to Q'' by means of the formula

$$Q''(a) = \begin{cases} -\frac{b(a)}{1-a}\frac{d\lambda(a)}{da} & u_0 < u \\ \\ \frac{b(a)}{a}\frac{d\lambda(a)}{da} & u_0 \geq u \end{cases} \tag{16}$$

almost everywhere in $(0, 1)$. The function $\frac{d\lambda(a)}{da}$ shall be termed the spectral density of $Q(u)$. From the above representation it follows that the set $\Psi^*_{u_0}$ can identified with the subset of positive measures $\mathcal{M}_{u_0}$ given by

$\mathcal{M}_{u_0} = \{x : (0, 1) \to R, \ \ x(a) > 0, \ \ x(a)$ continuous at all points $a \neq u_0\}$.

For the L-moment constraints, we have an entirely analog result to corollary 1 in section 3.

Corollary 2. *Let $Q(u) \in \Psi^*_{u_0}$ and λ be the spectral measure associated to Q. Then the following equations hold:*

$$\int_0^1 P_{r-1}(u)Q(u)du = \int_0^1 P^*_{r-1}(a)\, d\lambda(a) \quad r = 1, \ldots, k$$

where

$$P^*_r(a) = \begin{cases} \sum_{i=0}^{r-1} p_{i,r-1} \frac{(a^{i+1} + a^i + \ldots + a - i - 1)b(a)}{(i+1)(i+2)} & a \geq u_0 \\ \\ \sum_{i=0}^{r-1} p_{i,r-1} \frac{a^{i+1}b(a)}{(i+1)(i+2)} & a < u_0. \end{cases} \tag{17}$$

Consequently, the L-moment problem in the set $\Psi^*_{u_0}$ becomes equivalent to finding a parametric family of positive measures Γ^*_λ satisfying:

$$\Gamma^*_\lambda \subset \mathcal{M}_{u_0}$$

$$\mathcal{R}(\Gamma^*_\lambda) = \mathcal{R}(\mathcal{M}_{u_0})$$

Following the same methodology developed in previous sections, we would solve the transformed problem and map the solution back to Ψ_{u_0}.

Remark 7. *¿From equation (17), it follows that the functions $P^*_r(a)$ are continuous in* $(0, 1)$ *for all $r \geq 0$. Furthermore, $P^*_r(a)$ will remain bounded on* $[0, 1]$ *if and only if* $\lim_{a\to 0}\ a \cdot b(a) < \infty$ *and* $\lim_{a\to 0}\ (1 - a) \cdot b(a) < \infty$. *From this point on, it shall be assumed that the growth function $b(a)$ satisfies these limiting conditions.*

5.2 Parametric Solutions

Motivated by the results obtained in previous sections for the set Ψ_0, we consider the exponential family of spectral densities given by

$$\Gamma^*_\lambda \;=\; \{x(a) \;=\; \exp(\sum_{r=1}^{k-1} \lambda_r P^*_r(a)\,)\,\}$$

Following the same approach as in section 3.1, one can show that all the assumptions required in Borwein and Lewis theorem are satisfied. As we know, this implies that the system,

$$\begin{array}{l} \int_0^1 P^*_{r-1}(a)x(a) = l_r, \quad r = 1,\ldots,k \\ x \in \Gamma^*_\lambda \end{array}$$

will have a unique solution, for all $(l_2,\ldots,l_k) \in \Psi^*_{u_0}$. From this, we obtain a result analogous to theorem 2, which we only state without proof.

Theorem 6. *Let $L = \{L_2,\ldots,L_k\}$ be L-moments functionals. Let $\Gamma^*(\lambda_1,\ldots,\lambda_{k-1})$ be the family of quantile functions having spectral densities in Γ^*_λ. Then*

1. $\Gamma^*(\lambda_1,\ldots,\lambda_{k-1}) \subseteq \Psi^*_{u_0}$.
2. $\mathcal{R}(\,\Gamma^*(\lambda_1,\ldots,\lambda_{k-1})\,) \;=\; \mathcal{R}(\Psi^*_{u_0})$.
3. *There exists a unique correspondence between the vectors $(\lambda_1,\ldots,\lambda_{k-1}) \in \mathbb{R}^{k-1}$ and $(l_2,\ldots,l_k) \in \mathcal{R}(\Psi^*_{u_0})$.*

To handle problem $\mathcal{P}$ for the larger set Ψ_{u_0}, we shall make use of the decomposition mentioned in section 5 to write $Q \;\in\; \Psi_{u_0}$ as follows:

$$Q(u) = Q_0(u) \;+\; \alpha(u - u_0)$$

where $Q_0(u) \in \Psi^*_{u_0}$ and α is a positive constant satisfying: $\alpha = Q^{'}(u_0)$. This simple decomposition allows us to embed a parametric family contained in the set $\Psi^*_{u_0}$ into a larger family of quantile functions in the set Ψ_{u_0}, by simply adding the term $\alpha(u - u_0)$. A useful property of this embedding comes from the fact that the good interpolating properties are preserved by the extended family. This is shown in the next result.

Let $\Gamma(\alpha, \mu, \lambda_1, \ldots, \lambda_{k-1})$ be the extension of the family $\Gamma^*(\lambda_1, \ldots, \lambda_{k-1})$ to the set Ψ_{u_0}, given by

$$\Gamma(\alpha, \mu, \lambda_1, \ldots, \lambda_{k-1}) \ = \ \Gamma(\lambda_1, \ldots, \lambda_{k-1}) \ + \ \alpha(u \ - \ u_0) \ + \ \mu$$

Proposition 3. *Let $L_1, \ldots, L_k$ be a finite number of L-moment functionals. The family $\Gamma(\alpha, \mu, \lambda_1, \ldots, \lambda_{k-1})$ satisfies*

$$\mathcal{R}(\Gamma(\alpha, \mu, \lambda_1, \ldots, \lambda_{k-1})) = \mathcal{R}(\Psi_{u_0})$$

Proof. Let $(l_2, \ldots, l_k) \in \mathcal{R}(\Psi_{u_0})$. Since $L_r(u - u_0) = 0$, for all $r \geq 3$, it follows that the vector

$$(l_2 - \alpha \cdot L_2(u - u_0), \ l_3, \ \ldots, \ l_k)$$

is in the set $\mathcal{R}(\Psi^*_{u_0})$. By definition of $\Gamma(\alpha, \mu, \lambda_1, \ldots, \lambda_{k-1})$ the constraints

$$L_r(Q) = l_r, \qquad r = 1, \ldots, k$$

$$Q \in \Gamma(\alpha, \mu, \lambda_1, \ldots, \lambda_{k-1})$$

are feasible, if and only if the constraints

$$L_2(Q) = l_2 - \alpha L_2(u - u_0)$$

$$L_r(Q) = l_r, \qquad r = 3, \ldots, k$$

$$Q \in \Gamma(\lambda_1, \ldots, \lambda_k)$$

are feasible as well. Now, the result follows from theorem 6.

□

Remark 8. *Equation (16) shows that the asymptotic behavior of any member of the family $\Gamma(\alpha, \lambda_1, \ldots, \lambda_{k-1})$ is proportional to the solutions of the equation: $Q''(u) = b(u)$.*

Following the spirit of previous sections, we shall find it convenient to modify this class of models slightly, to add the scale-invariance property, without loosing, of course, its interpolating properties.

Define Ψ'_{u_0} to be the set

$$\Psi'_{u_0} = \{ \, Q \in \Psi_{u_0} : \ L_2(Q) = 1. \}$$

Definition 8. *We shall denote by* $\Gamma(\alpha, \mu, l_2, \lambda_1, \ldots, \lambda_{k-1})$ *the location scale invariant family*

$$\Gamma(\alpha, \mu, l_2, \lambda_1, \ldots, \lambda_{k-1}) = \{l_2 \cdot Q(u; \alpha, \lambda_1, \ldots, \lambda_k) \; + \; \mu\}$$

where $Q(u; \alpha, \lambda_1, \ldots, \lambda_k) \in \Psi'_{u_0}$ *is the parametric subfamily given by*

$$Q(u; \alpha, \lambda_1, \ldots, \lambda_{k-1}) = \begin{cases} \int_{u_0}^{u} \frac{b(a)(u-a)}{1-a} \; \exp\{\sum_{r=1}^{k-1} \lambda_r P_r^*(a)\}da \; + \alpha(u-u_0) & u \geq u_0 \\ \\ \int_{u}^{u_0} \frac{b(a)(a-u)}{a} \; \exp\{\sum_{r=1}^{k-1} \lambda_r P_r^*(a)\}da \; + \alpha(u-u_0) & u \, < \, u_0 \end{cases}$$

Further, the parameters: $\alpha, \lambda_1, \ldots, \lambda_{k-1}$*, are constrained to satisfy the "normalization" condition:*

$$L_2(\, Q(u; \alpha, \lambda_1, \ldots, \lambda_{k-1})\,) = 1 \tag{18}$$

Applying proposition 3 above, we obtain a result analogous to theorem 3 of section 3.4.

Theorem 7. *Let* $L_1, \ldots, L_k$ *be a finite number of L-moment functionals. For the family* $\Gamma(\alpha, \mu, l_2, \lambda_1, \ldots, \lambda_{k-1})$ *we have*

1. $\Gamma(\alpha, \mu, l_2, \lambda_1, \ldots, \lambda_{k-1}) \subseteq \Psi_{u_0}$
2. $\mathcal{R}(\Gamma(\alpha, \mu, l_2, \lambda_1, \ldots, \lambda_{k-1})) = \mathcal{R}(\Psi_{u_0})$

Remark 9. *Equation (18) turns* $\Gamma(\alpha, \mu, l_2, \lambda_1, \ldots, \lambda_{k-1})$ *into a* $k+1$*-parameter family of distributions. The parameters* μ *and* l_2 *are location and scale parameters respectively. Therefore, their range of values is the usual one for these parameters. Other features of the shape, namely those summarized by the functionals* $\tau_3(Q), \ldots, \tau_k(Q)$*, are described by the parameters,* $\lambda_1, \ldots, \lambda_{k-1}$*. The normalization restriction allows us to separate higher order features, such as skewness and fat tails, from any particular value of the scale parameter* l_2*. Finally, the parameter* α *stands for the value of* Q' *at* u_0*. The range of feasible values for the vector* $(\alpha, \lambda_1, \ldots, \lambda_{k-1})$ *is constrained only by the normalization equation and the positivity of* α*.*

Remark 10. *An important feature of the above model, lies on its ability to extend arbitrary location-scale families in the directions summarized by the higher order L-moments, while preserving the shape constraint of unimodality. This is possible due to the flexibility in choosing the growth function* $b(a)$*. To see this conclusion, consider a general unimodal location-scale family* $\Im = \{l_2 \cdot Q_0 + \mu\}$ *where* Q_0 *satisfies:* $Q_0(u_0) = 0$ *and* $L_2(Q_0) = 1$*. The family* $\Im$ *is recovered by choosing* $b(a)$ *as,*

$$b(a) = \begin{cases} \frac{(1-a)\cdot Q''(a)}{e} & a \geq u_0 \\ \frac{a \cdot Q''(a)}{e} & a \leq u_0 \end{cases}$$

and setting $\alpha = Q^{'}(u_0)$, $\lambda_r = 0$, $r = 1, \ldots, k-1$.

5.3 A parsimonious unimodal family

The Location-Scale family $\Gamma(\alpha, l_2, \lambda_1, \ldots, \lambda_{k-1})$ requires $k+1$ parameters to interpolate k descriptive features. A more parsimonious version of this model can be obtained, if we do not let the parameter α to vary freely, but rather it is assumed to be related to the other parameters of the model in a suitable way. Given the relationship: $\alpha = Q'(u_0)$, the parameter α should be understood as a measure of the local peakedness of the density function around the mode, $m_0 = Q(u_0)$, and therefore will provide no information about the extreme tails of the distribution -in fact, it can be easily shown that the asymptotic tail behavior of the distributions in the above model is independent of the value of α. Further, the model already incorporates other parameters such as the l-kurtosis, which seem to be more effective than α as descriptors of shape features of the sample, since they summarize both peakedness and tail behavior into a single quantity, and can always be suitably modified to describe a wide range of different sets of data. Based on these facts, we can conjecture that in many instances this simplification can be carried out without loosing much of the flexibility of this model

In order to keep the the L-moments matching property in the reduced model, some restrictions on the parametrization of α are required.

Let $\alpha : \mathcal{R}(\Psi'_{u_0}) \to [0, +\infty)$ be a function describing the dependence of α in terms of L-moments of the set Ψ'_{u_0}. Since $l'_2 = 1$, we shall find it more convenient to express α only in terms of higher L-moments, i.e., $\alpha(l'_3, \ldots, l'_k)$. Suppose $\alpha(l'_3, \ldots, l'_k)$ satisfy the following constraint qualification:

CQ. For all $(1, l'_3, \ldots, l'_k) \in \mathcal{R}(\Psi'_{u_0})$, there exists $Q \in \Psi'_{u_0}$ such that

- $Q'(u_0) = \alpha(l'_3, \ldots, l'_k)$
- $L_r(Q) = l'_r, \quad r = 3, \ldots k.$

For $(1, l'_3, \ldots, l'_k) \in \mathcal{R}(\Psi'_{u_0})$, let $\alpha^* = \alpha(l'_3, \ldots, l'_k)$. Using the same arguments as in theorem 6, it follows that the system

$$\begin{cases} L_2(\, Q(u; \alpha^*, \lambda_1, \ldots, \lambda_{k-1})\,) = 1, \\ L_r(\, Q(u; \alpha^*, \lambda_1, \ldots, \lambda_{k-1})\,) = l'_r, \quad r = 3, \ldots, k, \end{cases}$$

will have a unique solution, which for convenience we shall express in a functional form

$$\lambda_r = \lambda_r(l'_3, \ldots, l'_k), \quad r = 1, \cdots, k-1.$$

Definition 9. *Let $\alpha(l'_3, \ldots, l'_k)$ be a function satisfying **CQ**. For any $(l_2, \ldots, l_k) \in \mathcal{R}(\Psi_{u_0})$, take $l'_r = l_r/l_2, \ r = 3, \ldots, k$. We shall denote by $\Gamma(\mu, l_2, \ldots, l_k)$ the location-scale invariant family,*

$$\Gamma(\mu, l_2, \ldots, l_k) \; = \; \{\, l_2 \, Q(u; l'_3, \ldots, l'_k) \; + \; \mu \,\}$$

where $Q(u; l'_3, \ldots, l'_k) \subseteq \Psi'_{u_0}$ is the parametric subfamily given by

$$Q(u; l'_3, \ldots, l'_k) \;\; = \;\; \int_{u_0}^{u} \frac{b(a)(u-a)}{1-a} \exp\{ \sum_{r=1}^{k-1} \lambda_r(l'_3, \ldots, l'_k) \, P^*_r(a) \, \} \, da \; + \; \alpha(l'_3, \ldots, l'_k)(u - u_0)$$

$u \geq u_0$

$$Q(u; l'_3, \ldots, l'_k) \;\; = \;\; \int_{u}^{u_0} \frac{b(a)(u-a)}{1-a} \exp\{ \sum_{r=1}^{k-1} \lambda_r(l'_3 \ldots, l'_k) \, P^*_r(a) \, \} \, da \; + \; \alpha(l'_3, \ldots, l'_k)(u - u_0)$$

$u < u_0.$

We have the following result, whose proof is already trivial.

Theorem 8. *For the family $\Gamma(\mu, l_2, \ldots, l_k)$ the following statements hold*

1. $\mathcal{R}(\Gamma(\mu, l_2 \ldots, l_k) \;=\; \mathcal{R}(\Psi_{u_0})$.
2. *The L-moments equations*

$$L_r(Q) = l_r, \quad Q \in \Gamma(\mu, l_2, \ldots, l_k), \quad r = 1, \ldots, k.$$

establish a unique correspondence between the L-moments vectors $(l_1, \ldots, l_k)$ and the distributions of the family.

Remark 11. *For any function $\alpha(l_3', \ldots, l_k')$, consistent with* ***CQ****, the family $\Gamma(\mu, l_2, \ldots, l_k)$ is a k-parameter family of distributions that solves the L-moments problem in the set Ψ_{u_0}. The remaining of this section shall be devoted to the selection of $\alpha(l_3', \ldots, l_k')$.*

For each vector of higher order l-moments, $l = (l_3', \ldots, l_k') \in \mathcal{R}(\Psi_{u_0}')$, let A_l be the subset of positive numbers

$$A_l = \{Q'(u_0) : \; Q \in \Psi_{u_0}', \;\; L_r(Q) = l_r' \;\; r = 3, \ldots, k.\}$$

It follows trivially that the sets A_l are closed intervals of the form $[0, M]$, where M takes the functional form

$$M(l_3', \ldots, l_k') = \sup \{Q'(u_0) : \;\; Q \in \Psi_{u_0}', \;\; L_r(Q) = l_r' \;\; r = 3, \ldots, k.\}$$

A general and computationally feasible method to obtain the value of this constant is described in Appendix B. Two plausible candidates for the function $\alpha(l_3', \ldots, l_k')$ are discussed next.

1. $\alpha(l_3', \ldots, l_k')$ is defined as the middle point of the range of possible values that it can achieve,

$$\alpha(l_3', \ldots, l_k') \;=\; \frac{M(l_3', \ldots, l_k')}{2}$$

2. Quite often, we wish to define the function α in such a way that the above model contains a specified location-scale invariant family of distributions $\Im \subset \Psi_{u_0}$ (for instance, when the gaussian model represents the null model and $u_0 = \frac{1}{2}$). In such a case, the selection of α as the middle point is likely to exclude $\Im$ from this model.

   Clearly, for this model to contain the family $\Im$ it is enough that it contains one member of $\Im$. Choose Q_0 as the only member of $\Im$ satisfying: $Q_0 \in \Psi'_{u_0}$. Denote by $(l_3^*, \ldots, l_k^*)$ the vector of higher l-moments of Q_0. It follows that for any function $\alpha(l'_3, \ldots, l'_k)$ satisfying

   i) $\alpha(l_3^*, \ldots, l_k^*) \; = \; Q'_0(u_0)$

   ii) $0 \; \leq \; \alpha(l'_3, \ldots, l'_k) \; \leq \; M(l'_3, \ldots, l'_k)$,

   the family $\Im$ will be contained in $\Gamma(\mu, l_2, \ldots, l_k)$. A natural choice for $\alpha(l_3, \ldots, l_k)$ may be

$$\alpha(l'_3, \ldots, l'_k) \; = \; \frac{Q'_0(u_0)}{M(l_3^*, \ldots, l_k^*)} \cdot M(l'_3, \ldots, l'_k)$$

6 Calibration of Unimodal Families

Despite the fact that the the original motivation to introduce the L-moments interpolation problem on the sets Ψ_{u_0} was rather technical, the above families themselves may actually be adequate for modelling purposes, in situations where the parameter u_0 may be safely assumed to be known. That is the case of the assumption of symmetry, resulting in the set $\Psi_{\frac{1}{2}}$. In such contexts, the parsimonious family $\Gamma(\mu, \sigma, l_3, \ldots, l_k)$ becomes a rather flexible and easy to calibrate model. In this section we explore certain novel issues that arises in the feasibility of the L-moments estimation methodology due to the unimodality constraint.

It becomes evident that in general, samples drawn from a unimodal distribution, will not give rise to unimodal empirical distributions. For closely related reasons, sample L-moments will fall out of the set $\mathcal{R}(\Psi_{u_0})$ with a positive, although rather small, probability. Without going into a formal proof of this general fact, take the case of the fourth L-moment. Following Hosking (1996), it is not difficult to show that for all $Q \in \Psi_{u_0}$, we have: $L_4(Q) \geq 0$. On the other hand, it can be easily seen that for samples of sizes as small as three, the quantity $\hat{l}_4$ may take negative values. This fact clearly invalidates the possibility of extending theorem 4 to the setting of unimodal distributions. The feasibility of the method can only be assured with a high probability.

Proposition 4. *Let $X_1, \ldots, X_n$, be an iid sample from a distribution F having quantile function $Q \in \Psi_{u_0}$. Let $\alpha(l'_3, \ldots, l_k)$ be a continuous positive function satisfying **CQ**, and $\Gamma(\mu, l_2, \ldots, l_k)$ be the parametric model defined in section 5.3. Then the probability that the system*

$$L_k(Q) = \hat{l}_r, \quad Q \in \Gamma(\mu, l_2, \ldots, l_k), \quad r = 1, \ldots, k$$

is feasible, will tend to 1, as $n \to \infty$.

Proof. In view of theorem 8, it is sufficient to show that

$$\mathbf{P}\left((\hat{l}_1, \ldots, \hat{l}_k) \in \mathcal{R}(\Psi_{u_0})\right) \to 1, \quad \text{as } n \to \infty.$$

Let $\mathbf{l} = (l_1, \ldots, l_k)$, be the L-moments of Q. Following the same steps as in the proof of proposition 1, one can show that $\mathbf{l} \in \operatorname{int} \mathcal{R}(\Psi_{u_0})$. The rest of the proof follows from the convergence in probability of sample L-moments to the vector $\mathbf{l}$ and the continuity of the function $\alpha(\cdot)$. □

6.1 Calibration Algorithm

In this section we describe a numerically tractable algorithm to calibrate the parsimonious family introduced in section 5.3.

Steps of Calibration

1. Select the following inputs of the model:
   - A unimodal growth function $b(a)$.
   - A functional relationship $\alpha(l'_3, \ldots, l'_k)$ satisfying **CQ**.
2. Compute the empirical median and the empirical L-moments.

3. Normalize the sample to obtain $\hat{l}'_2 = 1$.

4. Compute $\hat{\alpha} = \alpha(\hat{l}'_3, \ldots, \hat{l}'_k)$.

5. Find the unique solution of the equations:

$$\int_0^1 P_1^*(a) \exp\{ \textstyle\sum_{r=1}^{k-1} \lambda_r P_r^*(a) \} \, da \; = \; 1 \; - \; \hat{\alpha}\sigma(U_{u_0})$$

$$\int_0^1 P_{r-1}^*(a) \exp\{ \textstyle\sum_{r=1}^{k-1} \lambda_r P_r^*(a) \} \, da \; = \; \hat{l}'_r, \quad r = 3, \ldots, k.$$

   where $U_{u_0} = (u - u_0)$.

6. Estimate the parameter l_2 as the empirical second L-moment.

7. Estimate the parameter μ solving the equation

$$L_1(Q) = \hat{l}_1.$$

   For the estimation of the parameter μ, we could choose a general functional of location m such that its empirical version, $m_n = m(Q_n)$ is a consistent estimate of $m(Q)$, for all $Q \in \Psi_{u_0}$, and find the unique value of the parameter μ solving the equation

$$m(Q) = m_n.$$

6.2 Unimodal Quantiles with unknown mode

In a number of practical situations, particularly when data are clearly far from being symmetric, uncertainty about the value of u_0 must be incorporated in the model. In this section, we shall be concerned with the L-moment problem for the general set Ψ_U, and show how the results from previous sections can be combined to provide a parsimonious solution.

For each $u_0 \in (0,1)$, let $\Gamma_{u_0} \subset \Psi_{u_0}$ be a parametric family of quantile functions that solves the L-moment problem for the set Ψ_{u_0}. It becomes evident that the union family $\Gamma_U = \{Q : Q \in \Gamma_{u_0}, \text{ for all } u_0 \in (0,1)\}$, offers a solution to the L-moments problem for the set Ψ_U. Based on the results obtained for the three types of families considered in the previous sections, it is not hard to build a "pasting" model $\Gamma_U \subseteq \Psi_U$ that is capable of interpolating any vector of L-moments contained in the range of Ψ_U.

However, as we argued in section 5.3, the number of parameters of Γ_U will exceed the number of descriptive features that we wish the model to reflect. A parsimonious subclass of Γ_U preserving all its interpolating properties, can be built if we establish a suitable functional relationship of the mode u_0 with the parameters $l_1, \ldots, l_k$. A justification for this approach might be argued based on similar arguments to those given in section 5.3 when only uncertainty about the parameter α was allowed.

Define Ψ'_U to be the set

$$\Psi'_U \;=\; \{\, Q \in \Psi_U : \; L_2(Q) = 1 \,\}.$$

Notice that Ψ'_U is the union of the sets $\Psi'_{u_0},\; u_0 \in (0,1)$ defined in section 3.3. Following the approach of section 5.3 we shall require two functions $\alpha(\cdot)$ and $u(\cdot)$ defined on the set $\mathcal{R}(\Psi'_U)$. Since $l'_2 = 1$, as before, we find it more convenient to express them in terms of higher order L-moments, i.e., $\alpha(l'_3, \ldots, l'_k)$ and $u(l'_3, \ldots, l'_k)$. Suppose they satisfy the following constraint qualifications:

(**CQ**). For all $(l'_3, \ldots, l'_k) \in \mathcal{R}(\Psi'_U)$, there exists $Q \in \Psi'_U$ such that

- $u_0 = u(l'_3, \ldots, l'_k)$ is the mode of Q
- $Q'(u_0) = \alpha(l'_3, \ldots, l'_k)$
- $L_r(Q) = l_r, \;\; r = 3, \ldots, k$

Consider the general family $\Gamma(\alpha, \mu, l_2, \lambda_1, \ldots, \lambda_{k-1}) \subseteq \Psi_{u_0}$, introduced in section 3.3. There, we showed that the system

$$L_r(Q) = l_r, \quad r = 1, \ldots k$$
$$Q \in \Gamma(\alpha, \mu, l_2, \lambda_1, \ldots, \lambda_{k-1})$$

has always a solution, although not necessarily unique, for all $(l_1, \ldots, l_k) \in \mathcal{R}(\Psi_{u_0})$. Define Γ_U as the union of these families when u_0 varies in $(0,1)$. It is clear that Γ_U represents a solution for problem $\mathcal{P}$ in the set Ψ_U. By replacing the "free" parameters u_0 and α for $u(l'_3, \ldots, l'_k)$ and $\alpha(l'_3, \ldots, l'_k)$ respectively in the definition of Γ_U, we obtain a k-parameter family that solves $\mathcal{P}$ in the set Ψ_U. Next, we define it more formally.

Definition 10. *For* $(l_2, \ldots, l_k) \in \mathcal{R}(\Psi_U)$, *let* $l'_r = l_r/l_2, \;\; r = 2, \ldots, k$.

Let $u(l'_3, \ldots, l'_k)$ *and* $\alpha(l'_3, \ldots, l'_k)$ *be two functions satisfying* ***CQ***.

We shall denote by $\Gamma_U(\mu, l_2, \ldots, l_k)$ *the location-scale invariant family,*

$$\Gamma(\mu, l_2, \ldots, l_k) \;=\; \{\, l_2\, Q(u; l'_3, \ldots, l'_k) \;+\; \mu \,\}$$

where $Q(u;\, l'_3, \ldots, l'_k) \in \Psi'_U$ *is the parametric subfamily defined as follows*

If $u \geq u^*$

$$\begin{aligned} Q(u;\, l'_3, \ldots, l'_k) \;\; &= \;\; \int_{u^*}^{u} \frac{b(a)(u-a)}{1-a} \exp\{\sum_{r=1}^{k-1} \lambda_r(l'_3, \ldots, l'_k)\, P^*_r(a)\}\, da \\ & \quad + \;\alpha^*(u - u_0) \end{aligned}$$

If $u < u^*$

$$\begin{aligned} Q(u;\, l'_3, \ldots, l'_k) \;\; &= \;\; \int_{u}^{u^*} \frac{b(a)(u-a)}{1-a} \exp\{\sum_{r=1}^{k-1} \lambda_r(l'_3 \ldots, l'_k)\, P^*_r(a)\}\, da \\ & \quad + \;\alpha^*(u - u_0)\,, \end{aligned}$$

where

1. $u^* = u(l'_3, \ldots, l'_k)$ *and* $\alpha^* = \alpha(l_3, \ldots, l'_k)$.
2. *The functions* $\lambda_1(l'_3, \ldots, l'_k),\; \ldots,\; \lambda_{k-1}(l'_3, \ldots, l'_k)$ *are defined as the unique solution of the equations,*

   $$\int_0^1 P^*_{r-1}(a) \exp\{\textstyle\sum_{r=1}^{k-1} \lambda_r P^*_r(a)\} da \;=\; l'_r$$

   $$\int_0^1 P^*_1(a) \exp\{\textstyle\sum_{r=1}^{k-1} \lambda_r P^*_r(a)\}\, da \;=\; 1 \;-\; \alpha^* L_2(U_{u_0})$$

   where $U_{u_0} = (u - u_0)$.

Calibration Algorithm

An extension of the algorithm given in section 6.1 for the calibration of the family $\Gamma(\mu, l_2, \ldots, l_k)$ calls for a tractable numerical procedure to obtain the functions $\alpha(l_1, \ldots, l_k)$ and $u_0(l_1, \ldots, l_k)$. The important issue regrading the selection of these functions will be addressed in further work.

Steps of Calibration.

1. Select the following inputs of the model:

   - A unimodal growth function $b(a)$.
   - Functions $u(l'_3, \ldots, l'_k)$ and $\alpha(l'_3, \ldots, l'_k)$ satisfying **CQ**.

2. Compute the empirical L-moments.

3. Normalize the sample to obtain $\hat{l}'_2 = 1$.

4. Compute $\hat{u} = u(\hat{l}'_3, \ldots, \hat{l}'_k)$ and $\hat{\alpha} = \alpha(\hat{l}'_3, \ldots, \hat{l}'_k)$.

5. Find the unique solution of the equations:

$$\int_0^1 P_1^*(a) \exp\{ \sum_{r=1}^{k-1} \lambda_r P_r^*(a) \} \, da \;=\; 1 \;-\; \hat{\alpha} L_2(U_{u_0})$$

$$\int_0^1 P_{r-1}^*(a) \exp\{ \sum_{r=1}^{k-1} \lambda_r P_r^*(a) \} da \;=\; \hat{l}'_r \quad r = 3, \ldots, k,$$

   where $U_{u_0} = (u - u_0)$ and the functions $P_r^*(u)$ were defined in corollary 2 of section 5.1, now taking $u_0 = \hat{u}$.

6. Estimate the parameter l_2 as the empirical second L-moment.

7. Estimate the parameter μ solving the equation

$$L_1(Q) = \hat{l}_1.$$

   For the estimation of the parameter μ, we could choose a general functional of location m such that its empirical version, $m_n = m(Q_n)$, is a consistent estimate of $m(Q)$ for all $Q \in \Psi_U$, and find the unique value of the parameter μ solving the equation

$$m(Q) = m_n.$$

Appendix A

Appendix B