Estimating the Spectral Measure of a Multivariate Stable Distribution via Spherical Harmonic Analysis*

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December 13, 2002

*This research partially supported by MITACS Canada.

Abstract

A new method is developed for estimating the spectral measure of a multivariate stable probability measure, by representing the measure as a sum of spherical harmonics.

Keywords: *Multivariate; Stable Distribution; Lévy Distribution; Spherical Harmonic; Gegenbauer Polynomial; Noncommutative Harmonic Analysis; Deconvolution.*

Introduction

Stable probability distributions are the natural generalizations of the normal distribution, and share with it two key properties:

- **Stability:** The normal distribution is **stable** in the sense that, if $\mathbf{X}$ and $\mathbf{Y}$ are independent random variables, with identical normal distributions, then $\mathbf{X}+\mathbf{Y}$ is also normal, and

$$\frac{1}{2^{1/2}}(\mathbf{X}+\mathbf{Y}) \underset{distr}{\cong} \mathbf{X} \underset{distr}{\cong} \mathbf{Y}.$$

  In a similar fashion, if $\mathbf{X}$ and $\mathbf{Y}$ are independent, identically distributed (i.i.d) stable random variables, then $\mathbf{X}+\mathbf{Y}$ is also stable, and its distribution is the same as $\mathbf{X}$ and $\mathbf{Y}$ when renormalized by $2^{-1/\alpha}$. The **stability exponent** α ranges from 0 to 2 . When $\alpha = 2$, we have the familiar normal distribution.

- **Renormalization Limit:** The Central Limit Theorem says that the normal distribution is the natural limiting distribution of a suitably renormalized infinite sum of independent random variables with finite variance. If $\mathbf{X}_1, \mathbf{X}_2, \ldots$ is a sequence of such variables, then the random variables

$$\frac{1}{N^{1/2}} \sum_{n=1}^{N} \mathbf{X}_n,$$

  converge, in density, to a normal distribution. Similarly, if $\{\mathbf{Y}_k\}_{k=1}^{\infty}$ are independent random variables whose distributions decay according to a power law with exponent $-1-\alpha$, then the random variables

$$\frac{1}{N^{1/\alpha}} \sum_{n=1}^{N} \mathbf{Y}_n,$$

  converge, in distribution, to an α-stable distribution

Thus, stable distributions model random aggregations of many small, independent perturbations. For example, stable distributions model the motions of Markovian stochastic processes whose increments exhibit power laws. Stable distributions arise with surprising frequency in certain systems, especially those involving many independent interacting units with sensitive dependencies between them. They have appeared in mathematical finance [32, 17, 16, 33, 51, 18, 44, 22, 23, 14, 35, 8], Internet traffic statistics [47, 37, 36, 60], and arise in mathematical models of random scalar fields [61, 26], radar [40], and signal processing [49, 11, 12], telecommunications [7], and even the power distribution of ocean waves [41].

For further examples, see [61, 50, 20]. The definitive reference on univariate stable distributions is [61]; the definitive reference on multivariate distributions and stable processes is [50]. Other recent references are [21, 45, 3], and a forthcoming book by Nolan [39]; slightly older references are [20, 1].

Although one-dimensional stable distributions are well-understood, there are many open questions in the multivariate regime. The simplicity of the multivariate Gaussian universe does not extend to non-Gaussian multivariate stable distributions. An N-dimensional Gaussian distribution is completely determined by its $N \times N$ covariance matrix, which transforms nicely under linear changes of coordinates. In particular, by orthogonally diagonalizing the matrix, we can find an orthonormal basis for $\mathbb{R}^N$; with respect to this basis, the coordinates of the multivariate normal variable are independent univariate normal variables —this is *Principle Component Analysis*.

For a general multivariate stable distribution, however, the situation is much more complex. Since the marginals do not have finite variance, it does not make sense to define a "covariance matrix" in the usual way; none of the integrals would converge. Various modified notions of "covariance" have been proposed (see, for example, [50]), but these do not transform in any simple way under changes of coordinates. In particular, there is nothing analogous to a "principle components analysis". Instead, the correlation structure of a stable distribution on $\mathbb{R}^D$ is determined by an arbitrary measure, Γ, on the sphere $\mathbb{S}^{D-1} = \left\{\vec{x} \in \mathbb{R}^D \ ; \ \|\vec{x}\| = 1\right\}$, called the **spectral measure**, as follows.

For any $\alpha \in [0, 2)$, define the constant

$$\mathcal{B}_\alpha \quad = \quad \begin{cases} \tan\left(\frac{\pi\alpha}{2}\right) & \text{if} \quad \alpha \neq 1; \\ -\frac{2}{\pi} & \text{if} \quad \alpha = 1. \end{cases}$$

For any real number $r \in \mathbb{R}$, define

$$r^{\langle\alpha\rangle} \quad = \quad \begin{cases} \mathsf{sign}(r) \cdot |r|^\alpha & \text{if} \ \ \alpha \neq 1; \\ r \cdot \log|r| & \text{if} \ \ \alpha = 1; \end{cases} \qquad \text{and} \quad \eta^{(\alpha)}(r) \ = \ -|r|^\alpha \ - \ \mathcal{B}_\alpha \cdot r^{\langle\alpha\rangle}\mathbf{i}. \quad (1)$$

Finally, for any $\vec{\xi} \in \mathbb{R}^D$ and $\mathbf{s} \in \mathbb{S}^{D-1}$, let $\eta^{(\alpha)}\left\langle\vec{\xi}, \mathbf{s}\right\rangle \ = \ \eta^{(\alpha)}\left(\left\langle\vec{\xi}, \mathbf{s}\right\rangle\right)$.

Theorem 1 *Let $\alpha \in [0, 2)$, and let ρ be an α-stable probability measure on $\mathbb{R}^D$, with center $\vec{\mu} \in \mathbb{R}^D$. Then ρ has characteristic function*

$$\chi[\vec{\xi}] \quad = \quad \exp\left(\Phi[\vec{\xi}]\right),$$

where the **log characteristic function** Φ *is given:*

$$\Phi[\vec{\xi}] \quad = \quad \left\langle\vec{\mu}, \vec{\xi}\right\rangle \cdot \mathbf{i} \quad + \quad \int_{\mathbb{S}^{D-1}} \eta^{(\alpha)}\left\langle\vec{\xi}, \mathbf{s}\right\rangle \, d\Gamma[\mathbf{s}], \qquad (2)$$

and where Γ is a nonnegative Borel measure on $\mathbb{S}^{D-1}$.

Proof: See [50, §2.3, p.65], or [30]. □

Γ is called the **spectral measure** of the distribution[1], and is essentially an "infinite-dimensional" data-structure, so it is clear that, in general, no $N \times N$ matrix can possibly be adequate for representing it. A "principle components" type decomposition is only valid when the spectral measure consists of $2D$ antipodaly positioned atoms.

Estimating Γ is much difficult than estimating a covariance matrix. Whereas the terms of a covariance matrix can be directly computed by estimating the correlation between coordinates, Γ is only indirectly visible; the image of Γ under a sort of "spherical convolution" appears in the *logarithm* of the characteristic function of the distribution; there is no more direct way to observe it.

In this paper, we develop an method for estimating Γ from the log-characteristic function Φ. Assume for simplicity that the distribution is centered at the origin, and let the **spherical log-characteristic function** be the function $\mathbf{g} : \mathbb{S}^{D-1} \longrightarrow \mathbb{C}$ determined by restricting Φ to the sphere. Then, for all $\vec{\xi} \in \mathbb{S}^{D-1}$, we have

$$\mathbf{g}[\vec{\xi}] \quad = \quad \int_{\mathbb{S}^{D-1}} \eta^{(\alpha)} \left\langle \vec{\xi}, \mathbf{s} \right\rangle \, d\Gamma[\mathbf{s}]. \tag{3}$$

The characteristic function of a distribution is easy to estimate from empirical data, and thus, we assume we have a good estimate of $\mathbf{g}$ on some suitably fine mesh over $\mathbb{S}^{D-1}$ (the estimation of $\mathbf{g}$ is discussed in detail in [42, Prop. 25, §4.4, p. 48]). Hence, the problem is to recover Γ from $\mathbf{g}$.

Abusing notation, we might rewrite equation (3) as "$\mathbf{g} \;=\; \eta^{(\alpha)} * \Gamma$". If $D = 2$ or $D = 4$, then $\mathbb{S}^{D-1}$ is a topological group, and this "convolution" can be interpreted literally, via the formula:

$$\eta^{(\alpha)} * \Gamma(\vec{\xi}) \;=\; \int_{\mathbb{S}^{D-1}} \eta^{(\alpha)}(\vec{\xi}.\mathbf{s}^{-1}) \, d\Gamma[\mathbf{s}].$$

In other dimensions, however, $\mathbb{S}^{D-1}$ is not a topological group, and therefore, convolution *per se* is not well-defined. We must instead think of $\mathbb{S}^{D-1}$ as a homogeneous manifold under the action of $\mathbb{SO}^D(\mathbb{R})$, and define a kind of "convolution" in terms of this group action.

The eigenfunctions of the Laplacian operator on $\mathbb{S}^{D-1}$ are called **spherical harmonics**, and form an orthonormal basis for $\mathbf{L}^2(\mathbb{S}^{D-1})$, analogous to the Fourier basis for $\mathbf{L}^2(\mathbb{S}^1)$ from classical harmonic analysis. The expression of a function on $\mathbb{S}^{D-1}$ in terms of this basis is called its **spherical Fourier series**. A function $f \in \mathbf{L}^2(\mathbb{S}^{D-1})$ is called **zonal** if it is rotationally invariant around a particular coordinate axis —for example, $\eta^{(\alpha)}$ is zonal. There is a way of 'convolving' arbitrary functions by zonal functions, and, just as in classical harmonic analysis, convolution of a function f by η translates into componentwise multiplication of their respective Fourier coefficients.

[1]This terminology is standard, but somewhat unfortunate, since Γ is unrelated to any one of half a dozen other "spectra" and "spectral measures" currently existent in mathematics. Perhaps it would be more appropriate to call Γ a **Feldheim measure**, since Feldheim [19] was the first to define it.

Thus, to deconvolve f and η, it suffices to divide the Fourier coefficients of $\eta * f$ by those of η. If Γ is reasonably smooth, then the spherical Fourier series converges rapidly in $\mathbf{L}^2$ (Theorem 14). This, in turn, implies rapid convergence of the estimated stable probability density function in $\mathbf{L}^p$, for $1 \leq p \leq \infty$

Our main result is as follows:

Theorem 2 *Let $\alpha \in [0, 2)$, $\alpha \neq 1$, and suppose ρ is an α-stable probability measure on $\mathbb{R}^D$ with density function $F : \mathbb{R}^D \longrightarrow [0, \infty)$, spectral measure Γ, and spherical log-characteristic function $\mathbf{g} : \mathbb{S}^{D-1} \longrightarrow \mathbb{C}$. Suppose that Γ is absolutely continuous relative to the spherical Lebesgue measure $\mathfrak{L}$, and that $d\Gamma = \gamma\ d\mathfrak{L}$, where $\gamma \in \mathbf{L}^2(\mathbb{S}^{D-1};\ \mathfrak{L})$.*

There exists a sequence of functions $\mathcal{Z}_n : \mathbb{S}^{D-1} \times \mathbb{S}^{D-1} \longrightarrow \mathbb{C}$ (for $n \in \mathbb{N}$) and a sequence of constants $\{A_n\}_{n=1}^{\infty}$ with the following properties:

1. *For all $n \in \mathbb{N}$, define $\gamma_n : \mathbb{S}^{D-1} \longrightarrow \mathbb{C}$ by*

$$\gamma_n(\mathbf{s}) \quad = \quad \frac{1}{A_n} \int_{\mathbb{S}^{D-1}} \mathcal{Z}_n(\mathbf{s}, \sigma) \mathbf{g}(\sigma)\ d\mathfrak{L}[\sigma], \qquad \text{for any } \mathbf{s} \in \mathbb{S}^{D-1}.$$

   Then $\{\gamma_n\}_{n=1}^{\infty}$ are orthogonal in $\mathbf{L}^2(\mathbb{S}^{D-1})$, and $\gamma \ = \ \sum_{n=1}^{\infty} \gamma_n$.

2. *For all $N \in \mathbb{N}$, let $\gamma^{[N]} \ = \ \sum_{n=1}^{N} \gamma_n$, let $\Gamma^{[N]} = \gamma^{[N]}\,\mathfrak{L}$, and let $\rho^{[N]}$ be the corresponding α-stable probability measure, with density function $F^{[N]} : \mathbb{R}^D \longrightarrow [0, \infty)$. If $\gamma \in \mathbf{C}^{2M}(\mathbb{S}^{D-1})$, then, for all $p \in [1, \infty]$, $\lim_{k \to \infty} \left\| F - F^{[n]} \right\|_p \ = \ 0$, and furthermore, $\left\| F - F^{[n]} \right\|_\infty$ is of order less than $\mathcal{O}\left(n^{-2M}\right)$.*

Proof: Part (1) is Theorem 12 and Corollary 13. Part (2) is Corollary 16. □

This approach to estimating Γ has three advantages:

1. It is relatively fast, computationally. Computing a spherical Fourier coefficient with precision ϵ is a numerical integration of complexity $\mathcal{O}\left(N^{2(D-1)}\right)$ (where $N \sim 1/\epsilon$), to be contrasted with the $\mathcal{O}\left(N^{3(D-1)}\right)$ required by an explicit matrix-inversion approach such as [29] (see §1).
2. Part (2) of Theorem 2 provides a good convergence rate for the partial sums of the spherical Fourier series, especially when γ is smooth.
3. A spherical Fourier series explicitly represents Γ as a *continuous* object on $\mathbb{S}^{D-1}$, rather than as a sum of atoms. If Γ is, in reality, discrete, this representation might be misleading. In many cases, however, Γ is absolutely continuous, relative to the Lebesgue measure —for example, if the stable distribution is *sub-Gaussian* [50, §2.5]. Also, physical intuition suggests that a continuous spectral measure is more 'natural' than a discrete one.

According to a theorem of Araujo and Giné [1, Corollary 6.20(b), Chapt. 3]) the radial distribution of a stable probability distribution decays most slowly in those angular directions with the heaviest concentration of mass in the spectral measure. Thus, if Γ is continuous, then a discrete approximation of Γ may introduce anomalous asymptotic behaviour to the estimated distribution; a continuous approximation is preferable for this reason.

Organization of this Paper: In §1, we summarize previous work on this problem. In §2, we develop some background material, treating $\mathbb{S}^{D-1}$ as homogeneous manifold under the action of $\mathbb{SO}^{D}(\mathbb{R})$, and reviewing zonal functions, the eigenfunctions of the Laplacian, and a suitable notion of convolution, and provide explicit formulae for the spherical harmonics. In §3, we define the spherical Fourier transform and show how to compute "deconvolution" using this transform. In §4, we characterize the rate of convergence of the spherical Fourier series as an estimate of the spectral measure, and relate this to convergence of the underlying stable distribution.

1 Summary of previous Work

Early on, Press [43] developed an estimation scheme for multivariate stable distributions, through a straightforward generalization of his one-dimensional method. Press's method, however, only works for "pseudo-Gaussian" distributions, with log-characteristic functions of the form:

$$\Phi_{\mathbf{X}}(\vec{\xi}) \quad = \quad \left\langle \vec{\xi}, \vec{\mu} \right\rangle \mathbf{i} \ + \ \left\langle \vec{\xi}, \Omega\vec{\xi} \right\rangle^{\alpha/2},$$

where Ω is some symmetric, positive semidefinite "covariance matrix". If Ω has unit eigenvectors $\vec{\omega}_1, \ldots, \vec{\omega}_D$, with eigenvalues $\lambda_1, \ldots \lambda_D$ (ie. as a covariance matrix, we have "principle components" $\lambda_1\vec{\omega}_1, \ \ldots, \ \lambda_1\vec{\omega}_1$), then the spectral measure of this distribution is symmetric and atomic, with atoms at each of $\pm\vec{\omega}_1, \ldots, \pm\vec{\omega}_D$, with masses $\lambda_1, \ldots \lambda_D$ —in other words:

$$\Gamma \quad = \quad \sum_{d=1}^{D} \lambda_d \left(\delta_{\vec{\omega}_d} + \delta_{-\vec{\omega}_d} \right), \qquad \text{where } \delta_{\vec{\omega}} \text{ is the point mass at } \vec{\omega}.$$

Press proposes to solve for the components of the matrix Ω by empirically estimating the log characteristic function at some collection of frequencies $\{\vec{\xi}_1, \ldots, \vec{\xi}_N\}$, where $N = D(D+1)/2$, and then solving a system of N linear equations. He claims that his method will generalize to a *sum* of pseudo-Gaussians:

$$\Phi_{\mathbf{X}}(\vec{\xi}) \quad = \quad \left\langle \vec{\xi}, \vec{\mu} \right\rangle \mathbf{i} \ + \ \sum_{m=1}^{M} \left\langle \vec{\xi}, \Omega_m\vec{\xi} \right\rangle^{\alpha/2}.$$

(where $\Omega_1, \ldots, \Omega_M$ are linearly independent, symmetric, positive semidefinite matrices). However, in this case, one no longer ends up with a system of linear equations, so it is not clear that the method is tractable. In any event, Press's method only applies to multivariate distributions with particularly simple atomic spectral measures, which furthermore must be symmetrically distributed. Empirical evidence (see, for example, [5]) suggests that the stable distributions found in financial data are significantly skewed; symmetry is not a reasonable assumption.

Cheng, Rachev and Xin [53, 6] develop a more sophisticated method, by expressing a stable random vector in spherical polar coordinates, and then examining the order statistics of the radial component, as a function of the angular component. They utilize the aforementioned theorem of Araujo and Giné [1] stating that the radial distribution decays most slowly in those angular directions with the heaviest concentration of spectral mass; these differences in decay rate are then used to estimate the density distribution of the spectral measure.

More generally, Hurd *et al.* [34] consider any multivariate, infinitely-divisble distribution ρ whose Lévy-Khintchine measure λ takes the form

$$d\lambda[r \cdot \mathbf{s}] \;=\; f(r)\; dr\; d\Gamma[\mathbf{s}],$$

where $\mathbf{s} \in \mathbb{S}^{D-1}$ and Γ is some "spectral measure" on $\mathbb{S}^{D-1}$, while $r \in [0, \infty)$, and $f : [0,\infty) \longrightarrow [0,\infty)$ is some function asymptotically of order $f(r) \sim \mathcal{O}\left(r^{-\alpha-1}\right)$. A result similar to that of Araujo and Giné [1] is shown for this class of distributions, providing a mechanism for estimating Γ from empirical data by looking at the angular distribution of extremal events.

Nolan, Panorska, and McCulloch [29, 28], develop a method based upon a discrete approximation of the spectral measure. If the spectral measure is treated as a sum of a finite number of atoms,

$$\Gamma \;=\; \sum_{\mathbf{a} \in \mathcal{A}} \gamma_{\mathbf{a}} \delta_{\mathbf{a}},$$

then, for any fixed $\vec{\xi} \in \mathbb{S}^{D-1}$, the function $\eta^{(\alpha)}_{\vec{\xi}}(\mathbf{s}) = \eta^{(\alpha)}\left\langle \vec{\xi}, \mathbf{s} \right\rangle$ of Theorem 1 can be restricted to a function $\eta^{(\alpha)}_{\vec{\xi}} : \mathcal{A} \longrightarrow \mathbb{C}$. The set of all discrete measures supported on $\mathcal{A}$ is a finite-dimensional vector space, which we can identify with $\mathbb{C}^{\mathcal{A}}$, and $\eta^{(\alpha)}_{\vec{\xi}}$ is just a linear functional on this vector space. If $\Xi \subset \mathbb{S}^{D-1}$ is some finite set, then we can define a linear map $F : \mathbb{C}^{\mathcal{A}} \longrightarrow \mathbb{C}^{\Xi}$, where, for each $\vec{\xi} \in \Xi$,

$$F(\Gamma)_{\vec{\xi}} \;=\; \mathbf{g}(\vec{\xi}) \;=\; \int_{\mathbb{S}^{D-1}} \eta^{(\alpha)}_{\vec{\xi}}\; d\Gamma.$$

The method of Nolan *et al.* then comes down to *inverting* this linear transformation to recover Γ from an empirical estimate of $\mathbf{g}$. They explicitly implemented their method in the two-dimensional case (ie. when the spectral measure lives on a circle), and tested it against a variety of distributions. They found that it worked fairly well for a variety of measures on the circle, and consistently outperformed the method of Chen *et al.* The methods of Chen *et al.* and Nolan *et al.* are also discussed in [38, §5].

2 Zonal functions, Laplacians, and Convolution on Spheres[2]

$\mathbb{S}^{D-1}$ is a compact Riemannian manifold, and $\mathbb{G} = \mathbb{SO}^D(\mathbb{R})$ is a (nonabelian) compact Lie group, acting transitively and isometrically on $\mathbb{S}^{D-1}$ by rotations. We will develop a version of harmonic analysis on $\mathbb{S}^{D-1}$ as a homogeneous Riemannian manifold (this theory is actually applicable to any homogeneous Riemannian manifold; it may be helpful to keep this in mind).

Let $\mathfrak{L}$ be the canonical volume measure induced on $\mathbb{S}^{D-1}$ by its Riemann structure. For example, on $\mathbb{S}^2$, $\mathfrak{L}$ is the usual "surface area" measure. $\mathbb{S}^{D-1}$ is compact, so $\mathfrak{L}$ is finite —assume $\mathfrak{L}$ is normalized to have total mass 1. Let

$$\mathbf{L}^2(\mathbb{S}^{D-1}) \quad = \quad \left\{ f : \mathbb{S}^{D-1} \longrightarrow \mathbb{C} \; ; \; \int_{\mathbb{S}^{D-1}} |f(\mathbf{s})|^2 \, d\mathfrak{L}[\mathbf{s}] \; < \; \infty \right\}.$$

The action of $\mathbb{G}$ on $\mathbb{S}^{D-1}$ induces a linear $\mathbb{G}$-action on $\mathbf{L}^2(\mathbb{S}^{D-1})$ in the obvious way: if $\phi \in \mathbf{L}^2(\mathbb{S}^{D-1})$ and $g \in \mathbb{G}$, then $g.\phi : \mathbb{S}^{D-1} \longrightarrow \mathbb{C}$ is defined: $g.\phi(m) \; = \; \phi(g.m)$.

Let $\mathbf{C}^\infty(\mathbb{S}^{D-1})$ be the space of smooth, complex-valued functions on $\mathbb{S}^{D-1}$. $\mathfrak{L}$ is finite, so $\mathbf{C}^\infty(\mathbb{S}^{D-1})$ is a linear subspace of $\mathbf{L}^2(\mathbb{S}^{D-1})$ (though not a closed subspace). $\mathbb{G}$ acts smoothly on $\mathbb{S}^{D-1}$, so $\mathbf{C}^\infty(\mathbb{S}^{D-1})$ is $\mathbb{G}$-invariant. We consider the restricted action of $\mathbb{G}$ on $\mathbf{C}^\infty(\mathbb{S}^{D-1})$.

Let $\triangle : \mathbf{C}^\infty(\mathbb{S}^{D-1}) \longrightarrow \mathbf{C}^\infty(\mathbb{S}^{D-1})$ be the Laplacian operator.

Theorem 3 (The Laplacian on $\mathbb{S}^D$) [54]

Endow the circle $\mathbb{S}^1$ with the angular coordinate system $\theta \in (0, 2\pi)$, so that any point on $\mathbb{S}^1_* \; = \; \mathbb{S}^1 - \{(1,0)\}$ has coordinates $(\cos(\theta), \; \sin(\theta))$.

If $f : \mathbb{S}^1_* \longrightarrow \mathbb{C}$, then, in this coordinate system, $\triangle_{\mathbb{S}^1} f \; = \; \dfrac{\partial^2 f}{\partial \theta^2}$.

For $D \geq 2$, let $\mathbb{S}^D_* \; = \; \mathbb{S}^D \setminus \left(\mathbb{R}^{D-1} \times [0, \infty) \times \{0\}\right)$, and define the diffeomorphism

$$\begin{array}{rcl} \mathbb{S}^{D-1}_* \times (0, \pi) & \longrightarrow & \mathbb{S}^D_* \\ (\mathbf{s}, \phi) & \mapsto & [\cos(\phi); \;\; \sin(\phi) \cdot \mathbf{s}] \end{array}$$

Then we have the following inductive formula:

$$\triangle_{\mathbb{S}^D} f \quad = \quad \frac{\partial^2 f}{\partial \phi^2} \; + \; (D-1)\cot(\phi) \frac{\partial f}{\partial \phi} \; + \; \frac{1}{\sin(\phi)^2} \triangle_{\mathbb{S}^{D-1}} f.$$ □

$\triangle$ commutes with the isometric $\mathbb{G}$ action: for all $g \in \mathbb{G}$,

$$\triangle(g.\phi) \; = \; g.(\triangle \phi).$$

Let $\Lambda := \{\lambda \in \mathbb{C} \; ; \; -\lambda \text{ is an eigenvalue of } \triangle\}$, and for each $\lambda \in \Lambda$, let

$$\mathbb{V}_\lambda \quad = \quad \left\{ \phi \in \mathbf{C}^\infty(\mathbb{S}^{D-1}) \; ; \; \triangle \phi = -\lambda \phi \right\}$$

[2] This review of background material loosely follows [56, §3.3]. A friendlier approach is [42, §5.1-5.2].

be the corresponding eigenspace. Thus, $\mathbb{V}_\lambda$ is a $\mathbb{G}$-invariant subspace.

The eigenfunctions of the Laplacian on $\mathbb{S}^{D-1}$ are called **spherical harmonics**. Further information on spherical harmonics can be found in [56, §4.3], [4, ch.II], [25, ch.3,ch.5], [31, ch.7-8], [54, §11,§12], and also in [46, 48, 24, 2, 9, 52, 55, 58].

Let $\mathbf{e} = (1, 0, \ldots, 0) \in \mathbb{S}^{D-1}$, and define

$$\mathbb{G}_{\mathbf{e}} \;=\; \{g \in \mathbb{G} \,;\, g.\mathbf{e} \;=\; \mathbf{e}\},$$

the set of all orthogonal transformations of $\mathbb{R}^D$ fixing the $\mathbf{e}$-axis. In other words, $\mathbb{G}_{\mathbf{e}}$ is the set of all "rotations" of the remaining $(D-1)$ dimensions about this axis; hence, there is a natural isomorphism $\mathbb{G}_{\mathbf{e}} \cong \mathbb{SO}^{D-1}(\mathbb{R})$. $\mathbb{G}_{\mathbf{e}}$ is thus a connected, compact subgroup of $\mathbb{G}$. The action of $\mathbb{G}$ upon $\mathbf{C}^\infty(\mathbb{S}^{D-1})$ restricts to an action of $\mathbb{G}_{\mathbf{e}}$, and the spaces $\mathbb{V}_\lambda$ remain invariant under this new action.

A function $\zeta \in \mathbf{C}^\infty(\mathbb{S}^{D-1})$ is called **zonal** (relative to $\mathbb{G}$ and the fixed point $\mathbf{e} \in \mathbb{S}^{D-1}$) if it is invariant under the action of $\mathbb{G}_{\mathbf{e}}$. Formally, for any $\mathbb{G}_{\mathbf{e}}$-invariant subspace $\mathbb{V} \subset \mathbf{C}^\infty(\mathbb{S}^{D-1})$, define

$$\mathcal{Z}_{\mathbf{e}}(\mathbb{V}) \quad = \quad \{\zeta \in \mathbb{V} \,;\, \forall g \in \mathbb{G}_{\mathbf{e}}, \;\; g.\zeta \;=\; \zeta\}.$$

Thus, the zonal elements of $\mathbf{C}^\infty(\mathbb{S}^{D-1})$ are smooth functions which are *rotationally invariant* about the $\mathbf{e}$ axis. Clearly, any zonal function must be of the form $\zeta(x_1, x_2, \ldots, x_D) \;=\; \zeta_1(x_1)$ where $\zeta_1 : [-1, 1] \longrightarrow \mathbb{C}$.

Proposition 4 *1. If $\mathbb{V} \subset \mathbf{C}(\mathbb{S}^{D-1})$ is a nontrivial $\mathbb{G}$-invariant subspace, then $\mathcal{Z}_{\mathbf{e}}(\mathbb{V})$ is nontrivial.*

2. If $\dim(\mathcal{Z}_{\mathbf{e}}(\mathbb{V})) \;=\; 1$, then $\mathbb{V}$ is an irreducible $\mathbb{G}$-module.

Proof:

Proof of Part 1:

Claim 1: *$\mathbb{V}$ contains an element ϕ such that $\phi(\mathbf{e}) \neq 0$.*

Proof: Since $\mathbb{V}$ is nontrivial, there is some $\psi \in \mathbb{V}$ which is nonzero *somewhere* –say $\psi(x) \neq 0$. Since $\mathbb{G}$ acts transitively on $\mathbb{S}^{D-1}$, find $g \in \mathbb{G}$ so that $g.\mathbf{e} = x$. Thus, if $\phi = g.\psi$, then $\phi(\mathbf{e}) = \psi(g.\mathbf{e}) = \psi(x) \neq 0$. Since $\mathbb{V}$ is $\mathbb{G}$-invariant, $\phi \in \mathbb{V}$ is the element we seek. .... □ [Claim 1]

Now, $\mathbb{G}_{\mathbf{e}}$ is a closed subgroup of the compact group $\mathbb{G}$; thus, $\mathbb{G}_{\mathbf{e}}$ is compact, so it has a finite Haar measure $\mathfrak{H}$. Define

$$\zeta \;:=\; \int_{\mathbb{G}_{\mathbf{e}}} g.\phi \; d\mathfrak{H}[g].$$

Since $\mathfrak{H}$ is finite, this integral is well-defined. Since $\mathbb{V}$ is a closed, $\mathbb{G}$-invariant subspace, the element ζ is in $\mathbb{V}$. Furthermore, since $\zeta(\mathbf{e}) = \phi(\mathbf{e})$, and $\phi(\mathbf{e}) \neq 0$, we conclude that ζ is nontrivial. Finally, note that ζ is $\mathbb{G}_{\mathbf{e}}$-invariant by construction —in other words, it is zonal.

Proof of Part 2: Suppose $\mathbb{V} = \mathbb{V}_1 \oplus \mathbb{V}_2$, where $\mathbb{V}_1, \mathbb{V}_2$ are $\mathbb{G}$-invariant. Then by **Part 1**, we can construct linearly independent zonal functions $\zeta_1 \in \mathcal{Z}_{\mathbf{e}}(\mathbb{V}_1)$ and $\zeta_2 \in \mathcal{Z}_{\mathbf{e}}(\mathbb{V}_2)$. Since $\zeta_1, \zeta_2 \in \mathcal{Z}_{\mathbf{e}}(\mathbb{V})$, this contradicts the hypothesis that $\dim(\mathcal{Z}_{\mathbf{e}}(\mathbb{V})) = 1$. □

For any $r > 0$, let $\mathbb{B}(\mathbf{e}, r)$ be the ball of radius r about $\mathbf{e}$ in $\mathbb{S}^{D-1}$, relative to the intrinsic Riemannian metric.

Lemma 5 *For all $r > 0$, $\mathbb{G}_{\mathbf{e}}$ acts transitively on $\partial\mathbb{B}(\mathbf{e}, r)$ in $\mathbb{S}^{D-1}$.*

Proposition 6 *Each eigenspace $\mathbb{V}_\lambda$ of $\triangle_{\mathbb{S}^{D-1}}$ is an irreducible $\mathbb{G}$-module.*

Proof: By Proposition 4, it suffices to show that $\dim[\mathcal{Z}_{\mathbf{e}}(\mathbb{V}_\lambda)] = 1$. So, suppose that $\zeta_1, \zeta_2 \in \mathcal{Z}_{\mathbf{e}}(\mathbb{V}_\lambda)$ are linearly independent. Since they are zonal, $\zeta_1(\mathbf{s})$ and $\zeta_2(\mathbf{s})$ are functions only of the distance from $\mathbf{s}$ to $\mathbf{e}$. So, for some $\mathbf{s} \in \mathbb{S}^{D-1}$ with $\mathsf{distance}(\mathbf{s}, \mathbf{e}) = r$, let $z_1 = \zeta_1(\mathbf{s})$ and $z_2 = \zeta_2(\mathbf{s})$, and let $\zeta := z_2\zeta_1 - z_1\zeta_2$. Thus, ζ is also zonal. We aim to show that ζ is the zero function; thus, ζ_1 and ζ_2 are just scalar multiples of one another.

Now, by construction, $\zeta(\mathbf{s}) = 0$, and thus, $\zeta \equiv 0$ on $\partial\mathbb{B}(\mathbf{e}; r)$. At the same time, however, ζ is a linear combination of two elements of $\mathbb{V}_\lambda$; hence, it is also in $\mathbb{V}_\lambda$ —ie. ζ is a $(-\lambda)$-eigenfunctions of $\triangle$. Fix λ, and let r get small. If r is made small enough, then the homogeneous Dirichlet boundary condition $\zeta_{|\partial\mathbb{B}(\mathbf{e};r)} \equiv 0$ forces the smallest eigenvalue of $\triangle$ to be larger in absolute value than λ, creating a contradiction. □

One consequence of this irreducibility is

Theorem 7 (Schur's Lemma) [57]
Let $\mathbb{V}$ be a complex Banach space and an irreducible $\mathbb{G}$-module. If $\phi : \mathbb{V} \longrightarrow \mathbb{V}$ is a continuous $\mathbb{C}$-linear map that commutes with the $\mathbb{G}$-action, then ϕ is multiplication by a scalar. □

Now consider the D-torus $\mathbb{T}^D$, equipped with the standard equivariant metric. The eigenfunctions of the Laplacian on are the periodic functions of the form

$$\mathcal{E}_{\mathbf{n}}(\mathbf{x}) \quad = \quad \exp\left(2\pi\mathbf{i} \cdot \langle \mathbf{n}, \mathbf{x} \rangle\right),$$

with $\mathbf{n} \in \widehat{\mathbb{T}^D} \cong \mathbb{Z}^D$, where $\mathbf{x} \in [0,1)^D$ and $[0,1)^D$ is identified with $\mathbb{T}^D$ in the obvious way. These eigenfunctions form an orthonormal basis for $\mathbf{L}^2(\mathbb{T}^D)$. The same is true for arbitrary homogeneous Riemannian manifolds, and in particular, for the sphere:

Theorem 8 *$\mathbf{L}^2(\mathbb{S}^{D-1})$ is an orthogonal direct sum of the eigenspaces of $\triangle$. In other words,*

$$\mathbf{L}^2(\mathbb{S}^{D-1}) \quad = \quad \bigoplus_{\lambda \in \Lambda} \mathbb{V}_\lambda,$$

where the subspaces $\mathbb{V}_{\lambda_1}$ and $\mathbb{V}_{\lambda_2}$ are orthogonal whenever $\lambda_1 \neq \lambda_2$.

Proof: See for example [59, ch.6, p.255], [10, Thm 3.21, p.156]. Or treat $\triangle$ as an elliptic differential operator, and use [15, §6.5, Thm.1], Alternately, employ the Spectral Theorem for unbounded self-adjoint operators [13, §X.4]. $\square$

If $\eta : \mathbb{S}^{D-1} \times \mathbb{S}^{D-1} \longrightarrow \mathbb{C}$, then say that η is a $\mathbb{G}$-**equivariant** if, for all $\sigma, \mathbf{s} \in \mathbb{S}^{D-1}$ and $g \in \mathbb{G}$, $\eta(g.\sigma, g.\mathbf{s}) = \eta(\sigma, \mathbf{s})$. Since $\mathbb{G}$ acts isometrically and transitively on $\mathbb{S}^{D-1}$, this is equivalent to saying that $\eta(\mathbf{s}, \sigma)$ is a function only of the inner product $\langle \mathbf{s}, \sigma \rangle$. We will thus often write $\eta(\mathbf{s}, \sigma)$ as '$\eta\langle \mathbf{s}, \sigma \rangle$'. For instance, the function $\eta^{(\alpha)} : \mathbb{S}^{D-1} \times \mathbb{S}^{D-1} \longrightarrow \mathbb{C}$ defined by equation (1) is $\mathbb{G}$-equivariant.

If η is $\mathbb{G}$-equivariant, $\phi : \mathbb{S}^{D-1} \longrightarrow \mathbb{C}$, and both are $\mathfrak{L}$-integrable, then we define the **convolution** $\eta * \phi : \mathbb{S}^{D-1} \longrightarrow \mathbb{C}$ by

$$(\eta * \phi)(\mathbf{s}) \quad = \quad \int_{\mathbb{S}^{D-1}} \eta(\mathbf{s}, \sigma) \phi(\sigma) \; d\mathfrak{L}[\sigma].$$

For example, if Γ is a measure on $\mathbb{S}^{D-1}$, with Radon-Nikodym derivative $\gamma : \mathbb{S}^{D-1} \longrightarrow \mathbb{C}$, then $\eta * \gamma : \mathbb{S}^{D-1} \longrightarrow \mathbb{C}$ is defined

$$\eta * \gamma(\mathbf{s}) \quad = \quad \int_{\mathbb{S}^{D-1}} \eta(\mathbf{s}, \sigma) \gamma(\sigma) \; d\mathfrak{L}[\sigma] \quad = \quad \int_{\mathbb{S}^{D-1}} \eta(\mathbf{s}, \sigma) \; d\Gamma[\sigma].$$

In particular, if Γ is a spectral measure and $\eta = \eta^{(\alpha)}$, then this formula is identical to equation (3). In other words,

$$\eta^{(\alpha)} * \gamma \quad = \quad \mathbf{g},$$

where $\mathbf{g}$ is the spherical log-characteristic function.

Recall again the case of $\mathbb{T}^D$. The eigenfunctions of the Laplacian, $\{\mathcal{E}_{\mathbf{n}} \; ; \; \mathbf{n} \in \mathbb{Z}^D\}$, are well-behaved under convolution: classical harmonic analysis tells us that

$$\left(\sum_{\mathbf{n} \in \mathbb{Z}^D} a_{\mathbf{n}} \mathcal{E}_{\mathbf{n}}(\mathbf{x}) \right) * \left(\sum_{\mathbf{n} \in \mathbb{Z}^D} b_{\mathbf{n}} \mathcal{E}_{\mathbf{n}}(\mathbf{x}) \right) \quad = \quad \sum_{\mathbf{n} \in \mathbb{Z}^D} (a_{\mathbf{n}} \cdot b_{\mathbf{n}}) \, \mathcal{E}_{\mathbf{n}}(\mathbf{x}).$$

A similar formula holds for zonal spherical harmonics.

Proposition 9 (Convolution and Eigenfunctions)

Let $\eta : \mathbb{S}^{D-1} \times \mathbb{S}^{D-1} \longrightarrow \mathbb{C}$ be $\mathbb{G}$-equivariant. Fix $\lambda \in \Lambda$ and $\zeta \in \mathcal{Z}_{\mathbf{e}}(\mathbb{V}_\lambda)$, and define complex constant $A_\lambda = \dfrac{(\eta * \zeta)(\mathbf{e})}{\zeta(\mathbf{e})}$. Then for any $\phi \in \mathbb{V}_\lambda$, $\eta * \phi = A_\lambda \cdot \phi$.

Proof: Let $T_\eta : \mathcal{C}^\infty(\mathbb{S}^{D-1}) \longrightarrow \mathcal{C}^\infty(\mathbb{S}^{D-1})$ be defined: $T_\eta(\phi) = \eta * \phi$.

Claim 1: *T_η commutes with the $\mathbb{G}$-action: for all $g \in \mathbb{G}$, $T_\eta[g.\phi] = g.T_\eta[\phi]$.*

Proof: For any $\sigma \in \mathbb{S}^{D-1}$,

$$\begin{aligned} T_\eta[g.\phi](\sigma) &= [\eta * (g.\phi)](\sigma) \\ &= \int_{\mathbb{S}^{D-1}} \eta(\sigma, \mathbf{s})\phi(g.\mathbf{s}) \ d\mathfrak{L}[\mathbf{s}] \\ &\underset{(1)}{=} \int_{\mathbb{S}^{D-1}} \eta(\sigma, g^{-1}.\mathbf{s}')\phi(\mathbf{s}') \ d\mathfrak{L}[\mathbf{s}'] \\ &\underset{(2)}{=} \int_{\mathbb{S}^{D-1}} \eta(g.\sigma, \mathbf{s}')\phi(\mathbf{s}') \ d\mathfrak{L}[\mathbf{s}'] \\ &= (\eta * \phi)(g.\sigma) \quad = \quad g.(\eta * \phi)(\sigma). \end{aligned}$$

(1) where $\mathbf{s}' := g.\mathbf{s}$. (2) Because η is $\mathbb{G}$-equivariant. ...... □ `[Claim 1]`

Claim 2: *T_η commutes with $\triangle$.*

Proof: For each $\mathbf{s} \in \mathbb{S}^{D-1}$, define $\eta_\mathbf{s} : \mathbb{S}^{D-1} \longrightarrow \mathbb{C}$ by $\eta_\mathbf{s}(\sigma) = \eta(\mathbf{s}, \sigma) = \eta(\sigma, \mathbf{s})$. Thus,

$$(\eta * \phi)(\sigma) \quad = \quad \int_{\mathbb{S}^{D-1}} \eta(\sigma, \mathbf{s}) \cdot \phi(\mathbf{s}) \ d\mathfrak{L}[\mathbf{s}] \quad = \quad \int_{\mathbb{S}^{D-1}} \phi(\mathbf{s}) \cdot \eta_\mathbf{s}(\sigma) \ d\mathfrak{L}[\mathbf{s}].$$

Hence,

$$\triangle(\eta * \phi)(\sigma) \quad = \quad \triangle \int_{\mathbb{S}^{D-1}} \phi(\mathbf{s}) \cdot \eta_\mathbf{s}(\sigma) \ d\mathfrak{L}[\mathbf{s}] \quad = \quad \int_{\mathbb{S}^{D-1}} \phi(\mathbf{s}) \cdot \triangle\eta_\mathbf{s}(\sigma) \ d\mathfrak{L}[\mathbf{s}], \tag{4}$$

because $\triangle$ is a linear operator.

Claim 2.1: $\triangle\eta_\mathbf{s}(\sigma) = \triangle\eta_\sigma(\mathbf{s})$.

Proof: Find some $g \in \mathbb{G}$ so that $g.\sigma = \mathbf{s}$ and $g.\mathbf{s} = \sigma$. Thus for any $\sigma \in \mathbb{S}^{D-1}$,
$\eta_\sigma(\sigma) \ = \ \eta(\sigma, \sigma) \ = \ \eta(g.\sigma, \ g.\sigma) \ = \ \eta(\mathbf{s}, \ g.\sigma) \ = \ \eta_\mathbf{s}(g.\sigma) \ = \ (g.\eta_\mathbf{s})(\sigma)$.

$$\text{In other words,} \quad \eta_\sigma \quad = \quad (g.\eta_\mathbf{s}).$$
$$\text{Thus,} \quad \triangle\eta_\sigma \quad = \quad \triangle(g.\eta_\mathbf{s}) \ = \ g.(\triangle\eta_\mathbf{s}).$$

In particular, $\triangle\eta_\sigma(\mathbf{s}) \ = \ g.(\triangle\eta_\mathbf{s})(\mathbf{s}) \ = \ \triangle\eta_\mathbf{s}(g.\mathbf{s}) \ = \ \triangle\eta_\mathbf{s}(\sigma)$. □ `[Claim 2.1]`

Hence, we can rewrite expression (4) as:

$$\int_{\mathbb{S}^{D-1}} \phi(\mathbf{s}) \cdot \triangle\eta_\sigma(\mathbf{s}) \ d\mathfrak{L}[\mathbf{s}].$$

But $\mathbb{S}^{D-1}$ is a manifold without boundary, so $\triangle$ is self-adjoint [59, ch.6]. Hence,

$$\begin{aligned} \int_{\mathbb{S}^{D-1}} \phi(\mathbf{s}) \cdot \triangle\eta_\sigma(\mathbf{s}) \ d\mathfrak{L}[\mathbf{s}] \quad &= \quad \int_{\mathbb{S}^{D-1}} \triangle\phi(\mathbf{s}) \cdot \eta_\sigma(\mathbf{s}) \ d\mathfrak{L}[\mathbf{s}] \\ = \quad \int_{\mathbb{S}^{D-1}} \eta(\sigma, \mathbf{s}) \cdot \triangle\phi(\mathbf{s}) \ d\mathfrak{L}[\mathbf{s}] \quad &= \quad \eta * (\triangle\phi)(\sigma). \end{aligned}$$

.... □ `[Claim 2]`

It follows from **Claim 2** that T_η must leave invariant all eigenspaces of $\triangle$; in other words, for all $\lambda \in \Lambda$, $\mathbb{V}_\lambda$ is invariant under T_η.

But by **Claim 1**, the restricted map $(T_\eta)_{|\mathbb{V}_\lambda} : \mathbb{V}_\lambda \longrightarrow \mathbb{V}_\lambda$ is then an isomorphism of linear $\mathbb{G}$-modules. Since $\mathbb{G}$ acts *irreducibly* on $\mathbb{V}_\lambda$ (by Proposition 6), it follows from Schur's Lemma that T_η must act on $\mathbb{V}_\lambda$ by scalar multiplication: thus, there is some $A_\lambda \in \mathbb{C}$ so that, for all $\phi \in \mathbb{V}_\lambda$,

$$T_\eta(\phi) \;=\; A_\lambda \cdot \phi.$$

In other words, $\eta * \phi = A_\lambda \cdot \phi$. In particular, if $\zeta \in \mathcal{Z}_{\mathbf{e}}(\mathbb{V}_\lambda)$, then $\eta * \zeta \;=\; A_\lambda \cdot \zeta$; hence we must have $A_\lambda = \frac{\eta*\zeta(\mathbf{e})}{\zeta(\mathbf{e})}$. □

Corollary 10 *Let $\zeta \in \mathcal{Z}_{\mathbf{e}}(\mathbb{V}_\lambda)$ be a zonal eigenfunction, normalized so that $\|\zeta\|_2 = 1$. Define $Z : \mathbb{S}^{D-1} \times \mathbb{S}^{D-1} \longrightarrow \mathbb{C}$ by*

$$Z(\sigma, \mathbf{s}) \;=\; \zeta(g_\sigma^{-1}.\mathbf{s}),$$

where $g_\sigma \in \mathbb{G}$ is any element so that $g_\sigma.\mathbf{e} = \sigma$. Then Z is well-defined, independent of the choice of g_σ, and is $\mathbb{G}$-equivariant. Define $\mathbb{P}_\lambda : \mathbf{L}^2(\mathbb{S}^{D-1}) \longrightarrow \mathbf{L}^2(\mathbb{S}^{D-1})$ by

$$\mathbb{P}_\lambda(\phi) \;=\; \zeta(\mathbf{e}) \cdot (Z * \phi)\,.$$

Then $\mathbb{P}_\lambda$ is the **orthogonal projection** *from $\mathbf{L}^2(\mathbb{S}^{D-1})$ onto the eigenspace $\mathbb{V}_\lambda$.*

Proof:

Proof of "Well Defined": If $g_1, g_2 \in \mathbb{G}$ so that $g_1.\mathbf{e} = g_2.\mathbf{e} = \sigma$, then $g_1^{-1}.g_2.\mathbf{e} = \mathbf{e}$; thus, $g_1^{-1}.g_2 \in \mathbb{G}_{\mathbf{e}}$. But ζ is zonal about $\mathbf{e}$, so $\zeta(g_2^{-1}.\mathbf{s}) \;=\; \zeta(g_1^{-1}.g_2.g_2^{-1}.\mathbf{s}) \;=\; \zeta(g_1^{-1}.\mathbf{s})$.

Proof of "Equivariant": Let $\sigma, \mathbf{s} \in \mathbb{S}^{D-1}$, and $h \in \mathbb{G}$. Note that we can pick $g_{(h.\sigma)} = h.g_\sigma$. Thus,

$$\begin{aligned} Z(h.\sigma,\ h.\mathbf{s}) \;&=\; \zeta(g_{(h.\sigma)}^{-1}.h.\mathbf{s}) \;=\; \zeta((h.g_\sigma)^{-1}.h.\mathbf{s}) \;=\; \zeta(g_\sigma^{-1}.h^{-1}.h.\mathbf{s}) \;=\; \zeta(g_\sigma^{-1}.\mathbf{s}) \\ &=\; Z(\sigma,\ \mathbf{s}). \end{aligned}$$

Proof of "Orthogonal Projection": Since $\mathbb{P}_\lambda$ is defined by a convolution integral, it is clearly a linear operator. It suffices to show that $\mathbb{P}_\lambda$ fixes $\mathbb{V}_\lambda$, and annihilates $\mathbb{V}_\lambda^\perp$.

If $\phi \in \mathbb{V}_\lambda$, then by Proposition 9,

$$Z * \phi \quad=\quad \frac{(Z * \zeta)(\mathbf{e})}{\zeta(\mathbf{e})} \cdot \phi.$$

Thus, $\mathbb{P}_\lambda(\phi) \;=\; (Z * \zeta)(\mathbf{e}) \cdot \phi$, so it suffices to show that $(Z * \zeta)(\mathbf{e}) = 1$. But:

$$\begin{aligned} Z * \zeta(\mathbf{e}) \;&=\; \int_{\mathbb{S}^{D-1}} Z(\mathbf{e}, \mathbf{s}) \zeta(\mathbf{s}) \; d\mathfrak{L}[\mathbf{s}] \\ &=\; \int_{\mathbb{S}^{D-1}} \zeta(g_{\mathbf{e}}^{-1}.\mathbf{s}) \cdot \zeta(\mathbf{s}) \; d\mathfrak{L}[\mathbf{s}] \\ &=\; \int_{\mathbb{S}^{D-1}} \zeta(\mathbf{s}) \cdot \zeta(\mathbf{s}) \; d\mathfrak{L}[\mathbf{s}] \qquad (\text{since } \; g_{\mathbf{e}} \;=\; \mathbf{Id}) \\ &=\; \|\zeta\|_2^{\,2} \quad=\quad 1, \quad \text{by hypothesis.} \end{aligned}$$

On the other hand, if $\phi \in \mathbb{V}_\lambda^\perp$, then for all $\mathbf{s} \in \mathbb{S}^{D-1}$,

$$\begin{aligned} Z * \phi(\mathbf{s}) &= \int_{\mathbb{S}^{D-1}} \zeta(g_{\mathbf{s}}^{-1}.\mathbf{s}) \cdot \phi(\mathbf{s})\ d\mathfrak{L}[\mathbf{s}] \\ &= \int_{\mathbb{S}^{D-1}} \left(g_{\mathbf{s}}^{-1}.\zeta\right)(\mathbf{s}) \cdot \phi(\mathbf{s})\ d\mathfrak{L}[\mathbf{s}] \\ &= \left\langle g_{\mathbf{s}}^{-1}.\zeta, \quad \phi \right\rangle \quad = \quad 0, \end{aligned}$$

because $g_{\mathbf{s}}^{-1}.\zeta \in \mathbb{V}_\lambda \perp \phi$. $\square$

Proposition 11 (Zonal Eigenfunctions of $\triangle$ on $\mathbb{S}^{D-1}$)
The eigenvalues of $\triangle$ on $\mathbb{S}^{D-1}$ are all of the form

$$\lambda_N \quad = \quad N \cdot (N + D - 2),$$

for some $N \in \mathbb{N}$. Let ζ_N be a corresponding eigenfunction, and assume that ζ_N is zonal (relative to $\mathbb{SO}^D(\mathbb{R})$ and $\mathbf{e}$).

Case $D = 2$: *Modulo multiplication by some normalizing constant,*

$$\zeta_N(\theta) \ = \ \cos(N \cdot \theta)$$

where we use the coordinate system $(0, 2\pi) \ni \theta \mapsto (\cos(\theta), \sin(\theta)) \in \mathbb{S}^1$. If we write ζ_N in terms of Cartesian coordinates (x_1, x_2) on $\mathbb{R}^2$, we get the **Chebyshev polynomials:**

$$\zeta_N(x_1, x_2) \quad = \quad 2^{(N-1)} x_1^N \ + \ \sum_{n=1}^{\lfloor \frac{N}{2} \rfloor} (-1)^n 2^{(N-1-2n)} \frac{N}{n} \binom{N-n-1}{n-1} x_1^{(N-2n)}. \quad (5)$$

Case $D = 3$: *Modulo multiplication by some constant, ζ_N is a* **Legendre Polynomial:**

$$\zeta_N(x_1, x_2, x_3) \quad = \quad \sum_{n=0}^{\lfloor N/2 \rfloor} (-1)^n \, 2^{N-2n} \frac{\Gamma\left[\frac{1}{2} + N - n\right]}{\Gamma\left[\frac{1}{2}\right] \cdot n! \cdot (N-2n)!} \cdot x_1^{N-2n}.$$

Case $D \geq 4$: *Let $\nu \ = \ \frac{D-2}{2}$. For any $N \in \mathbb{N}$ and $n \in [0..N/2]$, define coefficients $c_{N;n}^{(\nu)} \ = \ \dfrac{\Gamma(\nu + (N-n))}{\Gamma(\nu) \cdot n! \cdot (N-2n)!}$, and define the (N, ν)th* **Gegenbauer polynomial:**

$$C_N^{(\nu)}(x) \quad = \quad \sum_{n=0}^{\lfloor N/2 \rfloor} (-1)^n 2^{N-2n} \cdot c_{N;n}^{(\nu)} \cdot x^{N-2n}.$$

$$\textit{Let}\quad K_N^{(\nu)} \;=\; \left\|C_N^{(\nu)}\right\|_2 \;=\; \sqrt{\int_{\mathbb{S}^{D-1}} \left|C_N^{(\nu)}(x_1)\right|^2 d\mathbf{x}}$$

$$= \frac{\sqrt{2}\cdot\pi^{(D-1)/4}}{\Gamma(\nu)}\cdot\sqrt{\sum_{k=0}^{2\cdot\lfloor N/2\rfloor}(-1)^k\cdot 2^{2N-2k}\cdot\frac{\Gamma\left(N-k+\frac{1}{2}\right)}{\Gamma\left(N-k+\frac{D}{2}\right)}\cdot\left(\sum_{n=0}^{k} c_{N;n}^{(\nu)} c_{N;(k-n)}^{(\nu)}\right)}.$$

Assume that ζ_N is of unit norm. Then ζ_N is a normalized Gegenbauer polynomial*:*

$$\zeta_N(x_1,x_2,\ldots,x_D) \;=\; \frac{1}{K_N^{(\nu)}}\, C_N^{(\nu)}(x_1).$$

Proof:

Proof of Characterization of Eigenvalues: See [59, ch.6], [56, ch.3], or [42, Cor.42, §5.2].

Proof of Case $D = 2$: It is clear from the definition of the Laplacian on $\mathbb{S}^1$ that the function ζ_N is an eigenfunction of $\triangle\mathbb{S}^1$. The subgroup of $\mathbb{SO}^2(\mathbb{R})$ fixing $\mathbf{e}$ is just the two-element group of maps $(x_1, x_2) \mapsto (x_1, \pm x_2)$; since the function ζ_N is symmetric relative to the x_2 variable, it is zonal relative to these maps.

The formula (5) is then just a standard trigonometric identity, where we identify $x_1 = \cos(\theta)$; see, for example [27, §1.33 (3), p.27],

Proof of Case $D = 3$: This is just the Gegenbauer polynomial when $D = 3$. For a direct proof, see, for example [4, Thm.1, §2.1, p.90], where there is unfortunately an error in the definition of the Legendre functions —see [54, §1, p.2] for a correct definition).

Proof of Case $D \geq 4$: This is just a big computation. See [42, Prop.44, §5.2] or [56] ——— □

3 Spherical Fourier Series

Theorem 12 (Spherical Fourier Analysis)

For all $n \in \mathbb{N}$, let $\zeta_n : \mathbb{S}^{D-1} \longrightarrow \mathbb{C}$ be the zonal harmonic polynomials defined by Proposition 11, and then define $\mathcal{Z}_n : \mathbb{S}^{D-1} \times \mathbb{S}^{D-1} \longrightarrow \mathbb{C}$ by

$$\mathcal{Z}_n(\mathbf{s},\sigma) \;=\; \zeta_n(e)\cdot\zeta_n\langle\mathbf{s},\sigma\rangle.$$

Then $\mathcal{Z}_n$ is rotationally equivariant.

Now, suppose $\gamma \in \mathbf{L}^2(\mathbb{S}^{D-1};\ \mathbb{C})$. If we define $\gamma_n := \mathcal{Z}_n * \gamma$ then $\gamma_n \in \mathbb{V}_{(\lambda_n)}$, and γ has the orthogonal decomposition:

$$\gamma \;=\; \sum_{n=1}^{\infty}\gamma_n. \tag{6}$$

Proof: This follows from Theorem 8 and Corollary 10, using the zonal functions provided by Proposition 11. □

Corollary 13 ((De)convolution on Spheres)

Suppose $\eta : \mathbb{S}^{D-1} \times \mathbb{S}^{D-1} \longrightarrow \mathbb{C}$ is rotationally equivariant, and suppose that $\mathbf{g} := \eta * \gamma$. If, for all $n \in \mathbb{N}$, ζ_n and $\mathcal{Z}_n$ are as in Theorem 12, and we define

$$\mathbf{g}_n \quad := \quad \mathcal{Z}_n * \mathbf{g}, \qquad \textit{and} \qquad A_n \quad := \quad \frac{(\eta * \zeta_n)(\mathbf{e}_1)}{\zeta_n(\mathbf{e}_1)},$$

then $\mathbf{g}_n = A_n \cdot \gamma_n$.

Conversely, suppose that γ is unknown, but we know η and $\mathbf{g}$. We can reconstruct γ via the formula:

$$\gamma \;=\; \sum_{n=1}^{\infty} \frac{1}{A_n} \mathbf{g}_n.$$

Proof: Combine Theorem 13 and Proposition 9. □

If $\gamma \in \mathbf{L}^2(\mathbb{S}^{D-1})$, then the **spherical Fourier Coefficients** of γ are the functions $\gamma_n := \mathcal{Z}_n * \gamma$, for $n \in \mathbb{N}$. (Notice that these "coefficients" are themselves functions, not numbers). The **spherical Fourier series** for γ is then the orthogonal decomposition $\gamma = \sum_{n=1}^{\infty} \gamma_n$.

Example: (Spherical Fourier series on $\mathbb{S}^1$)

Let for $N \in \mathbb{N}$, let $\zeta_N : \mathbb{S}^1 \longrightarrow \mathbb{C}$ be as in **Part 1** of Proposition 11:

$$\zeta_N(\theta) \quad = \quad \cos(N\theta) \quad = \quad \frac{1}{2}\left(\mathcal{E}_N(\theta) + \mathcal{E}_{(-N)}(\theta)\right),$$

where we identify $\mathbb{S}^1 \cong [0, 2\pi)$, and define $\mathcal{E}_K(\theta) := \exp(K\theta \cdot \mathbf{i})$. Let $\mathcal{Z}_N : \mathbb{S}^1 \times \mathbb{S}^1 \longrightarrow \mathbb{C}$ be defined from ζ_N as in Theorem 13. Then, for any $\gamma : \mathbb{S}^1 \longrightarrow \mathbb{R}$,

$$\begin{aligned}
\gamma_N \quad &= \quad \mathcal{Z}_N * \gamma \\
&\underset{(1)}{=} \quad \gamma * \zeta_N \;=\; \frac{1}{2}\left(\gamma * \mathcal{E}_N \;+\; \gamma * \mathcal{E}_{(-N)}\right) \\
&\underset{(2)}{=} \quad \frac{1}{2}\left(\widehat{\gamma}(N) \cdot \mathcal{E}_N \;+\; \widehat{f}(-N) \cdot \mathcal{E}_{(-N)}\right) \\
&\underset{(3)}{=} \quad \frac{1}{2}\left(\widehat{\gamma}(N) \cdot \mathcal{E}_N \;+\; \overline{\widehat{\gamma}(N) \cdot \mathcal{E}_N}\right) \\
&= \quad \mathbf{re}\left[\widehat{\gamma}(N) \cdot \mathcal{E}_N\right].
\end{aligned}$$

(1) Here, convolution is meant in the "usual" sense on the group $\mathbb{S}^1 = \mathbb{T}^1$.
(2) Here, $\widehat{\gamma}$ is the (classical) Fourier transform of γ as a function on the circle.
(3) because γ is real-valued.

Now, if we write $\widehat{\gamma}(N) = r_N \exp(\phi_N \cdot \mathbf{i})$, where $r_N \in [0, \infty)$ and $\phi_N \in [0, 2\pi)$, then, for any $\theta \in \mathbb{S}^1 \cong [0, 2\pi)$, we have:

$$\begin{aligned}
\gamma_N(\theta) &= \mathbf{re}\left[r_N \cdot \exp(\phi_N \mathbf{i}) \cdot \mathcal{E}_N(\theta)\right] \\
&= r_N \cdot \mathbf{re}\left[\exp(\phi_N \mathbf{i}) \cdot \exp((N \cdot \theta \cdot \mathbf{i})\right] \\
&= r_N \cdot \mathbf{re}\left[\exp\left(N \cdot \left(\theta + \frac{\phi_N}{N}\right) \cdot \mathbf{i}\right)\right] \\
&= r_N \cdot \mathbf{re}\left[\mathcal{E}_N\left(\theta + \frac{\phi_N}{N}\right)\right] \\
&= r_N \cdot \zeta_N\left(\theta + \frac{\phi_N}{N}\right).
\end{aligned}$$

In other words, convolving ζ_N by γ is equivalent to multiplying the magnitude of ζ_N by r_N, and rotating the phase by ϕ_N/N.

4 Asymptotic Decay and Convergence Rates

In classical harmonic analysis, the infinitesimal properties of a function f are reflected in the asymptotic behaviour of its Fourier transform, and vice versa. Generally, the smoother f is, the more rapidly $\widehat{f}$ decays near infinity. Conversely, if f is very "jaggy", undifferentiable, or discontinuous, then $\widehat{f}$ decays slowly or not at all near infinity, reflecting a concentration of the "energy" of f in high frequency Fourier components.

Hence, when approximating f by partial Fourier sums, the more jaggy f is, the more slowly the sum converges, and the more terms we must include in the sum to obtain a good approximation.

A similar phenomenon manifests when approximating a function $\gamma : \mathbb{S}^{D-1} \longrightarrow \mathbb{C}$ by a spherical Fourier series. By relating the decay rate of the spherical Fourier series to the smoothness of γ, we will be able to estimate the error introduced by approximating γ with a partial spherical Fourier sum.

If $\alpha > 0$, then we say that a sequence of functions $[\gamma_n|_{n=1}^{\infty}]$ is of **order** less than or equal to $\mathcal{O}(n^{-\alpha})$ if

$$0 \quad \leq \quad \lim_{n\to\infty} n^{\alpha} \cdot \|\gamma_n\|_2 \quad < \quad \infty.$$

Theorem 14 *Let $\gamma : \mathbb{S}^{D-1} \longrightarrow \mathbb{C}$, and suppose that γ is continuously $2M$-differentiable. Then the sequence $[\gamma_n|_{n=1}^{\infty}]$ is of order less than or equal to $\mathcal{O}\left(n^{-(2M+1)}\right)$.*

Proof: First suppose that γ is twice continuously differentiable. Thus, using the inductive formula from Theorem 3 we can apply $\triangle_{\mathbb{S}^{D-1}}$ to γ. Let $\alpha = \triangle_{\mathbb{S}^{D-1}}\gamma$. Since α is a continuous function, it is in $\mathbf{L}^2(\mathbb{S}^{D-1})$, and we can compute the spherical Fourier coefficients $\alpha_n = \mathcal{Z}_n * \alpha$, for all n, and conclude that $\alpha = \sum_{n=1}^{\infty} \alpha_n$.

In particular, since this sum converges absolutely in $\mathbf{L}^2(\mathbb{S}^{D-1})$, we know that the sequence $[\alpha_n|_{n=1}^\infty]$ is of order less than $\mathcal{O}(n^{-1})$.

By construction, we know that $\gamma_n = \mathcal{Z}_n * \gamma$ is an eigenfunction of $\triangle_{\mathbb{S}^{D-1}}$, with eigenvalue $\lambda_n = n(n+D-2)$. By Claim 2 of Proposition 9, the Laplacian operator commutes with convolution operators. Thus,

$$\begin{aligned} n(n+D-2)\gamma_n &= \triangle_{\mathbb{S}^{D-1}}\gamma_n \quad = \quad \triangle_{\mathbb{S}^{D-1}}(\mathcal{Z}_n * \gamma) \\ &= \mathcal{Z}_n * (\triangle_{\mathbb{S}^{D-1}}\gamma) \quad = \quad \mathcal{Z}_n * \alpha \\ &= \alpha_n. \end{aligned}$$

Since this is true for all n, we conclude that $[\gamma_n|_{n=1}^\infty]$ is of order less than or equal to $\mathcal{O}\left(\frac{1}{n(n+D-2)}\right) \cdot \mathcal{O}(n^{-1}) \;=\; \mathcal{O}(n^{-3})$.

Proceed inductively to prove the general case. ______________________ □

If $f, g : \mathbb{R}^D \longrightarrow \mathbb{C}$, then we define $\|f-g\|_p \;=\; \operatorname*{ess\,sup}_{\mathbf{x}\in\mathbb{R}} \left|f(\mathbf{x}) - g(\mathbf{x})\right|$, and, for any $p \in [1,\infty)$, we define

$$\|f-g\|_p \quad = \quad \left(\int_{\mathbb{R}^D} \left|f(\mathbf{x}) - g(\mathbf{x})\right|^p \; d\mathbf{x}\right)^{1/p}.$$

The following lemma is technical, but not difficult to prove [42, Cor.14-15, §3.2].

Lemma 15 *Suppose $\alpha \neq 1$, and that $[\rho_k|_{k=1}^\infty]$ is a sequence of α-stable probability measures on $\mathbb{R}^D$, with density functions $[F_k|_{k=1}^\infty]$, spectral measures $[\Gamma_k|_{k=1}^\infty]$, and spherical log-characteristic functions $[\mathbf{g}_k|_{k=1}^\infty]$. Let ρ be some other α-stable measure with density F, spectral measure Γ, and spherical log-characteristic function $\mathbf{g}$. Suppose that $\liminf_{k\to\infty} \min_{\mathbf{s}\in\mathbb{S}^{D-1}} \mathbf{g}_k(\mathbf{s}) \;>\; 0$, and $\min_{\mathbf{s}\in\mathbb{S}^{D-1}} \mathbf{g}(\mathbf{s}) \;>\; 0$. Then:*

1. *If Γ_k (resp. Γ) has Radon-Nikodym derivative γ_k (resp. γ), and $\lim_{k\to\infty} \|\gamma - \gamma_k\|_2 = 0$, then for every $q \in [1,\infty]$, $\lim_{k\to\infty} \|F - F_k\|_q \;=\; 0$.*
2. *There is a constant $C > 0$ so that for all $k \in \mathbb{N}$, $\|F - F_k\|_\infty < \; C \cdot \|\gamma - \gamma_k\|_2$.* □

Corollary 16 (Application to Spectral Measures)

Let $\alpha \in [0,2)$, $\alpha \neq 1$, and suppose ρ is an α-stable probability measure on $\mathbb{R}^D$ with density function $F : \mathbb{R}^D \longrightarrow [0,\infty)$, spectral measure Γ, and spherical log-characteristic function $\mathbf{g}$, with $\min_{\mathbf{s}\in\mathbb{S}^{D-1}} \mathbf{g}(\mathbf{s}) \;>\; 0$. Suppose that Γ is absolutely continuous relative to $\mathfrak{L}$, and that $d\Gamma = \gamma \; d\mathfrak{L}$, where $\gamma \in \mathbf{L}^2(\mathbb{S}^{D-1};\; \mathfrak{L})$ has spherical Fourier series $\gamma \;=\; \sum_{n=1}^\infty \gamma_n$.

For all $N \in \mathbb{N}$, let $\gamma^{[N]} = \sum_{n=1}^{N} \gamma_n$, let $\Gamma^{[N]} = \gamma^{[N]} \mathfrak{L}$, and let $\rho^{[N]}$ be the corresponding α-stable probability measure, with density function $F^{[N]} : \mathbb{R}^D \longrightarrow [0, \infty)$.

If $\gamma \in \mathbf{C}^{2M}(\mathbb{S}^{D-1})$, then, for all $p \in [1, \infty]$, $\lim_{k\to\infty} \left\| F - F^{[n]} \right\|_p = 0.$

Furthermore, $\left\| F - F^{[n]} \right\|_\infty$ is of order less than $\mathcal{O}\left(n^{-2M}\right)$.

Proof: By Theorem 14, we know that $\left\| \gamma - \gamma^{[n]} \right\|_2$ is of order less than $\mathcal{O}\left(n^{-2M}\right)$. Thus, applying Lemma 15, we conclude that $\left\| F - F^{[n]} \right\|_p$ is of order less than $\mathcal{O}\left(n^{-2M}\right)$. □

Conclusion

By expressing the log characteristic function $\mathbf{g}$ of equation (3) as a spherical Fourier series via Theorem 12, and then applying the "deconvolution" formula from Corollary 13, we can reconstruct a spherical Fourier series for the spectral measure Γ.

The advantages of this approach are threefold. First, once we have expressed $\mathbf{g}$ in terms of its spherical Fourier series, computing Γ is extremely straightforward; we need only divide the spherical Fourier coefficients of $\mathbf{g}$ by the constants A_n of Corollary 13. Computation of the Fourier coefficients, in turn, involves convolution with Gegenbauer polynomials. A closed-form expression for these polynomials is given (Theorem 11). This convolution can be computed by numerical integration over $\mathbb{S}^{D-1}$. To obtain a precision of ϵ requires a computation of complexity $\mathcal{O}\left(N^{2(D-1)}\right)$ (where $N \sim 1/\epsilon$), to be contrasted with the $\mathcal{O}\left(N^{3(D-1)}\right)$ required by an explicit matrix-inversion approach.

Second, if Γ is absolutely continuous with a C^{2M} Radon-Nikodym derivative, then the spherical Fourier series converges in $\mathbf{L}^2$ at a rate of $\mathcal{O}(N^{2M})$ (Theorem 14) so that the estimated stable probability density function in converges at a rate of $\mathcal{O}(N^{2M})$ in $\mathbf{L}^p$, for $1 \leq p \leq \infty$ (Corollary 16).

Finally, a spherical Fourier series explicitly represents Γ as a *continuous* object on $\mathbb{S}^{D-1}$, rather than as a sum of atoms, thereby avoiding the introduction of anomalous asymptotic behaviour to the estimated probability distribution.

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