

Principal Component Value-at-Risk

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Abstract

Value-at-Risk (VaR) is an industrial standard for monitoring financial risk in an investment portfolio. It measures potential losses within a given confidence interval. The implementation, calculation and interpretation of VaR contains a wealth of mathematical issues which are not fully understood. In this paper we present a methodology for an approximation to Value-at-Risk (VaR) that is based on the principal components of a sensitivity-adjusted covariance matrix. The result is an explicit expression in terms of portfolio deltas, gammas, and the variance/covariance matrix. It can be viewed as a non-linear extension of the linear model given by the delta-normal-VaR or RiskMetrics.

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1. Introduction

One of the standard measures of market risk is Value-at-Risk or VaR. It measures the maximum loss that the portfolio can experience with a certain probability, typically 95%, over a certain time horizon, one day for example; in this case, if VaR is given by x , one would expect to lose at least x dollars once every 20 days. Formally, if $\Pi(t, \mathbf{S}(t))$ is a random variable where $\mathbf{S}(t)$ represents a vector of risk factors $\mathbf{S}_1(t), \mathbf{S}_2(t), \dots, \mathbf{S}_n(t)$ over time t , then VaR will be given implicitly by the formula:

$$\text{Prob } \{ \Pi(0, \mathbf{S}(0)) - \Pi(t, \mathbf{S}(t)) > \mathbf{VaR} \} = 0.05.$$

The efficient calculation of Value-at-Risk (VaR) is one the contemporary challenges in the practice of risk management. Approaches for VaR calculations can be classified as historical, Monte Carlo or analytical. We refer the reader to Jorion (1997) for a detail discussion of Value-at-Risk methodologies and a number of historical cases. A good overview can also be found in Duffie and Pan (1997).

In this paper we are concerned with the analytical estimation of the losses that the portfolio faces due to market risk, as a function of the future values of \mathbf{S} . The RiskMetrics Technical Document (1996) presents the delta-normal VaR (δNVaR). In their definition, the risk factors are normally distributed with covariance matrix \mathbb{V} , and the portfolio is approximated by its linear component. The calculation of the δNVaR reduces to the integral of a Gaussian over a half plane that one can explicitly compute. In this case

$$\mathbf{VaR} = z_{0.95} \sqrt{\Delta^T \cdot \mathbb{V} \cdot \Delta}.$$

Here $z_{0.95}$ is the percentile of the univariate normal distribution, approximately 1.65, and Δ is the delta of the portfolio with components given by

$$\Delta^i = \left. \frac{\partial \Pi}{\partial S_i} \right|_{t=0}.$$

The good thing about δNVaR is that the formula is simple to calculate. What is not so good is that, in practice, portfolios tend to be well delta-hedged, which yields a non-realistic zero, or near zero, VaR.

In this paper, we refine this approximation to include quadratic terms given by the Γ or Hessian of the portfolio:

$$\Pi(t) - \Pi(0) \approx \Theta \cdot t + \sum_{i=1}^n \Delta^i \cdot \xi + \frac{1}{2} \xi^T \cdot \Gamma \cdot \xi, \quad \xi = [S_i(t) - S_i(0)]. \quad (1.1)$$

This approximation does not include the term $t^2 \cdot \partial^2 \Theta / \partial t^2$. In fact, ξ is a multi-variate random variable with covariance matrix proportional to t , so the expectation of its norm will be proportional to \sqrt{t} , and (1.1) thus contains all terms of order up to t . We will from now on suppose that $t = 1$, for simplicity; this is not unreasonable since the time-horizon for VaR is usually taken to be one day.

The approximation (1.1) will fail when we deal with a portfolio simultaneously hedged in its deltas and all gammas. This is a rare situation in practice for institutional portfolios, although it is possible to find such situations at the level of the trading desk. We refer the reader to Taleb (1997) for a practical discussion of instrument sensitivities.

Assuming a normal statistical distribution of ξ , one reduces the analysis of VaR to computing the integral of a Gaussian over a quadric, in a space of possibly very high dimension, as was shown by Albanese and Seco (2000). In practice, risk factors are usually log-normally distributed. Note that with a logarithmic change of variables we can still assume a portfolio with normal risk factors; therefore the formula (1.1) will still apply with the caveat that the deltas and gammas require an elementary transformation given by the chain rule.

The purpose of this article is to extend the δ NVaR to quadratic approximations of the portfolio as given by (1.1). In this case explicit, simple formulas do not exist; we mention, however, the result by Albanese and Seco (2000) that is an exact expression for VaR in the Fourier transform space that provides geometrical insight as to the way that VaR depends on portfolio sensitivities and the covariance structure. In this paper we depart from exact expressions and express the result in the form of an asymptotic expansion that works well in the tail of the distribution, where VaR lives. This yields an approximate expression for the integrals involved in the calculation of VaR. The asymptotic formula is of elementary form but it carries the notational burden resulting from a long sequence of elementary transformations; hence we defer the introduction of the result until theorem 4.3 below.

The basic idea, however, is easy to describe: the quadratic approximation given by (1.1) reduces the VaR calculation to the integral of Gaussians over quadrics. In the tail of the distribution, one can study them via asymptotic expansions using stationary phase techniques. The result uses only information about portfolio sensitivities and the variance/covariance matrix. In this paper we confine ourselves to the mathematical derivation of such formulas which are also of independent interest, together with their applications to VaR calculations. A systematic empirical study is postponed to later articles. We also postpone the extension to non-quadratic portfolios as well as to non-Gaussian risk factors, all of which can be dealt with by using similar, but more advanced, mathematical considerations.

There is a considerable literature concerning quadratic perturbations of the linear VaR. One of the most popular is the one that utilizes the Cornish-Fisher expansion for the quantile function of a non-Gaussian variable. It starts with the expansion (1.1) but instead of treating the resulting random variable in an exact manner, it approximates its quantile function using the third moment of the P&L function. This approximation is useful for portfolios that do not differ from linear by a large amount but it tends to fail as one gets deep in the tail of the P&L distribution and hence is of limited applicability. The result, however, is very simple to state and use. We refer the reader to the book of Hull (1999) for a detailed account of this expansion.

Some other delta-gamma approximations are discussed in Dowd (1998) They include the delta-gamma-normal approach in which the quadratic part of the approximation(1.1) is treated as an independent normal variable; Wilson's delta-gamma approach, which approximates the VaR by the solution of a suitable programming problem; and Zangari's moment-fitting approach which approximates the true probability distribution of the left hand side of (1.1) by matching its first four moments (which can be easily computed) to one of a suitably chosen four-parameter family of distributions (see also section 6.3 of RiskMetrics (1996)).

Quadratic approximations have also been the subject of a number of papers dedicated to numerical computations for VaR. We refer the reader to Cardenas, Fruchard, Koehler, Michel and Thomazeau (1997) for a numerical method to compute quadratic VaR using fast Fourier transform methods and to Duffie and Pan (1999) for its extension to jump-diffusion processes. In a related area, we also mention the work by Studer and Luthi (1997) who introduce the concept of *maximum loss*, related to VaR, and provide an algorithm to solve for quadratic portfolios.

The rest of our paper is organized as follows: the section on Portfolio Volatility provides the Gaussian integral over a quadric starting from the quadratic approximation (1.1) and the definition of VaR. The second section on Asymptotics for Gaussians over Quadrics, is the reduction to Gaussian integrals over hypersurfaces. The third, Principal Component VaR, deals with finding the asymptotics of the Gaussian integral by looking at the eigenvalues of the quadratic form. This will complete the discussion in the case that the lowest eigenvalue of the quadratic form is simple. In applications it could easily occur that the first, say, k eigenvalues are too close together. In the section on Uniform VaR we will present an alternative asymptotic series that will be useful when a clustering of eigenvalues near the lowest one takes place. Finally, some of the more technical calculations needed for the proofs are collected in appendices 1 and 2. Appendix 3 finishes the article with a related derivation of upper and lower bounds for VaR.

2. Portfolio Volatility

The covariance matrix of the risk factors is our basic measure of market risk. When dealing with a specific portfolio, we look at market risk in a biased manner as the incidence of market moves can affect our portfolio in different ways. We could easily imagine a portfolio that is insensitive to the principal components of market movements. Therefore, we need to search for a new volatility matrix adapted to our own portfolio. The quadratic approximation introduced earlier, together with elementary linear algebra, provides a covariance matrix adapted to our portfolio. We will refer to it as *portfolio volatility*. This was introduced by Albanese and Seco (2000), and we reproduce a sketch of their arguments for the convenience of the reader.

The quadratic approximation in (1.1) translates into

$$\text{Prob} \{ \Theta + \Delta \cdot \xi + \frac{1}{2} \xi \Gamma \xi^t \leq -\mathbf{VaR} \} = \alpha$$

where ξ is normally distributed with mean m and covariance matrix \mathbb{V} . Hence,

$$\int_{\Theta + \xi \cdot \Delta + \frac{1}{2} \xi \Gamma \xi^t \leq -\mathbf{VaR}} \exp \left\{ -\frac{1}{2} (\xi - m) \mathbb{V}^{-1} (\xi - m)^t \right\} \frac{1}{\sqrt{\det 2\pi \mathbb{V}}} d\xi = \alpha.$$

Using the Cholesky decomposition

$$\mathbb{V} = \mathbb{H}^t \cdot \mathbb{H},$$

we change variables in the integral to

$$(\xi - m) \mathbb{H}^{-1} = y,$$

to obtain

$$\int_{\tilde{\Theta} + y \cdot \tilde{\Delta} + \frac{1}{2} y \tilde{\Gamma} y^t \leq -\mathbf{VaR}} \exp \left\{ -\frac{1}{2} |y|^2 \right\} \frac{1}{(2\pi)^{n/2}} dy = \alpha.$$

with

$$\begin{aligned} \tilde{\Theta} &= \Theta + m \cdot \Delta + \frac{1}{2} m \Gamma m^t, \\ \tilde{\Delta} &= (\Delta + m \Gamma) \mathbb{H}^t \\ \tilde{\Gamma} &= \mathbb{H} \Gamma \mathbb{H}^t. \end{aligned}$$

Next, diagonalize $\tilde{\Gamma}$ into its principal components

$$\tilde{\Gamma} = \mathbb{P} \tilde{\mathbb{D}} \mathbb{P}^t,$$

and change variables to

$$y \mathbb{P} = z,$$

to obtain

$$\int_{\tilde{\Theta} + z \cdot \mathbb{P}^{-1} \tilde{\Delta} + \frac{1}{2} z \tilde{\mathbb{D}} z^t \leq -\mathbf{VaR}} \exp \left\{ -\frac{1}{2} |z|^2 \right\} \frac{\sqrt{\det \tilde{\Gamma}} dz}{(2\pi)^{n/2}} = \alpha.$$

We now complete the square

$$\tilde{\Theta} + z \cdot \mathbb{P}^{-1} \tilde{\Delta} + \frac{1}{2} z \tilde{\mathbb{D}} z^t = T + \frac{1}{2} (z + v) \mathbb{D} (z + v)^t,$$

for

$$\begin{aligned} v &= \tilde{\Delta} \mathbb{P} \tilde{\mathbb{D}}^{-1} \\ T &= \tilde{\Theta} - \frac{1}{2} v \mathbb{D} v^t. \end{aligned}$$

Finally, we obtain

$$\int_{\frac{1}{2} z \tilde{\mathbb{D}} z^t \leq -(\mathbf{VaR} + T)} \exp \left\{ -\frac{1}{2} |z - v|^2 \right\} \frac{dz}{(2\pi)^{n/2}} = \alpha.$$

We think of v as the effective delta of the portfolio, and of \mathbb{D} as the portfolio volatility. The quantity T is a deterministic number that increases or decreases (depending on its sign) our VaR.

As a result, we introduce the integral function

$$I(K) = \int_{\frac{1}{2}z\tilde{\mathbb{D}}z^t \leq -K} \exp\left\{-\frac{1}{2}|z-v|^2\right\} \frac{\sqrt{\det \Gamma} dz}{(2\pi)^{n/2}}.$$

The solution of the implicit equation

$$I(K) = \alpha \quad (\text{i.e. } \alpha = 0.05)$$

will therefore give us the VaR of our portfolio as

$$\mathbf{VaR} = K - T.$$

The signature of $\tilde{\mathbb{D}}$ is of importance. For that reason we distinguish between the negative and positive eigenspaces, and given a vector $x \in \mathbb{R}^n$, we consider its decomposition

$$x = x_+ + x_-,$$

in terms of its projections onto the positive and negative eigenspaces, respectively.

In this new form, we can rewrite

$$I(K) = \int_{\frac{1}{2}(|x_+|^2 - |x_-|^2) \leq -K} \exp\left\{-\frac{1}{2} \left\langle x - \tilde{v}, |\tilde{\mathbb{D}}|^{-1}(x - \tilde{v}) \right\rangle\right\} \frac{dx}{\sqrt{\det 2\pi \mathbb{V}}}$$

where $\tilde{v} = v |\mathbb{D}|^{1/2}$.

Note also that there is a relationship between the signature of $\tilde{\mathbb{D}}$ and the sign of K , neither of which need be positive or negative. Although VaR is always positive, recall that $K = \mathbf{VaR} + T$, and T can have either sign. Of all the possible combinations, a positive T and positive definite $\tilde{\mathbb{D}}$ leads to the situation of a portfolio that can only earn money and hence has a VaR of zero. Finally, note that in our setting the VaR is not discounted to a present value. Any discount methodology can be applied to our analysis in a trivial way.

In the sections to follow we will study the integral I , in the limit when $K \rightarrow \infty$, which is of interest when $\alpha \rightarrow 0$.

3. Asymptotics for Gaussians over Quadrics

In this section we will deal with the reduction to Gaussian integrals over hypersurfaces. We will start with an integral of the form

$$I(R^2) = \int_{Q(x-v) \leq -R^2} e^{-|x|^2/2} dx, \quad (3.1)$$

where

$$Q(x) = \frac{1}{2}x\mathbb{D}x^t,$$

is a quadratic form on \mathbf{R}^n which may be degenerate (although in applications to finance Q will typically be non-degenerate), and where v is a fixed vector in \mathbf{R}^n . Our results will apply to the approximate calculation of VaR, although these expansions are of independent interest. We refer the reader for example to Kotz, Johnson and Boyd (1995). We will rewrite $I(R^2)$ as a continuous sum of Laplace integrals, with large parameter R , over a fixed level set of Q , then we will derive the asymptotics of such Laplace integrals over general smooth hypersurfaces, expressed in geometrical data, such as the point of minimal distance to the origin and the principal curvatures at that point.

The first tool is provided by the following lemma:

Lemma 3.1. *The following formula holds in the sense of distributions:*

$$e^{-|x|^2/2} = (2\pi)^{n/2}\delta_0(x) + \frac{1}{2} \int_0^1 \Delta_x \left(\frac{e^{-|x|^2/2t}}{t^{n/2}} \right) dt.$$

Proof. Recall that the Fourier transform of $e^{-|x|^2/2}$ is $(2\pi)^{n/2}e^{-2\pi^2|\xi|^2}$. Now write

$$\begin{aligned} e^{-2\pi^2|\xi|^2} &= 1 + \int_0^1 \frac{d}{dt} e^{-2\pi^2 t|\xi|^2} dt \\ &= 1 - 2\pi^2 \int_0^1 |\xi|^2 e^{-2\pi^2 t|\xi|^2} dt \end{aligned}$$

and take the inverse Fourier transform of both sides. This proves the lemma. \square

Using the lemma, and observing that $0 \notin \{x : Q(x-v) < -R^2\}$ for sufficiently large R , ($R^2 > -Q(v)$), we can write $I(R^2)$ for those R as

$$I(R^2) = \frac{1}{2} \int_0^1 \frac{dt}{t^{n/2}} \int_{\{Q(x-v) \leq -R^2\}} \Delta_x(e^{-|x|^2/2t}) dx. \quad (3.2)$$

Replacing x by $Rx + v$, we obtain

$$I(R^2) = \frac{1}{2} R^{n-2} \int_0^1 \frac{dt}{t^{n/2}} \int_{\{Q(x) \leq -1\}} \Delta_x(e^{-|Rx+v|^2/2t}) dx. \quad (3.3)$$

The inner integral can now be converted into an integral over the boundary $\{Q(x) = -1\}$ by the Gauss divergence theorem, and the resulting boundary integral can be analyzed by classical techniques. We will first do this when $v = 0$, in which case the computations are a bit more transparent. In fact, we will carry out this part of the argument in a much more general setting, with an eye towards future applications that are more general than just quadratic portfolios:

Lemma 3.2. *Let D be a (not necessarily bounded) domain in \mathbf{R}^n with smooth boundary $\partial D = M$ and suppose that the function $|x|^2$ has a unique non-degenerate minimum on ∂D . Let $x_0 \in \partial D$ be this point of minimal distance to 0. Then we have an asymptotic expansion*

$$\int_D \Delta(\exp(-\lambda|x|^2/2)) dx = e^{(-\lambda|x_0|^2/2)} \sum_{\nu < N} c_\nu \lambda^{-\nu - (n-3)/2} + O(\lambda^{-(n-3)/2 - N}), \quad (3.4)$$

where λ is a large positive parameter. The main term of the expansion has coefficient

$$c_0 = (2\pi)^{(n-1)/2} \cdot |x_0| \cdot \det(I + |x_0|K)^{-1/2},$$

where K is the diagonal matrix whose diagonal entries are equal to the principal curvatures of $M = \partial D$ at x_0 .

Remark. This lemma has an immediate extension to the case where there are finitely many points of minimal distance to 0 on ∂D , all non-degenerate: it suffices to add the different asymptotic expansions for each point.

Proof: By the divergence theorem (whose application does not pose a problem for unbounded D , due to the exponential decay of the integrand) we have that

$$\begin{aligned} \int_D \Delta(\exp(-\lambda|x|^2/2)) dx &= \int_M \frac{\partial}{\partial n}(\exp(-\lambda|x|^2/2)) d\sigma(x) \\ &= -\lambda \int_M \langle x, n(x) \rangle \exp(-\lambda|x|^2/2) d\sigma(x), \end{aligned} \quad (3.5)$$

where $n(x)$ is the outward normal to M at the point x . It suffices to establish the asymptotic expansion (3.4) for the integral cut-off in an arbitrary neighborhood V of x_0 , since the contribution of the complement of such a neighborhood will be of the order $\exp(-\lambda(|x_0|^2/2 + c))$, for some $c > 0$. We will choose $V \subset M$ to be sufficiently small to have a parameterization $z : U \subset \mathbf{R}^{n-1} \rightarrow V \subset M$ of the form

$$z = z(x_1, \dots, x_{n-1}) = (x_1, \dots, x_{n-1}, f(x_1, \dots, x_{n-1}))$$

Here we are assuming that the normal vector of M at x_0 is $(0, \dots, 0, 1)$. We may also assume, without loss of generality, that $z(0) = x_0$, so that $|f(0)| = |x_0|$. Finally, we may arrange that x_1, \dots, x_{n-1} correspond to the directions of the principal curvatures of M at the point x_0 , so that the extrinsic curvature of ∂D at x_0 will be given by a diagonal matrix:

$$K(x_0) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{x=0} = \begin{pmatrix} k_1 & 0 & \dots & 0 \\ 0 & k_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & k_{n-1} \end{pmatrix}$$

Let $\chi \in C_c^\infty$ be some smooth cut-off function supported in V , which is identically 1 in some smaller neighborhood of x_0 . Write $x = (x', x_n)$. Then $|x|^2 = |x'|^2 + f(x')^2$ and $d\sigma(x) = \sqrt{1 + |\nabla f(x')|^2} dx'$ on $V \subset M$. Since $n(x) = (-\nabla f, 1)/\sqrt{1 + |\nabla f|^2}$, it follows that

$$\int_M \chi(x) \langle x, n(x) \rangle e^{-\lambda|x|^2/2} d\sigma(x) = \int_{\mathbf{R}^{n-1}} \chi(x', f(x')) (x' \cdot \nabla f - f) e^{-\lambda F(x')} dx',$$

with $F(x') = (|x'|^2 + |f(x')|^2)/2$.

The lemma follows using classical results on asymptotic integrals (cf. for example Wong (1989)), with

$$c_0 = (2\pi)^{n/2} |\det F''(0)|^{-1/2} |x_0|$$

In our case, the Hessian of F in $x' = 0$ is equal to $(Id + |x_0|K)$. This finishes the proof of the lemma, taking into account the extra factor of λ coming from (3.5). \square

For the case $v \neq 0$ we need an analogue of lemma 3.2 with exponent $|\sqrt{\lambda}x + v|^2$ instead of $\lambda|x|^2$. We state it in the special case that the hypersurface ∂D is given by the equation $Q(x) = -1$, which is all is needed in the present paper:

Lemma 3.3. *Suppose that Q has eigenvalues $-a_1^- < -a_2^- \leq \dots \leq -a_{n_-}^- < 0 \leq a_1^+ \leq \dots \leq a_{n_+}^+$, (so that a_1^- is of multiplicity 1) and let $v = (v_-, v_+) = (v_{-,1}, \dots, v_{-,n_-}, \dots, v_{+,n_+})$ be the coordinates of v in the corresponding eigenbasis. Define a constant $\gamma = \gamma(Q, v)$ by*

$$\gamma = \frac{1}{2} \left(\sum_{j \geq 2} -\frac{a_j^+}{a_j^+ + a_1^-} v_{+,j}^2 - v_{-,1}^2 + \sum_{k \geq 2} \frac{a_k^-}{a_1^- - a_k^-} v_{-,k}^2 \right). \quad (3.6)$$

Then we have an asymptotic expansion:

$$\int_{\{Q(x) \leq -1\}} \Delta_x \left(e^{-|\sqrt{\lambda}x+v|^2/2} \right) dx \simeq e^\gamma e^{-\lambda/2a_1^-} \sum_{j \geq 0} c_j \lambda^{-(n-3+j)/2}$$

with principal coefficient:

$$c_0 = 2(2\pi)^{\frac{n-1}{2}} (a_1^-)^{\frac{n}{2}-1} \prod_{j=1}^{n_+} (a_j^+ + a_1^-)^{-1/2} \prod_{k=2}^{n_-} (a_1^- - a_k^-)^{-1/2}.$$

The proof of this lemma, which is slightly more computational than the one of lemma 3.2, will be given in Appendix 1.

4. Principal component VaR.

In this section we will derive an asymptotic series for $I(R^2)$, first when $v = 0$ and then in general.

For the case $v = 0$ all we have to do is apply lemma 3.2 to the inner integral of (3.2) and then integrate over t . We stress that the quadratic form Q may have positive as well as negative eigenvalues. We have to introduce some notation. We may assume, without loss of generality, that $Q(x)$ is in diagonal form and we split \mathbf{R}^n as the orthogonal direct sum of the semi-positive and *strictly* negative subspaces of Q :

$$\mathbf{R}^n = \mathbf{R}^{n_+} \times \mathbf{R}^{n_-},$$

where Q restricted to \mathbf{R}^{n_+} (\mathbf{R}^{n_-}) is positive semi-definite (respectively negative definite). We will write $x = (y, z)$, $y \in \mathbf{R}^{n_+}$, $z \in \mathbf{R}^{n_-}$. If we denote by $0 \leq a_1^+ \leq \dots \leq a_{n_+}^+$ the positive eigenvalues of Q and by $a_1^- \geq \dots \geq a_{n_-}^-$ the *absolute values* of the *strictly* negative ones, then

$$\begin{aligned} Q(y, z) &= \sum_{j=1}^{n_+} a_j^+ y_j^2 - \sum_{k=1}^{n_-} a_k^- y_k^2 \\ &= Q^+(y) - Q^-(z). \end{aligned}$$

Note that we are just listing all eigenvalues in increasing order, starting with $-a_1^-$. Note also that Q^+ might be degenerate, but Q^- isn't, unless $n_- = 0$. We will also assume that $-a_1^-$ is different from all the other eigenvalues. Before stating the main theorems we need the following elementary lemma which can be easily proved by integration by parts:

Lemma 4.1. *Let $\alpha > 0$ and $R \in \mathbf{R}$. Then we have for all N that*

$$\int_0^1 t^{-\beta} e^{-R^2/2t\alpha} dt = e^{-R^2/2\alpha} \left(\sum_{j=0}^{N-1} \gamma_j R^{-2j-2} + O(R^{2N}) \right),$$

with constants γ_j depending on α and β , and $\gamma_0 = 2\alpha$.

Theorem 4.2. *Suppose that $n_- \geq 1$ and that the multiplicity of the most negative eigenvalue $-a_1^-$ is equal to one. Then:*

$$I(R^2) \simeq e^{-R^2/2a_1^-} \sum_{\nu \geq 0} C_\nu R^{-1-2\nu}, \quad (4.1)$$

with

$$C_0 = 2(2\pi)^{(n-1)/2} \frac{(a_1^-)^{n/2}}{\prod_{j \geq 1} (a_j^+ + a_1^-)^{1/2} \prod_{k \geq 2} (a_1^- - a_k^-)^{1/2}}. \quad (4.2)$$

Remarks. The other C_ν 's can, in principle, be easily read-off from the proof of theorem 4.2. However, in this paper we will mainly limit our computations to the principal term. Explicit estimates for the remainder terms for the asymptotic series above can also be easily obtained. These will in first instance depend on a choice of cut-off function (cf. the proof of lemma 3.2 above). For an optimal estimate, one would have to minimize over a suitable class of cut-offs. This is straightforward, but technically a little cumbersome, and we omit the details. A similar remark applies to theorems 4.3 and 5.1 below. Note also that the rate of exponential decay of $I(R)$ only depends on the most negative eigenvalue of Q .

Proof. We first determine the points of minimal norm on the surface $\{x : Q(x) = -1\}$ and their principle curvatures. A point of minimal distance must be a stationary point of the function $x \rightarrow |x|^2$ restricted to the surface. Hence for any such a point $x = (y, z)$ we must have that for some $\lambda \in \mathbf{R}$,

$$\lambda \cdot (y_1, \dots, y_{n_+}, z_1, \dots, z_{n_-}) = (a_1^+ y_1, \dots, a_{n_+}^+ y_{n_+}, -a_1^- z_1, \dots, -a_{n_-}^- z_{n_-}),$$

the vector on the right being the direction of the normal to the surface. It follows easily that either $y = 0$ or $z = 0$. The latter is not possible, since $Q(x) = -1$, so all stationary points will have $y = 0$. Furthermore, it follows that when $z_k \neq 0$, then $\lambda = -a_k^-$. Since a_1^- has multiplicity 1, a moment's thought shows that, letting $\{e_j^+\}$ and $\{e_j^-\}$ denote the standard basis of \mathbf{R}^{n_+} and \mathbf{R}^{n_-} , respectively, the stationary points will either be $(0, \pm e_1 / \sqrt{a_1^-})$ or will be of the form $z_j e_j + \dots + z_{j+p} e_{j+p}$, $j \geq 2$ being such that $a_{j-1}^- \neq a_j^-$ and p being the multiplicity of a_j^- . A point on the surface of this form will have distance squared $1/a_j^-$ to 0, and it follows that the points of minimal

distance are $(0, \pm e_1/\sqrt{a_1^-})$. We calculate the principal curvatures at these points as in the proof of lemma 3.2, using the local parameterizations

$$z_1 = f(y, z') = \pm \sqrt{(1 - a_2^- z_2^2 - \dots - a_{n_-}^- z_{n_-}^2 + a_1^+ y_1^2 + \dots + a_{n_+}^+ y_{n_+}^2)/a_1^-} \quad (4.3)$$

and find for the principal curvatures in the y and z' directions:

$$k_j^+ = a_j^+/\sqrt{a_1^-} \quad (j \geq 1) \text{ and } k_\nu^- = -a_\nu^-/\sqrt{a_1^-} \quad (\nu \geq 2).$$

Next, we apply lemma 3.2 for each fixed $t \in (0, 1)$, with $\lambda = R^2/t$ and then integrate over t : one finds that

$$I(R^2) = \sum_{\nu < N} A_\nu(R) R^{1-2\nu} + e_N(R) R^{1-2N},$$

where

$$A_\nu(R) = c_\nu \cdot \int_0^1 t^{-3/2+\nu} e^{-R^2/2ta_1^-} dt,$$

with c_ν being given by lemma 3.2, and with an error term $e_N(R)$ which can be estimated by

$$|e_N(R)| \leq C_N \cdot \int_0^1 t^{-3/2+N} e^{-R^2/2ta_1^-} dt.$$

Using lemma 4.1 for both the A_ν 's and the error term, we find the stated asymptotic expansion for $I(R^2)$. The principal coefficient can easily be calculated from the main terms in lemmas 3.2 and 4.1. This completes the proof of theorem 4.2. \square

For the case $v \neq 0$ we proceed as in the proof of theorem 4.2 and apply lemma 3.3 to (3.3). Since $|Rx + v|^2 = |(R/\sqrt{t})x + v/\sqrt{t}|^2$ we do not only have to put $\lambda = R^2/t$ in lemma 3.3, but also have to replace v by v/\sqrt{t} . The result is an expansion

$$I(R^2) \simeq \sum_{\nu} A_\nu(R) R^{1-\nu}$$

with

$$A_\nu(R) = c_\nu \int_0^1 e^{(\gamma - R^2/2a_1^-)/t} t^{(-3+\nu)/2} dt.$$

Treating these as in lemma 4.1, and finally expanding $((R^2/2a_1^-) - \gamma)^{-1}$ in decreasing powers of $1/R^2$, we find the asymptotic expansion for $I(R^2)$. With the notations introduced in lemma 3.3, this result is as follows:

Theorem 4.3. *Suppose that the smallest eigenvalue of Q has multiplicity 1. Then for $R^2 > 2a_1^- \gamma$:*

$$I(R^2) \simeq e^{-\frac{R^2}{2a_1^-}} e^\gamma \cdot \sum_{\nu \geq 0} C_\nu R^{-1-\nu} \quad (4.5)$$

with

$$C_0 = 2(2\pi)^{(n-1)/2} \frac{(a_1^-)^{n/2}}{\prod_{j=1}^{n_+} (a_j^+ + a_1^-)^{1/2} \prod_{k=2}^{n_-} (a_1^- - a_k^-)^{1/2}}. \quad (4.6)$$

Remark. Note that the successive terms in the expansion now decrease with inverse powers of R , instead of R^2 , as was the case in theorem 4.2. This, together with the overall factor of e^γ in front, constitutes the principal difference between (4.5) and (4.1). It thus becomes relevant to know the coefficient C_1 of the second term in the series (4.1) and (4.5). Elementary (but long) calculations show that these are equal to, respectively:

$$C_1^{v=0} = C_0 \left(\sum_{\nu=2}^{n_-} \beta_\nu^- a_\nu^- - \sum_{j=1}^{n_+} \beta_j^+ a_j^+ - 2a_1^- \right)$$

and (cf. appendix 2)

$$C_1^{v \neq 0} = C_0 \cdot v_{-,1} \cdot \left\{ \sqrt{a_1^-} - \frac{1}{2} \left(\sum_{j=1}^{n_+} (\beta_{+,j} + \beta_{+,j}^2 v_{+,j}^2) \frac{\partial^2 \psi}{\partial y_j^2}(0) + \sum_{k=2}^{n_-} (\beta_{-,k} + \beta_{-,k}^2 v_{-,k}^2) \frac{\partial^2 \psi}{\partial z_k^2}(0) \right) \right\},$$

where the β 's are defined by

$$\beta_{+,j} = \frac{a_1^-}{a_j^+ + a_1^-},$$

$$\beta_{-,k} = \frac{a_k^-}{a_1^- - a_k^-}.$$

5. Uniform VaR Estimates

In this section we will study an alternative asymptotic expansion when there is a clustering of eigenvalues near the lowest one. If for example $v = 0$, it follows from the expression for $C_1^{v=0}$ that

$$\begin{aligned} \left| \frac{C_1}{C_0} \right| &= \mathcal{O} \left(\|Q\| \cdot \left(\max_{j \geq 1, \nu \geq 2} \left(\frac{a_1^-}{a_1^- - a_j^-}, \frac{a_1^-}{a_1^- + a_j^+} \right) + 1 \right) \right) \\ &\leq C \cdot \|Q\| \cdot \left(\frac{a_1^-}{a_1^- - a_2^-} + 1 \right), \end{aligned}$$

where $\|Q\|$ denotes the norm of the symmetric matrix associated to the quadratic form Q ; one can make a similar observation if $v \neq 0$. Therefore one expects, assuming $\|Q\| \simeq 1$, that the main term will be a good approximation to $I(R^2)$, with small relative error

$$\frac{I(R^2) - C_0 R^{-1} e^{-R^2/2a_1^-}}{C_0 R^{-1} e^{-R^2/2a_1^-}}$$

if $R^2(a_1^- - a_2^-) \gg 1$, or $a_1^- - a_2^- \gg R^{-2}$. This will fail if a_1^- and a_2^- are too close together, relative to R^2 which are in the range of the VaR associated to our chosen confidence level. In such cases approximating by the first term of the asymptotic expansion would not be a good idea. More generally, the first k smallest eigenvalues $-a_1^-, \dots, -a_k^-$ might cluster, while there is a big gap $a_k^- - a_{k+1}^-$. Situations such as this are frequently encountered in multi-variate linear statistics and such a splitting of the eigenvalues of the covariance matrix into two subsets is the basis of dimensionality reduction in principal component analysis. In this section we will derive an asymptotic expansion for $I(R^2)$ which is *uniform* in the first $k - 1$ eigenvalue differences $a_{j+1}^- - a_j^-$, $1 \leq j \leq k - 1$. This will be done using a geometric construction, which we will first illustrate in the three-dimensional case, assuming Q to be negative definite. We thus suppose that $n = 3$ and that $Q(x) = -(a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2)$, with $a_1 \simeq a_2 \gg a_3 > 0$. We want to estimate the integral

$$I(R^2) = \int_{a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 \geq R^2} e^{-|x|^2} dx.$$

Again, by Lemma 3.1 and Gauss' theorem, we find that

$$I(R^2) = R^{n/2} \int_0^1 \frac{dt}{t^{n/2+1}} \int_{\Sigma} \langle x, n_{\Sigma}(x) \rangle e^{-R^2|x|^2/2t} d\sigma_{\Sigma}, \quad (5.1)$$

where we have written $\Sigma = \{x : a_1x_1^2 + a_2x_2^2 + a_3x_3^2 = 1\}$ and where $n_{\Sigma}(x)$ and $d\sigma_{\Sigma}(x)$ are, respectively, the outward normal and the surface measure on Σ at the point x . Let W denote the $x_1 - x_2$ plane. We let

$$\Sigma_2 := \Sigma \cap W = \{(x_1, x_2, 0) : a_1x_1^2 + a_2x_2^2 = 1\}.$$

Since $a_1 \simeq a_2$, Σ_2 is almost a circle. Fix $\zeta \in \Sigma_2$ and let V_{ζ} be the plane spanned by ζ and the x_3 -axis. Then one easily sees that $\Sigma \cap V_{\zeta}$ is an ellipse in the V_{ζ} -plane, with minor axis $|\zeta|^2$ and major axis $1/a_3$. We now write the $d\sigma_{\Sigma}(x)$ -integral in (5.1) as a double integral:

$$\int_{\Sigma_2} \int_{\Sigma \cap V_{\zeta}} \langle x, n_{\Sigma}(x) \rangle e^{-R^2|x|^2/2t} f(x, \zeta) d\sigma_{\Sigma \cap V_{\zeta}}(x) d\sigma_{\Sigma_2}(\zeta),$$

where $d\sigma_{\Sigma_2}$ denotes the surface measure on Σ_2 and $f(x, \zeta)$ is some appropriate Jacobian. We next apply, for each $\zeta \in \Sigma_2$, the asymptotic expansion of theorem 4.2 to the inner integral over $\Sigma \cap V_{\zeta}$. Since $a_3|\zeta|^2 \ll a_1\zeta_1^2 + a_2\zeta_2^2 = 1$ and therefore $1/|\zeta|^2 \gg a_3$ for $\zeta \in \Sigma_2$, we are in good shape to use theorem 4.2. Integrate this asymptotic expansion, first over t , as in the proofs of theorems 1 and 2, and finally term by term over $\zeta \in \Sigma_2$. The result will be an asymptotic expansion of $I(R^2)$ in decreasing powers of R , with coefficients which are expressed as integrals over Σ_2 .

This idea obviously generalizes to general n and Q and we limit ourselves here to stating the final result, referring to Appendix 2 for the technical details. First we have to introduce some notation. As before, we suppose that

$$Q(x) = \sum_{j=1}^{n_+} a_j^+ y_j^2 - \sum_{\nu=1}^{n_-} a_{\nu}^- z_{\nu}^2,$$

where the y_j and the z_{ν} are the coordinates with respect to the orthonormal basis $e_1^+, \dots, e_{n_+}^+$ and $e_1^-, \dots, e_{n_-}^-$ of \mathbf{R}^{n_+} and \mathbf{R}^{n_-} , respectively. We will write

$$\Sigma = \{x \in \mathbf{R}^n : Q(x) = -1\}.$$

Fix a k , $1 \leq k \leq n_-$. As explained in the introduction to this section, in applications k will be chosen such that $a_k^- - a_{k+1}^- \gg \max_{j < k} \{a_j^- - a_{j+1}^-\}$. Let $W_k = \text{Span} [e_1^-, \dots, e_k^-]$ and define

$$\Sigma_k = \Sigma \cap W_k.$$

Let $d\sigma_{\Sigma_k}$ be the surface measure on Σ_k . For $\zeta \in \Sigma_k$, let v_ζ be the orthogonal projection of v onto the linear subspace V_ζ spanned by ζ and the orthogonal complement of W_k and let

$$\Gamma(\zeta) = \Gamma(\zeta, v) = -\frac{1}{2}|v - v_\zeta|^2 + \gamma(v_\zeta),$$

with $\gamma(v_\zeta) = \gamma(v_\zeta, Q|_{V_\zeta})$ as in theorem 4.3. More explicitly,

$$\gamma(v_\zeta) = \frac{1}{2} \left(\sum_{j \geq 1} -\frac{|\zeta|^2 a_j^+}{1 + |\zeta|^2 a_j^+} v_{+,j}^2 - \frac{\langle v, \zeta \rangle^2}{|\zeta|^2} + \sum_{\nu \geq k+1} \frac{|\zeta|^2 a_\nu^-}{1 - |\zeta|^2 a_\nu^-} v_{-, \nu}^2 \right).$$

Then we will prove in Appendix 2 the following asymptotic expansion for $I(R^2)$:

Theorem 5.1. Suppose that $a_{k+1}^- < a_k^- \leq \dots \leq a_1^-$. Then for $R^2 > \max_{\zeta \in \Sigma_k} 2\gamma(v_\zeta)/|\zeta|^2$:

$$I(R^2) \simeq \sum_{\nu \geq 0} R^{k-2-\nu} \left(\int_{\Sigma_k} e^{-R^2 |\zeta|^2 / 2} e^{\Gamma(\zeta)} C_\nu(\zeta) d\sigma_{\Sigma_k}(\zeta) \right),$$

with the function $C_0 = C_0(\zeta)$ in the main term given by

$$C_0(\zeta) = (2\pi)^{(n-k)/2} \frac{1}{|\zeta|^2} \frac{\langle n_{\Sigma_k}(\zeta), \zeta \rangle}{\prod_{j=1}^{n_+} (1 + |\zeta|^2 a_j^+)^{1/2} \prod_{j=k+1}^{n_-} (1 - |\zeta|^2 a_j^-)^{1/2}}.$$

Remarks:

(i) This expansion is uniform in $a_1^- - a_2^-, \dots, a_{k-1}^- - a_k^-$. For example, if $v = 0$ then

$$\begin{aligned} |C_1(\zeta)/C_0(\zeta)| &\leq C \cdot \|Q\| \cdot \left(\frac{|\zeta|^2}{1 - |\zeta|^2 a_{k+1}^-} + 1 \right) \\ &\leq C \cdot \left(\frac{\|Q\|^2}{a_k^- - a_{k+1}^-} + \|Q\| \right) \end{aligned}$$

(ii) The asymptotic expansion of theorem 5.1 interpolates smoothly between the cases of an a_1^- of multiplicity 1 and an a_1^- of multiplicity $k > 1$. In fact, in the first case we again obtain theorem 4.3 simply by expanding each of the Σ_k -integrals above using stationary phase, while in the second case each of these integrals can be evaluated, leading to an expansion as in theorem 4.3, but multiplied by an extra factor of R^{n-k}

(iii) In practice one would first try to use only the first term of this expansion:

$$I(R^2) \simeq (2\pi)^{(n-k)/2} R^{k-2} \int_{\Sigma_k} \frac{\langle n_{\Sigma_k}(\zeta), \zeta \rangle e^{-R^2|\zeta|^2/2 + \Gamma(\zeta)}}{\prod_{j=1}^{n+} 1 + (|\zeta|^2 a_j^+)^{1/2} \prod_{j=k+1}^{n-} (1 - |\zeta|^2 a_j^-)^{1/2}} \frac{d\sigma_{\Sigma_k}(\zeta)}{|\zeta|^2},$$

Note that, contrary to (3.1), the integral is over a compact set, which will be low-dimensional if k is not too big, in which case it may be accurately evaluated numerically. Also note that the integrand is completely explicit in terms of the risk and portfolio data.

6. Conclusions

Under Gaussian assumptions for the underlying risk factors, the Value at Risk of a portfolio can be approximated by explicit asymptotic expressions. The nature of the approximation is twofold. On the one hand, the delta and gamma of the portfolio are used to monitor the movements of the P&L function. This constitutes an extension of the delta-normal methodology of RiskMetrics. On the other hand, we consider the confidence level (usually 95% or 99%) to be close to 100%, and an asymptotic expression is obtained which is valid in this limit. The methodology presented is fairly general, and can be extended to non-quadratic approximations, as well as non-Gaussian risk factors. Such extensions will be dealt with in future publications.

Appendix 1: proof of lemma 3.3

Before proving lemma 3.3 we need to establish the following lemmas:

Lemma A.1 *Given $g(x, \lambda)$ and $r_{i,j}(x)$ smooth, for $x \in \mathbb{R}^n$ and $\lambda > 0$, define*

$$a(x, \lambda) = g\left(\frac{x}{\sqrt{\lambda}}\right) \exp\left(-\frac{1}{\sqrt{\lambda}} \sum_{j,k} x_j x_k r_{jk}\left(\frac{x}{\sqrt{\lambda}}\right)\right)$$

We have that for all $K = 0, 1, 2, \dots$,

$$a(x, \lambda) = \sum_{j < K} \frac{P_j(x)}{\lambda^{j/2}} + \frac{1}{\lambda^{K/2}} R_K(x, \lambda), \quad (\text{A.1})$$

with $P_j(x)$ a polynomial in x and $R_K(x, \lambda)$ polynomially bounded in x , uniformly for $\lambda \geq 1$. Moreover,

$$P_0(x) = g(0), \quad P_1(x) = \langle x, \nabla g(0) \rangle - \frac{1}{2} g(0) \sum_{j,k} x_j x_k \frac{\partial^2 \psi}{\partial x_j \partial x_k}(0). \quad (\text{A.2})$$

Proof. Expand $g(x/\sqrt{\lambda})$ and the exponential in a Taylor series:

$$\begin{aligned} & g\left(\frac{x}{\sqrt{\lambda}}\right) \exp\left(-\frac{1}{\sqrt{\lambda}} \sum_{j,k} x_j x_k r_{jk}\left(\frac{x}{\sqrt{\lambda}}\right)\right) = \\ & = \left(\sum_{|\alpha| < K} \frac{\partial_x^\alpha g(0)}{\alpha!} \frac{x^\alpha}{\lambda^{|\alpha|/2}} + \frac{R_{1,K}(x, \lambda)}{\lambda^{K/2}} \right) \cdot \left(\sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \frac{(\sum_{j,k} x_j x_k r_{jk}(x/\sqrt{\lambda}))^j}{\lambda^{j/2}} \right) \end{aligned}$$

Next, expand $r_{jk}(x/\sqrt{\lambda})$,

$$\sum_{j,k} x_j x_k r_{jk}(x/\sqrt{\lambda}) = \sum_{2 \leq |\alpha| < K} \frac{x^\alpha \partial_x^\alpha \psi(0)}{\alpha!} \lambda^{-|\alpha|/2+1} + \lambda^{-K/2+1} R_{2,K}(x, \lambda).$$

Here the remainders $R_{1,K}$ and $R_{2,K}$ are both of the form

$$\sum_{|\alpha|=K} x^\alpha q_\alpha(x, \lambda)$$

with q_α uniformly bounded in $\lambda \geq 1$ and polynomially bounded in x . Rearranging terms, we obtain (A.1) and (A.2). \square

Lemma A.2 *Let $\langle Ax, x \rangle$ be a positive definite quadratic form on \mathbf{R}^N and let $\psi \in C^\infty(\mathbf{R}^N)$, $g \in C^\infty(\mathbf{R}^N)$, with all $\partial_x^\alpha g(x)$ polynomially bounded and ψ satisfying $\psi(0) = 0$ and $\|\partial_x^\alpha \psi\|_\infty \leq C_\alpha$, for all α . Let*

$$J(\lambda) = \int_{\mathbf{R}^N} e^{-\lambda(\langle Ax, x \rangle/2) - \sqrt{\lambda}\psi(x)} g(x) dx$$

Then

$$J(\lambda) \simeq \lambda^{-N/2} \sum_{j \geq 0} C_j \lambda^{j/2}, \quad (\text{A.3})$$

where, if we let $u = \nabla\psi(0)$,

$$C_0 = (2\pi)^{N/2} g(0) (\det(A))^{-1/2} e^{\langle A^{-1}u, u \rangle/2}.$$

Remarks: (i) Note that the asymptotic series decreases by powers of $\sqrt{\lambda}$ instead of by powers of λ , as was the case when $\psi = 0$. It is therefore interesting to compute the second coefficient C_1 . If we let $\psi''(0) = (\partial^2\psi(0)/\partial x_j \partial x_k)_{j,k}$ be the Hessian of ψ in 0, then one finds that

$$C_1 = - (2\pi)^{N/2} \frac{e^{\langle A^{-1}u, u \rangle/2}}{(\det A)^{1/2}} \cdot \left\{ \langle A^{-1}u, \nabla g(0) \rangle + \frac{1}{2} g(0) (\text{tr}(A^{-1}\psi''(0)) + \langle A^{-1}\psi''(0)u, A^{-1}u \rangle) \right\}.$$

(ii) The same type of expansion will hold if we replace the quadratic form $\langle Ax, x \rangle$ by a non-negative function $F = F(x)$ with non-degenerate minimum 0 in $x = 0$: this is a well-known consequence of the Morse lemma. The principal coefficient remains the same, with A replaced by the Hessian $F''(0)$; the next coefficient will of course change.

Proof. The proof closely follows the one of the standard case: replace the integration variable x by $x/\sqrt{\lambda}$. Then, using the second order Taylor expansion

$$\psi(x/\sqrt{\lambda}) = \frac{1}{\sqrt{\lambda}} \langle x, \nabla\psi(0) \rangle + \frac{1}{\lambda} \sum_{j,k} x_j x_k r_{jk} \left(\frac{x}{\sqrt{\lambda}} \right),$$

we obtain that

$$J(\lambda) = \lambda^{-N/2} \int_{\mathbf{R}^N} a(x, \lambda) e^{-\langle Ax, x \rangle / 2 - \langle x, \nabla \psi(0) \rangle} dx,$$

with amplitude $a(x, \lambda)$ given by

$$a(x, \lambda) = g\left(\frac{x}{\sqrt{\lambda}}\right) \exp\left(-\frac{1}{\sqrt{\lambda}} \sum_{j,k} x_j x_k r_{jk}\left(\frac{x}{\sqrt{\lambda}}\right)\right)$$

Using lemma A.1 we observe that $a(x, \lambda)$ has good asymptotic decay properties with respect to λ , while staying decent with respect to x . Substituting the asymptotic expansion (A.1) into the integral for $J(\lambda)$, we find the asymptotic expansion (A.3), with

$$C_j = \int_{\mathbf{R}^N} P_j(x) e^{-\langle Ax, x \rangle / 2 - \langle x, \nabla \psi(0) \rangle} dx,$$

from which one easily computes C_0 and C_1 ; we omit the details. This proves lemma A.2. \square

As in the proof of lemma 3.2, it suffices to establish lemma 3.3 for the integral cut-off in some small neighborhood of the points of minimal distance to the origin. We next compute

$$\frac{\partial}{\partial n} (e^{-|\sqrt{\lambda}x+v|^2/2}) = - \langle \lambda x + \sqrt{\lambda}v, n(x) \rangle e^{-(\lambda|x|^2/2 + \sqrt{\lambda}\langle x, v \rangle + |v|^2/2)}.$$

Again by the divergence theorem and using a local parameterization $x_n = f(x')$ of the boundary, as in the proof of lemma 3.2, we see that the integral we have to study is:

$$e^{-|v|^2/2} \int_{\mathbf{R}^{n-1}} \left(\lambda g_1(x') + \sqrt{\lambda} g_2(x') \right) e^{-\lambda F(x') - \sqrt{\lambda} \psi(x')} dx', \quad (A.4)$$

where

$$\begin{aligned} g_1(x') &= f(x') - \langle x', \nabla f(x') \rangle \\ g_2(x') &= v_n - \langle v', \nabla f(x') \rangle \end{aligned} \quad (A.5)$$

and

$$F(x') = \frac{1}{2} (|x'|^2 + f(x')^2)$$

$$\psi(x') = \langle x', v' \rangle + v_n f(x'),$$

where $v' = (v_1, \dots, v_{n-1})$. We will take the parameterizations (4.3) from the proof of theorem 4.2. Using the same notations as before, and with z_1 playing the rôle of x_n above, we have that

$$F(y, z') = \frac{1}{2} \left(\sum_{j \geq 1} \frac{a_j^+ + a_1^-}{a_1^-} y_j^2 + \sum_{k \geq 2} \frac{a_1^- - a_k^-}{a_1^-} z_k^2 + \frac{1}{a_1^-} \right), \quad (\text{A.6})$$

which is a constant plus a quadratic form in (y, z') , while, writing $v = (v_-, v_+) \in \mathbf{R}^{n_-} \times \mathbf{R}^{n_+}$ and $v_- = (v_{-,1}, v_-)$,

$$\psi(x') = \langle y, v_+ \rangle + \langle z', v_- \rangle + \frac{v_{-,1}}{\sqrt{a_1^-}} \cdot \sqrt{\sum_{j=1}^{n_+} a_j^+ y_j^2 - \sum_{k=2}^{n_-} a_k^- z_k^2}. \quad (\text{A.7})$$

Applying lemma A.2 to (A.4), with amplitudes and phases given by (A.5), (A.6) and (A.7), one easily finds lemma 3.3. An elementary but tedious calculation shows that the second coefficient equals:

$$c_1 = 2(2\pi)^{(n-1)/2} \cdot \frac{v_1^- (a_1^-)^{\frac{n}{2}-1}}{\prod_{j=1}^{n_+} (a_j^+ + a_1^-)^{1/2} \prod_{k=2}^{n_-} (a_1^- - a_k^-)^{1/2}}$$

$$\cdot \left\{ \sqrt{a_1^-} - \frac{1}{2} \left(\sum_{j=1}^{n_+} (\beta_{+,j} + \beta_{+,j}^2 v_{+,j}^2) \frac{\partial^2 \psi}{\partial y_j^2}(0) + \sum_{k=2}^{n_-} (\beta_{-,k} + \beta_{-,k}^2 v_{-,k}^2) \frac{\partial^2 \psi}{\partial z_k^2}(0) \right) \right\},$$

where the β 's are the inverses of the eigenvalues of the quadratic form in (A.3):

$$\beta_{+,j} = \frac{a_1^-}{a_j^+ + a_1^-}, \quad \beta_{-,k} = \frac{a_k^-}{a_1^- - a_k^-}$$

Note that to obtain lemma 3.3 one has to sum four asymptotic series: one for g_1 , one for g_2 and for each of these one according to the choice of sign in (4.3). Changing the

sign leaves F unchanged and has the effect of replacing $v_{-,1}$ by $-v_{-,1}$ in the expression (A.7) for ψ . Then both $u = \nabla\psi(0)$ and $\psi''(0)$ remain the same, as do the first two terms in the expansion (A.3), since the value in 0 of the amplitudes (A.5) and of their first derivatives will not change either. The net effect is an overall factor of 2.

Appendix 2: proof of the uniform VaR expansion

Recall that $W_k = \text{Span} [e_1^-, \dots, e_k^-]$. For $z \in \mathbf{R}^{n_-}$ we will write $z = (z', z'')$ with $z' = (z_1, \dots, z_k) \in W_k = \mathbf{R}^k$ and $z'' = (z_{k+1}, \dots, z_{n_-}) \in \mathbf{R}^{-k+n_-}$ in the orthogonal complement of W_k . Also recall that V_ζ is the subspace spanned by a fixed $\zeta \in \Sigma_k = \Sigma \cap W_k$ and the orthogonal complement of W_k . An orthonormal basis of V_ζ is given by

$$\zeta/|\zeta|, e_{k+1}^-, \dots, e_{n_-}^-, e_1^+, \dots, e_{n_+}^+. \quad (\text{A.8})$$

If we denote the coordinates with respect to this basis by $(s, z_{k+1}, \dots, z_{n_-}, y_1, \dots, y_{n_+})$, then $\Sigma \cap V_\zeta$ will be given by the equation

$$-\frac{s^2}{|\zeta|^2} - \sum_{j=k+1}^{n_-} a_j^- z_j^2 + \sum_{j=1}^{n_+} a_j^+ y_j^2 = 1. \quad (\text{A.9})$$

If $\zeta \in \Sigma_k$, then it is easily seen that

$$T_\zeta(\Sigma_k) = T_\zeta(\Sigma) \cap W_k,$$

where T_ζ denotes the tangent space at ζ . Likewise, using (A.9), one sees that

$$T_\zeta(\Sigma \cap V_\zeta) = T_\zeta(\Sigma) \cap \text{Span} [e_{k+1}^-, \dots, e_{n_-}^-, e_1^+, \dots, e_{n_+}^+].$$

In particular, we have an orthogonal decomposition

$$T_\zeta(\Sigma) = T_\zeta(\Sigma_k) \oplus T_\zeta(\Sigma \cap V_\zeta).$$

As a consequence of this, if $d\sigma_\Sigma(\zeta)$, $d\sigma_{\Sigma_k}(\zeta)$ and $d\sigma_{\Sigma \cap V_\zeta}(\zeta)$ denote the surface measures on Σ , Σ_k and $\Sigma \cap V_\zeta$, respectively. Then, for $\zeta \in \Sigma_k$ we have

$$d\sigma_\Sigma(\zeta) = d\sigma_{\Sigma_k}(\zeta) \otimes d\sigma_{\Sigma \cap V_\zeta}(\zeta) \quad (\text{A.10})$$

We can consider $\Sigma \setminus \Sigma \cap (\{0\} \times W_k)$ as a fiberbundle over Σ_k , with projection π given by

$$\pi(x) = p_W(x)/Q(p_W(x)),$$

p_W being the orthogonal projection onto $W = W_k$. Equivalently, if $(y, z) \in \Sigma$, then $\pi(y, z) = z'/Q_-(z')$. Note that we are using here the non-degeneracy of Q_- . The fiber over $\zeta \in \Sigma_k$ is $\Sigma \cap V_\zeta \setminus \{z' = 0\}$, so that

$$\Sigma \setminus \{z' = 0\} = \cup_{\zeta \in \Sigma_k} \{\zeta\} \times (\Sigma \cap V_\zeta \setminus \{z' = 0\}).$$

We can define a measure on $\Sigma \setminus \{z' = 0\}$ by first integrating along each fiber, with respect to $d\sigma_{\Sigma \cap V_\zeta}$, and subsequently integrating along Σ_k , with respect to $d\sigma_{\Sigma_k}$. The resulting measure will be absolutely continuous with respect to the surface measure $d\sigma_\Sigma$ and there will exist a C^∞ -function f on $\Sigma \setminus \{z' = 0\}$ such that, for any $u \in C_c(\Sigma \setminus \{z' = 0\})$,

$$\int_\Sigma u(x) d\sigma_\Sigma(x) = \int_{\Sigma_k} \left(\int_{\Sigma \cap V_\zeta} u(x) f(x) d\sigma_{\Sigma \cap V_\zeta}(x) \right) d\sigma_{\Sigma_k}(\zeta).$$

It follows from (A.10) that $f(\zeta) = 1$ for $\zeta \in \Sigma_k$.

We will apply this formula to our integral $I(R^2)$ given by (1.1). The starting point is again formula (3.3), with the inner integral, as before, converted to a boundary integral:

$$I(R^2) = \frac{1}{2} R^{n-2} \int_0^1 \frac{dt}{t^{n/2}} \int_\Sigma \frac{\langle n_\Sigma(x), R^2 x + Rv \rangle}{t} e^{-|Rx+v|^2/2t} d\sigma_\Sigma(x).$$

Recalling that v_ζ is the orthogonal projection of v onto V_ζ and writing $v = v_\zeta + (v - v_\zeta)$ we have for any $x \in V_\zeta$ that $|Rx + v|^2 = |Rx + v_\zeta|^2 + |v - v_\zeta|^2$. Hence $I(R^2)$ equals

$$\begin{aligned} & \frac{1}{2} R^{n-2} \int_0^1 \frac{dt}{t^{n/2}} \int_{\Sigma_k} d\sigma_{\Sigma_k}(\zeta) e^{-|v-v_\zeta|^2/2t} \\ & \int_{x \in \Sigma \cap V_\zeta} \frac{\langle n_\Sigma(x), R^2 x + Rv \rangle}{t} e^{-|Rx+v_\zeta|^2/2t} f(x) d\sigma_{\Sigma \cap V_\zeta}(x) \end{aligned}$$

Treating the integral over $\Sigma \cap V_\zeta$ as in (the proof of) lemma 3.3, with $\lambda = R^2/t$ and v replaced by v_ζ/\sqrt{t} , we find that

$$\begin{aligned} & \int_{\Sigma \cap V_\zeta} \frac{\langle n_\Sigma(x), R^2 x + Rv \rangle}{t} e^{-|Rx+v_\zeta|^2/2t} f(x) d\sigma_{\Sigma \cap V_\zeta}(x) \simeq \\ & e^{\gamma(v_\zeta)/t} e^{-R^2|\zeta|^2/2t} \left(\frac{R}{\sqrt{t}} \right)^{-n-k} \sum_{j \geq 0} c_{0,j}(\zeta) \left(\frac{R}{\sqrt{t}} \right)^{2-j} + \frac{c_{1,j}(\zeta)}{\sqrt{t}} \left(\frac{R}{\sqrt{t}} \right)^{1-j}, \end{aligned}$$

where the $c_{0,j}$ come from the term with $\langle n_\Sigma(x), x \rangle$ and the $c_{1,j}$ from the one with $\langle v_\zeta, n_\Sigma(x) \rangle$. Here we used that in the orthonormal basis (A.8), $\Sigma \cap V_\zeta$ is given by the equation (A.9). Also note that $1/|\zeta|^2 > a_{k+1}^- \geq \dots \geq a_{n-}^-$, since

$$a_{k+1}^- |\zeta|^2 = a_{k+1}^- (\zeta_1^2 + \dots + \zeta_k^2) < a_1^- \zeta_1^2 + \dots + a_k^- \zeta_k^2 = -Q(\zeta) = 1.$$

Next, we integrate over t and use Lemma 4.1 again to obtain an expansion in negative powers of R . Remembering that $\Gamma(\zeta) = \gamma(v_\zeta) - |v - v_\zeta|^2/2$ and integrating the resulting asymptotic expansion over $\zeta \in \Sigma_k$ we obtain theorem 5.1, apart from the formula for $C_0(\zeta)$. But using that $f(\zeta) = 1$ on Σ_k one shows easily, as in the proof of lemma 3.3, that

$$c_{0,0} = 2(2\pi)^{(n-k)/2} \langle n_{\Sigma(\zeta)}, \zeta \rangle \cdot (\prod_{j=1}^{n+} (1 + |\zeta|^2 a_j^+)^{1/2} \prod_{j=k+1}^{n-} (1 - |\zeta|^2 a_j^-)^{1/2})^{-1}.$$

One easily checks that $\langle n_{\Sigma(\zeta)}, \zeta \rangle = \langle n_{\Sigma_k}(\zeta), \zeta \rangle$. Finally, lemma 4.1 introduces an extra factor of $|\zeta|^2$ in the denominator. This proves theorem 5.1.

Appendix 3: Upper and Lower Bounds

We first make an elementary remark on the implication of lower and upper bounds for VaR estimates. The point is that a lower bound for the probability distribution function $I(V) = \text{Prob}(\Pi(t, \mathbf{S}(t)) - \Pi(0, \mathbf{S}(0)) < -V^2)$ of the P&L function will lead to a lower bound for the VaR, and an upper bound will lead to an upper bound. In fact, let $I_* = I_*(V)$ and $I^* = I^*(V)$ be an upper and a lower bound, respectively, for I :

$$I_*(V) \leq I(V) \leq I^*(V);$$

$I(V)$ will be a decreasing function of V but this won't be required neither for I_* nor I^* . Pick some level of risk $\alpha \in (0, 1)$ and let $V_*(\alpha), V^*(\alpha)$ be such that

$$I_*(V_*(\alpha)) = I^*(V^*(\alpha)) = \alpha,$$

these equations presumably being easier to solve than the one we're really interested in

$$I(V(\alpha)) = \alpha.$$

Then $\alpha = I_*(V_*(\alpha)) \leq I(V_*(\alpha))$ and $I(V^*(\alpha)) \leq \alpha$ so that, assuming strict monotonicity of $I(V)$,

$$V_*(\alpha) \leq V(\alpha) \leq V^*(\alpha)$$

If $I(V)$ is not strictly monotonous, the defining equation for $V(\alpha)$, the VaR at risk level α above, need not have a unique solution, and one should take the largest, or the sup, of all solutions as the Value-at-Risk. In that case, the lower bound $V_*(\alpha)$ for $V(\alpha)$ will still be valid, but the upper bound will not necessarily hold. However, such a situation will rarely present itself in practice.

We now turn to our quadratic portfolio and assume that our $I(V)$ is given by (3.1), with $V = R^2$. A lower bound can easily be obtained from the following recently established inequalities for the error function, cf. Ruskai and Werner, Theorem 20:

$$g_\pi(x)e^{-x^2/2} \leq E(x) := \int_x^\infty e^{-s^2/2} ds \leq g_4(x)e^{-x^2/2}, \quad x \in \mathbf{R},$$

where

$$g_k(x) = \frac{k}{(k-1)x + \sqrt{x^2 + 2k}}$$

and where these inequalities are optimal in the class of functions g_k considered (our g_k differ slightly from the ones of Ruskai and Werner, since these authors give their estimates for the functions $V_0(x) := 2 \exp(x^2) \int_x^\infty \exp(-u^2) du = \exp(x^2) E(\sqrt{2}x)/\sqrt{2}$).

This estimate can be used to obtain a very rough lower bound for (3.1) in case Q is negative semi-definite (so that $x = z$ in the notations of section 4) and $v = 0$: replace the domain of integration by the smaller one $\{x : x_1^2 \geq R^2/a_1^-\}$ and do the x_2, \dots, x_n integrations. Then $I(R^2) \geq 2(2\pi)^{(n-1)/2} E(R/\sqrt{a_1^-})$, so that the above bound gives

$$I(R^2) \geq 2(2\pi)^{(n-1)/2} \frac{\pi \sqrt{a_1^-} e^{-R^2/2a_1^-}}{(\pi-1)R + \sqrt{R^2 + 2\pi a_1^-}}.$$

Of course, this lower bound is obtained by throwing away all the information contained in Q except it's lowest eigenvalue, and it will be worse than the first term (4.2) of the asymptotic (4.1) if the product $\prod_{k \geq 2} (1 - (a_k^-/a_1^-))$ becomes too small (details can be easily worked out) but it might be useful if this product is close to 1 (corresponding to

a large gap between a_1^- and the other eigenvalues of Q). Note incidentally that it will always be better than the trivial lower bound $2(2\pi)^{(n-1)/2}\sqrt{a_1^-}R^{-1}\exp(-R^2/2a_1^-)$ for the first term of the expansion (4.1); of course, this does not imply anything mathematically speaking, since we're ignoring error terms in (4.1).

Extensions to non-definite Q and non-zero v are possible, but we won't enter into that here.

Upper bounds are somewhat more involved. The idea is to estimate (3.1) by an integral over a bigger domain, which can be written as a disjoint union of pieces on each of which we can use the Ruskai and Werner estimates for $E(x)$. For example, letting $0 \leq s \leq 1$ we have that

$$\{x : Q(x) \leq -R^2\} \subseteq \{x : a_1^- x_1^2 \geq sR^2\} \cup \{x = (x_1, x') : a_1^- x_1^2 \leq sR^2, \sum_{k \geq 2} a_k^- x_k^2 \geq (1-s)R^2\}$$

(recall that we assume Q to be negative semi-definite) where the union on the right is a disjoint one. The integral over the first set on the right can be estimated as before, using now the upper bound for $E(x)$ involving $g_4(x)$. As to the integral over the other part, we can estimate it *very roughly* by an integral over $\{x : x_1^2 \leq sR^2/a_1^-, \|x'\|_\infty \geq (1-s)R^2/N \max(a_2^-, \dots, a_n^-)\}$ ($\|\cdot\|_\infty$ being the l^∞ -norm on \mathbf{R}^{n-1}), which can be written as the disjoint union of 2^{n-1} unbounded rectangles in \mathbf{R}^n , on each of which one can apply the Ruskai and Werner estimates. (For example, one can estimate

$$\int_{x_1^2 \leq sR^2/a_1^-} e^{-x_1^2/2} dx_1 = \sqrt{2\pi} - 2E(R\sqrt{s/a_1^-}) \leq \sqrt{2\pi} - 2e^{-s^2 R^2/2a_1^-} g_\pi(R\sqrt{s/a_1^-}),$$

etc.) The final upper bound for $I(R^2)$ is easy to work out, but complicated to write down. As a final step, one might optimize over $0 < s < 1$.

Finally, we remark that a much better result would be obtained by proving a Ruskai and Werner type estimate for

$$e^{x^2/2} \int_x^\infty s^{n-1} e^{-s^2/2} ds,$$

in particular for large n .

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