THE DENSITY IN A ONE-DIMENSIONAL POTENTIAL

by

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INTRODUCTION

In [FS2] we proved precise estimates for the eigenvalues E_k and normalized eigenfunctions φ_k of an ordinary differential operator $H = -\frac{d^2}{dx^2} + V(x)$ on an interval I_{BVP}. In this paper, we apply the results of [FS2] to study the density

$$\rho(x) = \sum_{E_k \le 0} |\varphi_k(x)|^2 \,. \tag{1}$$

We will introduce a simple approximation $\rho_{sc}(x)$, and estimate $\rho - \rho_{sc}$. Our goal is to prove the asymptotic formula announced in [FS1] for the ground–state energy of an atom. This requires estimates for $\rho - \rho_{sc}$ in weighted Sobolev norms of order -1, which we will derive here. The complete proofs of the results of [FS1] are given here and in the papers [FS2. . . 7].

The potentials of interest to us are large and slowly varying. A basic example is

$$V(x) = \lambda^2 V_1(x)\,, \tag{2}$$

with $V_1(x)$ a fixed, smooth function, and λ a large parameter. We suppose $V_1(x)$ is defined on $[-1, 1]$ and satisfies

$$V_1(0) < 0,\ V_1'(0) = 0,\ V_1'' > c > 0 \text{ on } [-1,1],\ \{V_1 < 0\} \subset\subset [-1,1]\,. \tag{3}$$

For such potentials, a standard approximate formula for the density is

$$\rho(x) \approx \tfrac{1}{\pi}(-V(x))_+^{1/2}\,, \tag{4}$$

where $t_+^s = \begin{cases} t^s & \text{if } t > 0 \\ 0 & \text{if } t \le 0 \end{cases}$. Unfortunately, this is too crude for our purposes. In place of (4), we will use the sharper approximation

(5)

$$\rho_{sc}(x) = \frac{1}{\pi}(-V(x))_+^{1/2} - (-V(x))_+^{-1/2}\Big(\int_{I_{\mathrm{BVP}}} (-V(y))_+^{-1/2}\, dy\Big)^{-1} \chi_-\big(\frac{1}{\pi}\phi\big)\,, \quad \text{where}$$

$$\phi = \frac{\pi}{2} + \int_{I_{\mathrm{BVP}}} (-V(x))_+^{1/2}\, dx\,, \quad \text{and} \tag{6}$$

$$(7) \qquad \chi_-(t) = t - k - 1/2 \quad \text{for} \quad k = (\text{greatest integer } \le t)\,.$$

We call (5) the semiclassical approximation for ρ, even though the name usually refers to (4). The role of the last term in (5) is perhaps obscure, but we will explain it later in the introduction.

For potentials of the form (2), (3) our basic result on the density is as follows.

Theorem 1. *Suppose V is given by (2), (3), and let ϕ be defined by (6).*

(A) *If the distance from $\frac{\phi}{\pi}$ to the nearest integer is at least $C\lambda^{-1}$, then $\|\rho - \rho_{sc}\|_{H^{-1}} \le C'_\varepsilon \lambda^{\varepsilon - \frac{2}{43}}$ for any $\varepsilon > 0$.*

(B) *If the distance from $\frac{\phi}{\pi}$ to the nearest integer is less than $C\lambda^{-1}$, then $\|\rho - \rho_{sc}\|_{H^{-1}} \le C'$.*

Here C, C', C'_ε may depend on V_1 but not on λ.

Note that $\|\rho_{sc}\|_{H^{-1}} \sim \lambda$. The exponent $\varepsilon - \frac{2}{43}$ in (A) is surely not optimal. All we need is some negative power of λ.

The distinction between cases (A) and (B) above is easily explained. Theorem 1 is closely related to the WKB approximation, which says that the eigenvalues E_k are given approximately as the solutions to

$$(8) \qquad \int_{I_{\mathrm{BVP}}} (E_k - V(x))_+^{1/2}\, dx \approx \pi(k - 1/2), \quad k \ge 1 \qquad \text{(See Erdélyi [E].)}$$

In particular, there is an eigenvalue E_{k_0} close to zero if $\frac{\phi}{\pi}$ is near an integer. We cannot predict the sign of E_{k_0}, hence we don't know whether $|\varphi_{k_0}(x)|^2$ occurs in the sum (1). If $\frac{\phi}{\pi}$ is not near an integer, we meet no such difficulty.

To understand atoms, we need to deal with potentials more general than (2). Already the hydrogen atom leads to

$$(9) \qquad V(x) = \frac{\ell(\ell+1)}{x^2} - \frac{1}{x} + E^0 \quad \text{on} \quad (0,\infty)$$

which is certainly not given by (2), (3). In [FS2] we studied potentials $V(x)$ satisfying estimates

$$|(\frac{d}{dx})^{\alpha}V(x)| \le C_{\alpha}S(x)(B(x))^{-\alpha} \quad \text{when} \quad x \in I\,, \tag{10}$$

for suitable weight functions $S(x)$ and $B(x)$. Here I is an interval containing $\{V<0\}$. Hypothesis (10) lets us treat all the potentials we need, simply by picking the proper weight functions $S(x)$, $B(x)$ and interval I. For instance, when V is given by (2), (3), then (10) holds with $S(x)=\lambda^2$, $B(x)=1$, $I=[-1,1]$. If V is given instead by (9) for suitable $E^0>0$, then we take $S(x)=\frac{1}{x}$, $B(x)=x$, $I=[\frac{\ell(\ell+1)}{2},\frac{2}{E^0})$. The main result of this paper is a version of Theorem 1 that applies to potentials satisfying (10).

To state our main theorem, we need the analogue of hypothesis (3) in the setting of (10). We assume V takes its minimum at $x_0 \in I$ and satisfies

$$\begin{aligned} &V(x_0) < -cS(x_0);\; V''(x) > cS(x_0)(B(x_0))^{-2} \quad \text{if} \quad |x-x_0| \le c_1 B(x_0);\\ &|V'(x)| > cS(x)(B(x))^{-1} \quad \text{if} \quad x \in I, |x-x_0| \ge c_1 B(x_0). \end{aligned} \tag{11}$$

We need also a number Λ that plays the role of λ in (2). It is given by

$$\Lambda = \Big(\int_{V(x)<0} \frac{dx}{(S(x))^{1/2}(B(x))^2}\Big)^{-1}, \tag{12}$$

as explained in [FS2].

Finally, we need an H^{-1}–norm adapted to weight functions $S(x)$, $B(x)$. For a function f on $I_{\mathrm{BVP}}=(a,b)$ and a point $y\in I$, we set

$$\mathcal{N}(f,y) = \Big(\frac{1}{B(y)}\int_{|x-y|<\hat{c}B(y)} \Big|\int_a^x f(t)\,dt\Big|^2 dx\Big)^{1/2} \tag{13}$$

for a small constant $\hat{c}$. This is an H^{-1}–norm, since it is an L^2–norm of a primitive of f.

Now we can state our main result.

WKB Density Theorem. *Let $V(x)$ satisfy (10), (11) and various technical conditions, and suppose Λ is large.*

(A) *If the distance from $\frac{\phi}{\pi}$ to the nearest integer is at least $C\Lambda^{-1}$, then*

$$\mathcal{N}(\rho - \rho_{sc}, y) \le C_\varepsilon \Lambda^{\varepsilon - \frac{45}{43}} \int_a^{y+2\hat{c}B(y)} (-V(x))_+^{1/2}\, dx + C_N \Lambda^{-N}$$

for any positive N, ε .

(B) *If the distance from $\frac{\phi}{\pi}$ to the nearest integer is less than $C\Lambda^{-1}$, then*

$$\mathcal{N}(\rho - \rho_{sc}, y) \le C\Lambda^{-1} \int_a^{y+2\hat{c}B(y)} (-V(x))_+^{1/2}\, dx + C_N \Lambda^{-N} .$$

For additional conclusions and a careful statement of all the hypotheses, we refer the reader to the relevant section of this paper.

We prepare to sketch the proofs of our results on the density. For simplicity, we confine ourselves to Theorem 1. We start by describing how the eigenfunctions behave.

Let $\{x \in I_{\text{BVP}} \mid V(x) < E\} = (x_{\text{left}}(E), x_{\text{rt}}(E))$. The endpoints $x_{\text{left}}(E)$, $x_{\text{rt}}(E)$ are called "turning points". Define

$$\phi(E) = \frac{1}{\pi} \int_{I_{\text{BVP}}} (E - V(x))_+^{1/2}\, dx + \frac{1}{2}, \quad \text{and} \tag{14}$$

$$\eta(x, E) = -\frac{\pi}{4} + \int_{x_{\text{left}}(E)}^{x} (E - V(t))^{1/2}\, dt \quad \text{for} \quad x_{\text{left}}(E) < x < x_{\text{rt}}(E) . \tag{15}$$

Let E_k, φ_k be an eigenvalue and its corresponding normalized eigenfunction. Then φ_k is given on overlapping intervals by the following approximate formulas.
(I) For $x \in (x_{\text{left}}(E_k), x_{\text{rt}}(E_k))$ not too near the turning points,

$$\varphi_k(x) \approx (\pi \phi'(E_k))^{-1/2} (E_k - V(x))^{-1/4} \cos(\eta(x, E_k)) . \tag{16}$$

(II) For x near the turning point $x_{\text{left}}(E_k)$,

$$\varphi_k(x) \approx \lambda^{-1/3}(\pi\phi'(E_k))^{-1/2}\Big(\frac{\partial y(x,E_k)}{\partial x}\Big)^{-1/2} A(\lambda^{2/3}y(x,E_k))\,, \tag{17}$$

where $A(t)$ is a slight variant of the Airey function, and

$$\lambda^2\Big(\frac{\partial y(x,E)}{\partial x}\Big)^2 y(x,E) \approx E - V(x)\,. \tag{18}$$

The auxiliary function $y(x,E)$ depends smoothly on x, $\frac{E}{\lambda^2}$.

(III) For x near the turning point $x_{\text{rt}}(E_k)$,

$$\varphi_k(x) \approx \pm\lambda^{-1/3}(\pi\phi'(E_k))^{-1/2}\Big(-\frac{\partial}{\partial x}\tilde{y}(x,E_k)\Big)^{-1/2} A(\lambda^{2/3}\tilde{y}(x,E_k)) \tag{19}$$

for an auxiliary function $\tilde{y}(x,E)$ analogous to $y(x,E)$.

(IV) For x outside $(x_{\text{left}}(E_k), x_{\text{rt}}(E_k))$ and not too near the turning points, $\varphi_k(x)$ is negligibly small.

These formulas and (8) comprise the standard WKB approximations. (Again, see Erdélyi [E].) Our previous paper [FS2] proves precise theorems justifying and refining (8) and (I)...(IV). As E_k gets too near to the minimum of V, (I)...(IV) lose accuracy.

(V) For E_k near the minimum of the potential, the approximation (8) remains highly accurate, and the eigenfunctions $\varphi_k(x)$ are highly concentrated near the point x_0 where V attains its minimum.

This concludes our discussion of the individual eigenvalues and eigenfunctions of $\frac{-d^2}{dx^2} + V$.

Our task in proving Theorem 1 is to use our precise knowledge of the eigenfunctions φ_k to understand the density (1). Because the φ_k are described by different formulas in overlapping regions, it is natural to introduce the *microlocalized density*

$$\rho(x,g) = \sum_{E_k\le 0} |\varphi_k(x)|^2 g(x,E_k) \tag{20}$$

corresponding to a function $g(x,E)$. We compare $\rho(x,g)$ with the *semiclassical microlocalized density*, defined by

(21)

$$\rho_{sc}(x,g) = \frac{1}{2\pi}\int_{-\infty}^{0} (E - V(x))_+^{-1/2} g(x,E)\, dE - \frac{(-V(x))_+^{-1/2} g(x,0)}{(\int_{I_{\mathrm{BVP}}} (-V(y))_+^{-1/2}\, dy)} \chi_-(\frac{1}{\pi}\phi)\,.$$

When $g \equiv 1$, $\rho(x,g)$ and $\rho_{sc}(x,g)$ reduce to the density (1) and its semiclassical approximation (5). Theorem 1 is a special case of the general assertion that $\rho_{sc}(x,g)$ approximates $\rho(x,g)$ closely in H^{-1} for all reasonable functions g. By a partition of unity, we may assume that g is supported in one of the regions (I)...(V) above.

We deal with each case (I)...(V) in turn. Suppose first that $g(x,E)$ is supported in region (I). Thus, $E - V(x)$ is positive and rather large in supp $g(x,E)$. For simplicity, we assume here that $E - V(x) > c\lambda^2$ in supp $g(x,E)$. Then

$$\begin{aligned}\rho(x,g) &= \sum_{E_k \le 0} |\varphi_k(x)|^2 g(x,E_k) \\ &\approx \sum_{E_k \le 0} \tfrac{1}{\pi}(\phi'(E_k))^{-1} g(x,E_k)(E_k - V(x))^{-1/2}\cos^2(\eta(x,E_k)) \\ &= \frac{1}{2\pi}\sum_{E_k\le 0}\frac{(\phi'(E_k))^{-1}g(x,E_k)}{(E_k - V(x))^{1/2}} + \frac{1}{2\pi}\sum_{E_k\le 0}\frac{(\phi'(E_k))^{-1}g(x,E_k)}{(E_k - V(x))^{1/2}}\cos(2\eta(x,E_k)) \\ &\equiv F(x) + G(x)\,. \end{aligned} \tag{22}$$

Note that the negative powers of $(E_k - V(x))$ pose no problem, since $g(x,E)$ is supported in region (I). We prove that $F(x) \approx \rho_{sc}(x,g)$ and that G has negligibly small norm in H^{-1}.

To estimate G in H^{-1}–norm, we compute its indefinite integral. In general, when $f(x)$ is smooth and $\eta'(x)$ is large, the indefinite integral of $f(x)\cos(2\eta(x))$ is approximately $\frac{f(x)}{2\eta'(x)}\sin(2\eta(x))$. Applying this remark to the summands in the definition of G, we obtain the approximate formula

$$\mathcal{G}(x) \approx \frac{1}{2\pi} \sum_{E_k \le 0} \frac{(\phi'(E_k))^{-1} g(x, E_k)}{(E_k - V(x))^{1/2}} \, \frac{\sin(2\eta(x, E_k))}{(2\frac{\partial\eta}{\partial x}(x, E_k))}$$
$$= \frac{1}{2\pi} \sum_{E_k \le 0} \frac{(\phi'(E_k))^{-1} g(x, E_k)}{2(E_k - V(x))} \sin(2\eta(x, E_k))$$

for a primitive of G. The terms on the right are nearly orthogonal because of the rapidly oscillating factors $\sin(2\eta(x, E_k))$, so that

$$\|G\|_{H^{-1}}^2 \approx \|\mathcal{G}\|_{L^2}^2 \sim \sum_{E_k \le 0} \int_{I_{\mathrm{BVP}}} \frac{(\phi'(E_k))^{-2} |g(x, E_k)|^2}{4(E_k - V(x))^2} \, dx \, .$$

The sum on the right is easily dominated by a negative power of λ, so G is negligibly small in H^{-1}.

To analyze $F(x)$ in (22), we introduce the function $E(t)$ that solves the equation $\phi(E) = t$. As t increases from $t_{\min} = 1/2$ to $t_{\max} = \phi(0) = \frac{1}{\pi}\phi$, $E(t)$ increases from $(\min V)$ to 0.

The WKB approximation (8) says that $\phi(E_k) \approx k$, hence $E_k \approx E(k)$. So $F(x)$ is given approximately by

$$(23) \quad F(x) \quad \approx \quad \frac{1}{2\pi} \sum_{k \in [t_{\min}, t_{\max}]} \frac{g(x, E(k)) E'(k)}{(E(k) - V(x))^{1/2}} \quad = \quad \frac{1}{2\pi} \sum_{k \in [t_{\min}, t_{\max}]} f(k, x)$$

with

$$(24) \qquad f(t, x) = \frac{g(x, E(t)) E'(t)}{(E(t) - V(x))^{1/2}} \, .$$

For fixed x, $f(t, x)$ is nearly constant as t varies over an interval of length $O(1)$. Hence it is reasonable to replace the sum in (23) by an integral. This yields

$$(25) \quad F(x) \approx \frac{1}{2\pi} \int_{t_{\min}}^{t_{\max}} \frac{g(x, E(t)) E'(t)}{(E(t) - V(x))^{1/2}} \, dt = \frac{1}{2\pi} \int_{\min V}^{0} \frac{g(x, E) \, dE}{(E - V(x))^{1/2}}$$
$$= \frac{1}{2\pi} \int_{-\infty}^{0} g(x, E)(E - V(x))_+^{-1/2} \, dE$$

since g is supported in region (I). The right–hand side of (25) is the crude form of the semiclassical approximation to $\rho(x, g)$.

The most serious error in the derivation of (25) comes from replacing the Riemann sum (23) by an integral. To remedy it, we study $\sum_{k\in[a,b]} f(k) - \int_a^b f(t)\,dt$ for general slowly varying functions f. We find easily that

$$(26) \qquad \sum_{k\in[a,b]} f(k) \approx \int_a^b f(t)dt - \chi_-(b)f(b) - \chi_+(a)f(a)\,,$$

with χ_- as in (7), and $\chi_+(t) = k - t - 1/2$ for $k =$ (least integer $\geq t$). The last terms in (26) compensate for the fact that small changes in a and b typically change the integral in (26) but not the sum. Using (26) to evaluate the right–hand side of (23), we obtain

$$\begin{aligned} F(x) &\approx \frac{1}{2\pi}\int_{t_{\min}}^{t_{\max}} \frac{g(x,E(t))E'(t)}{(E(t)-V(x))^{1/2}}\,dt - \tfrac{1}{2\pi}\chi_-(t_{\max})\frac{g(x,E(t_{\max}))E'(t_{\max})}{(E(t_{\max})-V(x))^{1/2}} \\ &\qquad - \frac{1}{2\pi}\chi_+(t_{\min})\frac{g(x,E(t_{\min}))E'(t_{\min})}{(E(t_{\min})-V(x))^{1/2}} \\ &= \frac{1}{2\pi}\int_{-\infty}^{0} g(x,E)(E-V(x))_+^{-1/2}dE \\ (27) \qquad &\qquad - \chi_-(\frac{1}{\pi}\phi)g(x,0)(-V(x))_+^{-1/2}(\int_{I_{\rm BVP}} (-V(y))_+^{-1/2}\,dy)^{-1}\,, \end{aligned}$$

since $t_{\max} = \frac{1}{\pi}\phi$, $E(t_{\max}) = 0$, $E'(t_{\max}) = (\phi'(0))^{-1} = (\frac{1}{2\pi}\int_{I_{\rm BVP}}(-V(y))_+^{-1/2}\,dy)^{-1}$ and g is supported in region (I). Recalling that G is negligibly small in H^{-1}, we conclude from (22) and (27) that

(28)

$$\rho(x,g) \approx F + G \approx \tfrac{1}{2\pi}\int_{-\infty}^{0} g(x,E)(E-V(x))_+^{-1/2}\,dE - \chi_-(\tfrac{\phi}{\pi})\frac{g(x,0)(-V(x))_+^{-1/2}}{\int_{I_{\rm BVP}}(-V(y))_+^{-1/2}\,dy} \equiv \rho_{\rm sc}(x,g)\,.$$

Thus we have shown that $\rho(x,g) \approx \rho_{sc}(x,g)$ when g is supported in region (I). It is equation (28) that teaches us the corrected definitions (5), (21) for the semiclassical density.

Next we study $\rho(x,g)$ when g is supported in region (II). To deal with this case, we need to know something about the "Airey function" $A(t)$. First of all,

$$A(t) \sim t_+^{-1/4}\cos(\frac{2}{3}t^{3/2} - \frac{\pi}{4}) \qquad \text{for} \quad |t| >> 1\,. \tag{29}$$

In particular, $A(t)$ oscillates more and more rapidly as $t \to +\infty$. At a crucial point we will use the following estimate from our previous paper [FS2].

(30)

$$X \equiv \int_{-\infty}^{\infty} \theta(\frac{t}{R})[A^2(t) - \frac{1}{2}t_+^{-1/2}]\,dt = O(R^{-5/2}) \quad \text{for} \quad R >> 1 \quad \text{and} \quad \theta \in C_0^\infty\,.$$

Note that (29) yields only $X = O(1)$.

Now we can start computing $\rho(x,g)$ in case (II). From (II) and the definition of the microlocalized density, we have

$$\begin{aligned}\rho(x,g) &\approx \frac{1}{\pi}\sum_{E_k\le 0} g(x,E_k)(\phi'(E_k))^{-1}\lambda^{-2/3}\Big(\frac{\partial y}{\partial x}(x,E_k)\Big)^{-1}A^2(\lambda^{2/3}y(x,E_k)) \\ &\approx \frac{1}{\pi}\sum_{k\in[t_{\min},t_{\max}]} f(k,x)\,, \qquad \textit{with}\end{aligned} \tag{31}$$

$$f(t,x) = g(x,E(t))E'(t)\Big(\frac{\partial y}{\partial x}(x,E(t))\Big)^{-1}\lambda^{-2/3}A^2(\lambda^{2/3}y(x,E(t)))\,. \tag{32}$$

Since $g(x,E)$ is supported in region (II), $|x - x_{\text{left}}(E(t))|$ is small in supp $f(t,x)$. Hence $|y(x,E(t))|$ is small also, as we see from (18). Consequently, $\lambda^{2/3}y(x,E(t))$ is not too large, so $A^2(\lambda^{2/3}y(x,E(t)))$ does not oscillate too rapidly in supp $f(t,x)$. It follows that $f(t,x)$ is nearly constant when x stays fixed and t varies by $O(1)$. Thus we may approximate the sum in (31) by appealing to our previous result (26) on general Riemann sums. We obtain

$$\begin{aligned}\rho(x,g) \approx \frac{1}{\pi}&\int_{t_{\min}}^{t_{\max}} \frac{g(x,E(t))E'(t)\lambda^{-2/3}}{(\frac{\partial y}{\partial x}(x,E(t)))}A^2(\lambda^{2/3}y(x,E(t)))\,dt \\ &- \frac{2\chi_-(\frac{\phi}{\pi})g(x,0)(\int_{I_{\text{BVP}}}(-V(y))_+^{-1/2}\,dy)^{-1}}{\lambda^{2/3}(\frac{\partial y}{\partial x}(x,0))}A^2(\lambda^{2/3}y(x,0))\,,\end{aligned} \tag{33}$$

since $t_{\max} = \frac{\phi}{\pi}$, $E(t_{\max}) = 0$, $t_{\min} = 1/2$, $\chi_+(t_{\min}) = \chi_+(1/2) = 0$. Changing variable in the integral in (33), we get

$$\begin{aligned}\rho(x,g) \approx \frac{1}{\pi}\int_{\min V}^{0} \frac{g(x,E)}{\lambda^{2/3}(\frac{\partial y}{\partial x}(x,E))} A^2(\lambda^{2/3}y(x,E))dE \\ - \frac{2\chi_-(\frac{\phi}{\pi})g(x,0)}{\lambda^{2/3}(\frac{\partial y}{\partial x}(x,0))}\Big(\int_{I_{\mathrm{BVP}}}(-V(y))_+^{-1/2}\,dy\Big)^{-1} A^2(\lambda^{2/3}y(x,0))\,.\end{aligned} \tag{34}$$

Next, we want to replace $A^2(\lambda^{2/3}y(x,E))$ by $\frac{1}{2}(\lambda^{2/3}y(x,E))_+^{-1/2}$ in (34). This is plausible, since (29) gives

$$\begin{aligned}A^2(\lambda^{2/3}y(x,E)) \approx \frac{1}{2}(\lambda^{2/3}y(x,E))_+^{-1/2} \\ + \frac{1}{2}(\lambda^{2/3}y(x,E))_+^{-1/2}\cos(\frac{4}{3}\lambda y^{3/2}(x,E) - \frac{\pi}{2})\,.\end{aligned}$$

The last term oscillates rapidly about zero, so it should have small norm in H^{-1} and a small effect on the integral in (34).

To replace $A^2(\lambda^{2/3}y(x,E))$ by $\frac{1}{2}(\lambda^{2/3}y(x,E))_+^{-1/2}$ with a small error, we study integrals of the form

$$F_R(x) = \int_{-\infty}^{x} \theta(\frac{x}{R},\frac{t}{R})[A^2(t) - \frac{1}{2}t_+^{-1/2}]\,dt, \quad \text{for} \quad \theta \in C_0^\infty \quad \text{and} \quad R >> 1\,. \tag{35}$$

When x is bounded or large negative, (35) is easily understood in terms of the asymptotics of $A(t)$. Unfortunately, when x is large positive, the region of integration in (35) extends over $\{|t| \le C\}$, where $A^2(t) - \frac{1}{2}(t)_+^{-1/2}$ is neither small nor rapidly oscillating. To overcome this difficulty, we appeal to (30) and write

$$F_R(x) = \int_{-\infty}^{\infty} \theta(\frac{x}{R},\frac{t}{R})[A^2(t) - \frac{1}{2}t_+^{-1/2}]\,dt - \int_{x}^{\infty} \theta(\frac{x}{R},\frac{t}{R})[A^2(t) - \frac{1}{2}t_+^{-1/2}]\,dt\,.$$

The first term on the right is $O(R^{-5/2})$ by (30), and the second integral is easily analyzed in terms of the asymptotics of $A(t)$ when x is large positive. Hence we can prove sharp bounds for $|F_R(x)|$ for all x.

We are ready to replace $A^2(\lambda^{2/3}y(x,E))$ by $\frac{1}{2}(\lambda^{2/3}y(x,E))_+^{-1/2}$ in (34). This leads to errors

$$G(x) = \int_{E<0} \frac{g(x,E)}{\lambda^{2/3}(\frac{\partial y}{\partial x}(x,E))}\,[A^2(\lambda^{2/3}y(x,E)) - \frac{1}{2}\,(\lambda^{2/3}y(x,E))_+^{-1/2}]\,dE \qquad \text{and}$$

$$H(x) = \frac{g(x,0)}{\lambda^{2/3}(\frac{\partial y}{\partial x}(x,0))}\,[A^2(\lambda^{2/3}y(x,0)) - \frac{1}{2}(\lambda^{2/3}y(x,0))_+^{-1/2}]\,,$$

which we have to estimate in the H^{-1}–norm.

The primitives of G and H are

$$\mathcal{G}(z) = \int\!\!\int_{\substack{E<0\\ x<z}} \frac{g(x,E)}{\lambda^{2/3}(\frac{\partial y}{\partial x}(x,E))}\,[A^2(\lambda^{2/3}y(x,E)) - \frac{1}{2}(\lambda^{2/3}y(x,E))_+^{-1/2}]\,dE\,dx$$

and

$$\mathcal{H}(z) = \int_{x<z} \frac{g(x,0)}{\lambda^{2/3}(\frac{\partial y}{\partial x}(x,0))}\,[A^2(\lambda^{2/3}y(x,0)) - \frac{1}{2}(\lambda^{2/3}y(x,0))_+^{-1/2}]\,dx\,.$$

To estimate G and H in H^{-1}–norm, we bound the L^2–norms of $\mathcal{G}$ and $\mathcal{H}$. We can reduce $\mathcal{G}$ and $\mathcal{H}$ to the form (35) by making simple changes of variable. In fact, set $t = \lambda^{2/3}y(x,0)$ and we find that

$$\text{(36)} \qquad \mathcal{H}(z) = \int_{-\infty}^{\lambda^{2/3}y(z,0)} \Theta(t)[A^2(t) - \frac{1}{2}t_+^{-1/2}]\,dt, \qquad \text{with } \Theta \text{ defined by}$$

$$\Theta(t)\,dt = \frac{g(x,0)}{\lambda^{2/3}(\frac{\partial y}{\partial x}(x,0))}\,dx\,.$$

Similarly, putting $t = \lambda^{2/3}y(x,E)$ gives

$$\text{(37)} \qquad \mathcal{G}(z) = \int\!\!\int_{\substack{x<z\\ t<\lambda^{2/3}y(x,0)}} h(x,t)[A^2(t) - \frac{1}{2}t_+^{-1/2}]\,dt\,dx, \quad \text{with}$$

$$h(x,t)\,dt = \frac{g(x,E)}{\lambda^{2/3}(\frac{\partial y}{\partial x}(x,E))}\,dE \qquad \text{for fixed } x\,.$$

Then with $\tilde{x}(t)$ defined as the solution of $\lambda^{2/3}y(\tilde{x},0) = t$, we can rewrite (37) in the form

$$\text{(38)} \qquad \mathcal{G}(z) = \int_{-\infty}^{\lambda^{2/3}y(z,0)} \tilde{\Theta}(z,t)[A^2(t) - \frac{1}{2}t_+^{-1/2}]\,dt, \quad \text{where}$$

$$\tilde{\Theta}(z,t) = \int_{\tilde{x}(t)}^{z} h(x,t)\,dx\,.$$

Equations (36) and (38) reduce $\mathcal{G}(z)$ and $\mathcal{H}(z)$ to the form (35). Our pointwise bounds for integrals (35) therefore control the L^2–norms of $\mathcal{G}$ and $\mathcal{H}$. In other words, we can control the H^{-1}–norms of the errors that arise when we replace $A^2(\lambda^{2/3}y)$ by $\frac{1}{2}(\lambda^{2/3}y)_+^{-1/2}$ in (34).

Now we have

$$\rho(x,g) \approx \frac{1}{2\pi}\int_{E<0}\lambda^{-1}\Big(\frac{\partial y}{\partial x}(x,E)\Big)^{-1}(y(x,E))_+^{-1/2}g(x,E)dE - \frac{\chi_-(\frac{\phi}{\pi})\lambda^{-1}(\frac{\partial y}{\partial x}(x,0))^{-1}(y(x,0))_+^{-1/2}g(x,0)}{\int_{I_{\mathrm{BVP}}}(-V(y))_+^{-1/2}\,dy}. \tag{39}$$

Equation (18) shows that $\lambda^{-1}(\frac{\partial y}{\partial x}(x,E))^{-1}(y(x,E))_+^{-1/2} \approx (E-V(x))_+^{-1/2}$, with a small error in H^{-1}. Putting this into (39), we get

$$\rho(x,g) \approx \frac{1}{2\pi}\int_{E<0}(E-V(x))_+^{-1/2}g(x,E)dE - \frac{\chi_-(\frac{\phi}{\pi})(-V(x))_+^{-1/2}g(x,0)}{\int_{I_{\mathrm{BVP}}}(-V(y))_+^{-1/2}dy} \equiv \rho_{sc}(x,g).$$

This completes the proof that $\rho(x,g) \approx \rho_{sc}(x,g)$ for g suppported in region (II). The proof for g supported in region (III) is the same as for region (II).

It is trivial to show that $\rho(x,g) \approx \rho_{sc}(x,g)$ when g is supported in region (IV). In fact, from (IV) and (20) we see that $\rho(x,g)$ is negligibly small, while (21) gives $\rho_{sc}(x,g) \equiv 0$.

Finally, suppose $g(x,E)$ is supported in region (V), where $E \approx \min\ V$. A sharp analysis would compare $V(x)$ to the harmonic oscillator, but the following crude argument is enough for our purposes. According to (V), the relevant eigenfunctions φ_k are highly concentrated in a small interval $J = [x_-, x_+]$ about the point x_0 where V takes its minimum. Hence

$$\rho(x,g) = \sum_k |\varphi_k(x)|^2 g(x,E_k) \approx \sum_k |\varphi_k(x)|^2 g(x_0,E_k), \quad \text{and}$$

$$\int\limits_{x<z}\rho(x,g)dx \approx \begin{cases} 0 & \text{for } z < x_- \\ \sum\limits_k g(x_0,E_k) & \text{for } z > x_+ \\ O(\sum\limits_k |g(x_0,E_k)|) & \text{for } z \in J. \end{cases} \tag{40}$$

Equation (40) holds also for $\rho_{sc}(x,g)$, and therefore

$$\mathcal{G}(z) \equiv \int\limits_{x<z} (\rho(x,g) - \rho_{\rm sc}(x,g))dx = O(\sum_k |g(x_0, E_k)|) \quad \text{for} \quad z \in J,$$

$\mathcal{G}(z)$ is negligibly small outside J.

Since also J is small, it follows that $\mathcal{G}$ has small L^2–norm. Thus $\rho(x,g) - \rho_{sc}(x,g)$ has small norm in H^{-1}. This concludes our sketch of case (V). Later, when we fill in details, we will restrict attention to $\rho(x,g)$ for $g(x,E)$ that depend on E alone. This is enough for our purposes.

We have shown that $\|\rho(x,g) - \rho_{sc}(x,g)\|_{H^{-1}}$ is small in each of the cases (I)...(V), so the proof of Theorem 1 is complete. The proof of our general result, the WKB Density Theorem, is essentially the same, although the details are more elaborate.

We will need also crude variants of our density theorems in several degenerate cases. For instance, if $V(x)$ is as in (2), (3), then we need a crude understanding of the density associated to $-\frac{d^2}{dx^2} + V(x) + E_0$, where $-E_0$ is just slightly greater than the minimum of V. The relevant density results for degenerate cases are proved in this paper, but we omit them from the introduction.

We next make a small remark to confirm that the extra term in the definition of ρ_{sc} improves the accuracy of the approximation. If we calculate the integral of ρ_{sc} over the whole interval $I_{\rm BVP}$, then we obtain an approximation to $\int_{I_{\rm BVP}} \rho(x)dx$, which is the number of non–positive eigenvalues of $-\frac{d^2}{dx^2} + V(x)$. ¿From (5), (6), (7) we obtain easily that $\int_{I_{\rm BVP}} \rho_{sc}(x)dx$ is the greatest integer in $\frac{\phi}{\pi}$. According to the WKB approximation (8), this is precisely the number of non–positive eigenvalues, provided $\frac{\phi}{\pi}$ isn't too near to an integer. If we had used the standard formula (4) as our definition of ρ_{sc}, then $\int_{I_{\rm BVP}} \rho_{sc}(x)dx$ would differ from $\int_{I_{\rm BVP}} \rho(x)dx$ by an error of the order of magnitude 1. Thus the last term in (5) contributes to the accuracy of the approximation.

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REVIEW OF EARLIER RESULTS

For the reader's convenience, we gather here those results from [FS2] which we require for our present work.

The Airey Function.

We work with a variant of the standard Airey function. Our $A(t)$ is a real–valued solution of $(\frac{d^2}{dt^2}+t)A(t)=0$, satisfying:

(A1) $$\Big|\Big(\frac{d}{dt}\Big)^m A(t)\Big| \leq C_{mK}(1+|t|)^{-K} \quad \text{as} \quad t\to-\infty, \quad \text{for any } m \text{ and } K\,;$$

(A2) $$A(t)\sim \mathrm{Re}\Big[\frac{e^{\pm i\frac{\pi}{4}+\frac{2}{3}t^{3/2}i}}{t^{1/4}}\Big(1+\sum_{s=1}^{\infty}\frac{c_s}{t^{\frac{3}{2}s}}\Big)\Big] \quad \text{as} \quad t\to+\infty\ ,$$

in the following sense. Given m and K, we can find M so that

$$\Big(\frac{d}{dt}\Big)^{\ell}\Big\{A(t)-\mathrm{Re}\Big[\frac{e^{\pm i\frac{\pi}{4}+\frac{2}{3}t^{3/2}i}}{t^{1/4}}(1+\sum_{s=1}^{M}\frac{c_s}{t^{\frac{3}{2}s}})\Big]\Big\}=O(t^{-K})$$

for $0\leq\ell\leq m$ and $t>1$. We have $c_1=\pm\frac{5i}{48}$.

(A3) Suppose $|(\frac{d}{dt})^m\theta(t)|\leq C_m R^{-m}$ for all m, with $R\geq 10$. Assume also that $\theta(t)$ has compact support. Then

$$\Big|\int_{-\infty}^{\infty}\theta(t)\big[A^2(t)-\frac{1}{2}t_+^{-1/2}\big]\ dt\Big|\leq C'R^{-5/2},$$

where C' depends only on finitely many of the C_m.

(This is immediate from Lemmas 4 and 7 in the section on normalizing eigenfunctions in [FS2]. Recently, we have found a much simpler proof of this fact.)

The WKB Theorems.

Set-up. We are given the following: A potential $V(x)$ defined on a (possibly unbounded) interval I_{BVP}; two positive functions $S(x)$, $B(x)$ defined on a subinterval

$I \subset I_{\mathrm{BVP}}$; two real numbers $E_0 \leq E_\infty$; positive numbers $\varepsilon < \frac{1}{100}$, $K > 1$ and $N > K\varepsilon^{-10}$. We define $N' = [\varepsilon N/500]$ and $N'' = \frac{3}{2}\varepsilon N' - K - 33$.

Our goal is to understand the eigenvalues and eigenfunctions of the self–adjoint operator $H = \frac{-d^2}{dx^2} + V(x)$ on $L^2(I_{\mathrm{BVP}})$, with Dirichlet or Neumann conditions at the endpoints.

Hypotheses.

Assumptions on $V(x)$, $S(x)$, $B(x)$ in I

(Hyp0) If $x, y \in I$ and $|x - y| < cB(x)$, then $c < \frac{B(y)}{B(x)} < C$ and $c < \frac{S(y)}{S(x)} < C$.

(Hyp1) For $x \in I$ and $\alpha \geq 0$ we have $|(\frac{d}{dx})^\alpha V(x)| \leq C_\alpha S(x) B^{-\alpha}(x)$.

(Hyp2) The equation $V(x) = E_0$ has two solutions $x_{\mathrm{left}} < x_{\mathrm{rt}}$ in I, and they satisfy $\mathrm{dist}(x_{\mathrm{left}}, \partial I) > cB(x_{\mathrm{left}}), \mathrm{dist}(x_{\mathrm{rt}}, \partial I) > cB(x_{\mathrm{rt}})$.

(Hyp3) For $x_{\mathrm{left}} \leq x \leq x_{\mathrm{left}} + c_1 B(x_{\mathrm{left}})$ we have $-V'(x) > cS(x_{\mathrm{left}})B^{-1}(x_{\mathrm{left}})$, and for $x_{\mathrm{rt}} - c_1 B(x_{\mathrm{rt}}) \leq x \leq x_{\mathrm{rt}}$ we have $+V'(x) > cS(x_{\mathrm{rt}})B^{-1}(x_{\mathrm{rt}})$.

(Hyp4) For $x_{\mathrm{left}} + c_1 B(x_{\mathrm{left}}) \leq x \leq x_{\mathrm{rt}} - c_1 B(x_{\mathrm{rt}})$ we have $cS(x) < E_0 - V(x) < CS(x)$.

To state the remaining hypotheses, we establish some notation. Set $\lambda(x) = S^{1/2}(x)B(x)$ for $x \in I$. Then set

$$B_{\mathrm{left}} = B(x_{\mathrm{left}}), \quad S_{\mathrm{left}} = S(x_{\mathrm{left}}), \quad \lambda_{\mathrm{left}} = \lambda(x_{\mathrm{left}}) \ .$$

$$B_{\mathrm{rt}} = B(x_{\mathrm{rt}}), \quad S_{\mathrm{rt}} = S(x_{\mathrm{rt}}), \quad \lambda_{\mathrm{rt}} = \lambda(x_{\mathrm{rt}}) \ .$$

For $|E - E_0| < cS_{\mathrm{left}}$, let $x_{\mathrm{left}}(E)$ be the solution of $V(x) = E$ nearest to x_{left}, and for $|E - E_0| < cS_{\mathrm{rt}}$, let $x_{\mathrm{rt}}(E)$ be the solution of $V(x) = E$ nearest to x_{rt}. Define $S_{\min} = \inf_{x_{\mathrm{left}} < x < x_{\mathrm{rt}}} S(x)$ and $\Lambda = \left(\int_{x_{\mathrm{left}}}^{x_{\mathrm{rt}}} \frac{dx}{S^{1/2}(x)B^2(x)}\right)^{-1}$.

Our remaining hypotheses are as follows.

Assumptions on $V(x)$ in all of I_{BVP}.

(Hyp5) If $|E - E_0| < c_2 S_{\min}$ and $E \leq E_\infty$, then $V(x) > E$ for all $x \in I_{\mathrm{BVP}} \backslash [x_{\mathrm{left}}(E), x_{\mathrm{rt}}(E)]$.

(Hyp6) If $x \in I_{\rm BVP}$ satisfies $x < x_{\rm left} - \frac{1}{2}\lambda^K_{\rm left} B_{\rm left}$ then $V(x) \geq E_\infty + \frac{100}{|x - x_{\rm left}|^2}$, and if $x \in I_{\rm BVP}$ satisfies $x > x_{\rm rt} + \frac{1}{2}\lambda^K_{\rm rt} B_{\rm rt}$, then $V(x) \geq E_\infty + \frac{100}{|x - x_{\rm rt}|^2}$.

Technical Assumptions.

(Hyp7) $\max_{x \in I} S(x) \leq \lambda^K_{\rm left} S_{\rm left}$ and $\max_{x\in I} S(x) \leq \lambda^K_{\rm rt} S_{\rm rt}$

(Hyp8) $\int_{x_{\rm left}}^{x_{\rm rt}} \frac{dx}{S^{1/2}(x)} \leq \Lambda^K \cdot \min(S_{\rm left}^{-1/2} B_{\rm left}, S_{\rm rt}^{-1/2} B_{\rm rt})$

(Hyp9) $[\int_{x_{\rm left}}^{x_{\rm rt}} \frac{dx}{S^{1/2}(x) B^4(x)}] \cdot [\int_{x_{\rm left}}^{x_{\rm rt}} \frac{dx}{S^{1/2}(x)}] \leq \Lambda^K$

WKB Condition.

(Hyp10) Λ is bounded below by a positive constant depending only on ε, K, N, and on c, C, c_1, c_2, C_α in (Hyp0)...(Hyp4).

Definitions and Basic Properties of Phases.

Assume hypotheses (Hyp0)...(Hyp10). For $|E - E_0| < c_\# S_{\min}$, define

$$\phi(E) = \int_{x_{\rm left}(E)}^{x_{\rm rt}(E)} (E - V(x))^{1/2}\, dx \quad \text{and}$$

$$\psi(E) = \lim_{\delta \to 0+} \Big[\int_{x_{\rm left}(E)+\delta}^{x_{\rm rt}(E)-\delta} V''(x)(E - V(x))^{-\frac{3}{2}}\, dx - q(E)\delta^{-1/2} \Big]$$

with $q(E)$ uniquely specified by demanding the finiteness of the limit.

Lemma 1. *For* $|E - E_0| < c_\# S_{\min}$ *we have*

$$\Big| \Big(\frac{d}{dE}\Big)^\beta \phi(E) \Big| \leq C^\beta_\# \int_{x_{\rm left}}^{x_{\rm rt}} S^{\frac{1}{2}-\beta}(x)\, dx \quad \textit{and}$$

$$\Big| \Big(\frac{d}{dE}\Big)^\beta \psi(E) \Big| \leq C^\beta_\# \int_{x_{\rm left}}^{x_{\rm rt}} \frac{dx}{S^{\frac{1}{2}+\beta}(x) B^2(x)}\ .$$

Also $c_\# \int_{x_{\rm left}}^{x_{\rm rt}} \frac{dx}{S^{1/2}(x)} < \frac{d\phi(E)}{dE} < C_\# \int_{x_{\rm left}}^{x_{\rm rt}} \frac{dx}{S^{1/2}(x)}$.

The constants $c_\#$, $C_\#$, $C^\beta_\#$ *depend only on* ε, K, N, c, C, c_1, c_2, C_α *in the hypotheses (Hyp 0)...(Hyp 4).*

WKB Eigenvalue Theorem. *If (Hyp0)…(Hyp10) hold, then there is a function* $\Phi(E)$ *on* $[E_0 - c_\# S_{\min}, E_0 + c_\# S_{\min}]$ *and there are numbers* $E_{k_{\min}}, E_{k_{\min}+1}, \dots, E_{k_{\max}} \le E_\infty$ *with the following properties.*

(A) $\Phi(E) = \pm\frac{\pi}{2} + \phi(E) + \frac{1}{48}\psi(E) + \phi_{\text{error}}(E)$, *with*

$$\left|\left(\frac{d}{dE}\right)^\beta \phi_{\text{error}}\right| \le C_\#^\beta \Lambda^{-1} \int_{x_{\text{left}}}^{x_{\text{rt}}} \frac{dx}{S^{\frac{1}{2}+\beta}(x) B^2(x)}, \quad \text{all } \beta \ge 0 \ .$$

(B) $\{k_{\min}, k_{\min}+1, \dots, k_{\max}\}$ *is exactly the set of integers* k *for which* $|\Phi(E) - \pi k| < C_\# \Lambda^{-N''}$ *for some* $E \in [E_0 - \frac{1}{4}c_\# S_{\min}, E_0 + \frac{1}{4}c_\# S_{\min}] \cap (-\infty, E_\infty]$.

(C) *If* $k_{\min} \le k < k_{\max}$, *then* E_k *is an eigenvalue of* H.

(D) *If* $k = k_{\max}$, *then* E_k *is either an eigenvalue of* H *or equal to* E_∞.

(E) *For* $k_{\min} \le k \le k_{\max}$ *we have* $|E_k - E_0| < c_\# S_{\min}$ *and* $|\Phi(E_k) - \pi k| < C_\# \Lambda^{-N''}$.

(F) *Every eigenvalue of* H *in the interval* $[E_0 - \frac{1}{4}c_\# S_{\min}, E_0 + \frac{1}{4}c_\# S_{\min}] \cap (-\infty, E_\infty]$ *is one of the* E_k $(k_{\min} \le k \le k_{\max})$.

The constants $c_\#$, $C_\#^\beta$ depend only on ε, K, N, c, C, c_1 c_2, C_α in (Hyp0)…(Hyp5).

Remark. Perhaps $E_{k_{\min}}$ or $E_{k_{\max}}$ or both lie slightly outside

$$[E_0 - \frac{1}{4}c_\# S_{\min}, E_0 + \frac{1}{4}c_\# S_{\min}] \ .$$

We prepare to describe the eigenfunctions of H. Given an energy E with $|E - E_0| < c_\# S_{\min}$, define intervals:

$$I^E_{\text{far left}} = I_{\text{BVP}} \cap (-\infty, x_{\text{left}}(E) - \lambda_{\text{left}}^{\varepsilon - 2/3} B_{\text{left}}]$$

$$I^E_{\text{Airey left}} = [x_{\text{left}}(E) - \lambda_{\text{left}}^{-\varepsilon} B_{\text{left}}, x_{\text{left}}(E) + \lambda_{\text{left}}^{-\varepsilon} B_{\text{left}}]$$

$$I^E_{\text{medium left}} = [x_{\text{left}}(E) + \lambda_{\text{left}}^{\varepsilon - \frac{2}{3}} B_{\text{left}}, x_{\text{left}}(E) + c_\# B_{\text{left}}]$$

$$I^E_{\text{center}} = [x_{\text{left}}(E) + \frac{1}{2}c_\# B_{\text{left}}, x_{\text{rt}}(E) - \frac{1}{2}c_\# B_{\text{rt}}]$$

$$I^E_{\text{medium rt}} = [x_{\text{rt}}(E) - c_\# B_{\text{rt}}, x_{\text{rt}}(E) - \lambda_{\text{rt}}^{\varepsilon - 2/3} B_{\text{rt}}]$$

$$I^E_{\text{Airey rt}} = [x_{\text{rt}}(E) - \lambda_{\text{rt}}^{-\varepsilon} B_{\text{rt}}, x_{\text{rt}}(E) + \lambda_{\text{rt}}^{-\varepsilon} B_{\text{rt}}]$$

$$I^E_{\text{far rt}} = [x_{\text{rt}}(E) + \lambda_{\text{rt}}^{\varepsilon - 2/3} B_{\text{rt}}, \infty) \cap I_{\text{BVP}}.$$

Also define regions

$$U_{\text{Airey left}} = \{(x,E) \,\Big|\, |E - E_0| < c_\# S_{\min}, |x - x_{\text{left}}(E)| < \lambda_{\text{left}}^{-\varepsilon} B_{\text{left}}\},$$

$$U_{\text{Airey rt}} = \{(x,E) \,\Big|\, |E - E_0| < c_\# S_{\min}, |x - x_{\text{rt}}(E)| < \lambda_{\text{rt}}^{-\varepsilon} B_{\text{rt}}\}$$

$$U_{\text{oscil}} = \{(x,E) \,\Big|\, |E - E_0| < c_\# S_{\min}, x_{\text{left}}(E) < x < x_{\text{rt}}(E)\} \ .$$

If $Y(x,E)$ is any smooth function with $\frac{\partial Y}{\partial x} \neq 0$, then define $\{Y,x\} \equiv \frac{1}{2}\frac{\partial_x^3 Y}{\partial_x Y} - \frac{3}{4}\left(\frac{\partial_x^2 Y}{\partial_x Y}\right)^2$.

WKB Eigenfunction Theorem. *Assume (Hyp0)… (Hyp10). Then there exist real–valued functions $Y_{\text{left}}(x,E)$ on $U_{\text{Airey left}}$, $Y_{\text{rt}}(x,E)$ on $U_{\text{Airey rt}}$, and complex–valued functions $u_{\text{left}}(x,E)$, $u_{\text{rt}}(x,E)$ on U_{oscil}, for which the following hold.*

Properties of the Auxiliary Functions.

(1) On $U_{\text{Airey left}}$ we can write $Y_{\text{left}} = Y_0^{\text{left}} + \lambda_{\text{left}}^{-2} Y_1^{\text{left}}$, with $|\partial_x^\alpha \partial_E^\beta Y_i^{\text{left}}| \leq C_\#^{\alpha\beta} B_{\text{left}}^{-\alpha} S_{\text{left}}^{-\beta}$ and $\lambda_{\text{left}}^2 (\frac{\partial Y_0^{\text{left}}}{\partial x})^2 Y_0^{\text{left}} = E - V(x)$ to order at least N' at $x = x_{\text{left}}(E)$. More precisely, $|\lambda_{\text{left}}^2 (\frac{\partial Y_{\text{left}}}{\partial x})^2 \cdot Y_{\text{left}} + \{Y_{\text{left}}, x\} - (E - V(x))| \leq C_\# \lambda_{\text{left}}^{-N'} S_{\text{left}}$ on $U^{\text{left}}_{\text{Airey}}$. Similarly, on $U_{\text{Airey rt}}$ we can write $Y_{\text{rt}} = Y_0^{\text{rt}} + \lambda_{\text{rt}}^{-2} Y_1^{\text{rt}}$, with $|\partial_x^\alpha \partial_E^\beta Y_i^{\text{rt}}| \leq C_\#^{\alpha\beta} B_{\text{rt}}^{-\alpha} S_{\text{rt}}^{-\beta}$ and $\lambda_{\text{rt}}^2 (\frac{\partial Y_0^{\text{rt}}}{\partial x})^2 Y_0^{\text{rt}} = E - V(x)$ to order at least N' at $x = x_{\text{rt}}(E)$. More precisely, $|\lambda_{\text{rt}}^2 (\frac{\partial Y_{\text{rt}}}{\partial x})^2 Y_{\text{rt}} + \{Y_{\text{rt}}, x\} - (E - V(x))| \leq C_\# \lambda_{\text{rt}}^{-N'} S_{\text{rt}}$ on $U^{\text{rt}}_{\text{Airey}}$.

(2) For $|E - E_0| < c_\# S_{\min}$ and $x \in I^E_{\text{medium left}}$ we have $|\partial_x^\alpha u_{\text{left}}| \leq C_\#^\alpha \lambda_{\text{left}}^{-1} (\frac{x - x_{\text{left}}(E)}{B_{\text{left}}})^{-\frac{3}{2}} \cdot (x - x_{\text{left}}(E))^{-\alpha}$, and $|\text{Re}\ u_{\text{left}}| \leq C_\# \lambda_{\text{left}}^{-2} (\frac{x - x_{\text{left}}(E)}{B_{\text{left}}})^{-3}$. Similarly, for $|E - E_0| < c_\# S_{\min}$ and $x \in I^E_{\text{medium rt}}$

we have $|\partial_x^\alpha u_{\rm rt}| \le C_\#^\alpha \lambda_{\rm rt}^{-1}(\frac{x_{\rm rt}(E)-x}{B_{\rm rt}})^{-\frac{3}{2}} \cdot (x_{\rm rt}(E) - x)^{-\alpha}$ and $|\text{Re } u_{\rm rt}| \le C_\# \lambda_{\rm rt}^{-2}(\frac{x_{\rm rt}(E)-x}{B_{\rm rt}})^{-3}$.

(3) For $|E - E_0| < c_\# S_{\min}$ and $x \in I^E_{\rm center}$ we have

$$|\partial_x^\alpha u_{\rm left}| \le C_\#^\alpha \Lambda^{-1} B^{-\alpha}(x) \quad \text{and} \quad |\text{Re } u_{\rm left}| \le C_\# \Lambda^{-2} \ .$$

Similarly, on the same region we have $|\partial_x^\alpha u_{\rm rt}| \le C_\#^\alpha \Lambda^{-1} B^{-\alpha}(x)$ and $|\text{Re } u_{\rm rt}| \le C_\# \Lambda^{-2}$.

Description of Eigenfunctions.

Let $F(x)$ be a real–valued eigenfunction of H with eigenvalue E and norm 1. Assume $|E - E_0| < c_\# S_{\min}$ and $E \le E_\infty$. Then there exist real constants $b_{\rm left}$, $b_{\rm rt}$ for which we have:

(4) $\int_{I^E_{\text{far left}}} |F(x)|^2\, dx \le \Lambda^{-N''}$ and $\int_{I^E_{\text{far rt}}} |F(x)|^2\, dx \le \Lambda^{-N''}$

(5) $\int_{I^E_{\text{Airey left}}} |F(x) - b_{\rm left} \lambda_{\rm left}^{-1/3} \left(\frac{\partial Y_{\rm left}(x,E)}{\partial x}\right)^{-1/2} A(\lambda_{\rm left}^{2/3} Y_{\rm left}(x,E))|^2\, dx \le \Lambda^{-N''}$ and $\int_{I^E_{\text{Airey rt}}} |F(x) - b_{\rm rt} \lambda_{\rm rt}^{-1/3} \left(\frac{-\partial Y_{\rm rt}(x,E)}{\partial x}\right)^{-1/2} A(\lambda_{\rm rt}^{2/3} Y_{\rm rt}(x,E))|^2\, dx \le \Lambda^{-N''}$

(6)

$$\int_{I^E_{\text{medium left}} \cup I^E_{\rm center}} \left| F(x) - b_{\rm left} \text{Re}\Big[\frac{e^{\pm i \frac{\pi}{4}}\, e^{i \int_{x_{\rm left}(E)}^{x} (E - V(t))^{1/2}\, dt}}{(E - V(x))^{1/4}} \cdot (1 + u_{\rm left}(x,E)) \Big] \right|^2 dx \le \Lambda^{-N''}$$

and

$$\int_{I^E_{\text{medium rt}} \cup I^E_{\rm center}} |F(x) - b_{\rm rt} \text{Re}\Big[\frac{e^{\mp i \frac{\pi}{4}}\, e^{-i \int_x^{x_{\rm rt}(E)} (E - V(t))^{1/2}\, dt}}{(E - V(x))^{1/4}} \cdot (1 + u_{\rm rt}(x,E)) \Big] |^2\, dx \le \Lambda^{-N''}$$

(7) $c_\# < (|b_{\rm left}|^2 + |b_{\rm rt}|^2) \int_{x_{\rm left}}^{x_{\rm rt}} \frac{dx}{S^{1/2}(x)} < C_\#$, and

(8) $\Big| |b_{\rm left}| \cdot |b_{\rm rt}|^{-1} - 1 \Big| \le C_\# \Lambda^{-2}$.

What the Constants may Depend on.

The constants $c_\#$, $C_\#$, $C_\#^\alpha$ above depend only on ε, K, N, c, C, c_1, c_2, C_α in (Hyp0)...(Hyp5).

WKB Normalization Theorem. *The constants* b_{left}, b_{rt} *in the WKB Eigenfunction Theorem satisfy*

$$\left| b_{\text{left}}^2 \cdot \Big(\frac{1}{2} \int_{E-V(x)>0} \frac{dx}{(E-V(x))^{1/2}} \Big) - 1 \right| \le C_\# \Lambda^{3\varepsilon - 2} \quad \textit{and}$$

$$\left| b_{\text{rt}}^2 \cdot \Big(\frac{1}{2} \int_{E-V(x)>0} \frac{dx}{(E-V(x))^{1/2}} \Big) - 1 \right| \le C_\# \Lambda^{3\varepsilon - 2},$$

with $C_\#$ *depending only on* ε, K, N, c, C, c_1, c_2, C_α *in (Hyp 0)...(Hyp 5).*

The WKB Theorem on Low Eigenvalues.

Let $\varepsilon, K, N > 0$ be given, with $\varepsilon N \ge 100$. Let $V(x)$ be a potential defined on a (possibly unbounded) interval I_{BVP}. Let S, B be positive numbers, and let $x_0 \in I_{\text{BVP}}$ be given. Define $\lambda = S^{1/2} B$. Let E_∞ be a given energy, with $E_\infty > V(x_0)$. We make the following assumptions.

(H0*) $I = \{|x - x_0| < cB\} \subset I_{\text{BVP}}$

(H1*) $|(\frac{d}{dx})^\alpha V(x)| \le C_\alpha S B^{-\alpha}$ in I

(H2*) $\frac{d^2}{dx^2} V \ge c' S B^{-2}$ in I

(H3*) $V'(x_0) = 0$

(H4*) For $x \in I_{\text{BVP}} \backslash I$ we have $V(x) \ge \min\{E_\infty, V(x_0) + c'' \lambda^{-2\varepsilon} S\}$.

(H5*) For $x \in I_{\text{BVP}}$ with $|x - x_0| > \frac{1}{2}\lambda^K B$, we have $V(x) \ge E_\infty + \frac{1000}{|x-x_0|^2}$.

(H6*) λ is bounded below by a positive constant depending only on c, c', c'', C_α in (H0*) and (H1*), and on ε, K, N.

Let $H = -\frac{d^2}{dx^2} + V(x)$ on $L^2(I_{\text{BVP}})$ with Dirichlet or Neumann conditions at the endpoints.

For $V(x_0) < E < V(x_0) + \lambda^{-2\varepsilon} S$, define $x_{\text{left}}(E) < x_{\text{rt}}(E)$ to be the two values of $x \in I$ at which $V(x) = E$. Then define

$$\phi(E) = \int_{x_{\text{left}}(E)}^{x_{\text{rt}}(E)} (E - V(x))^{1/2}\, dx$$

$$\begin{aligned}\psi(E) &= \lim_{\delta \to 0+} \Big[\int_{\substack{x \in I \\ E - V(x) > \delta}} V''(x)(E - V(x))^{-3/2}\, dx - q(E)\delta^{-1/2} \Big] \\ &= \lim_{\delta_{\text{left}}, \delta_{\text{rt}} \to 0+} \Big[\int_{x_{\text{left}}(E)+\delta_{\text{left}}}^{x_{\text{rt}}(E)-\delta_{\text{rt}}} V''(x)(E - V(x))^{-3/2}\, dx - q_{\text{left}}(E)\delta_{\text{left}}^{-1/2} \\ &\qquad - q_{\text{rt}}(E)\delta_{\text{rt}}^{-1/2} \Big]\end{aligned}$$

with $q(E)$, $q_{\text{left}}(E)$, $q_{\text{ rt}}(E)$ uniquely specified by demanding the finiteness of the limits.

Lemma 1. *The phases $\phi(E)$, $\psi(E)$ satisfy the estimates*

$$\Big| \Big(\frac{d}{dE}\Big)^{\beta} \phi(E) \Big| \le C_{\#}^{\beta} \lambda S^{-\beta}$$

$$\Big| \Big(\frac{d}{dE}\Big)^{\beta} \psi(E) \Big| \le C_{\#}^{\beta} \lambda^{-1} S^{-\beta}$$

$$\frac{d}{dE}\phi(E) \ge c_{\#} \lambda S^{-1}$$

for $V(x_0) < E < V(x_0) + \lambda^{-2\varepsilon} S$.

The constants $c_{\#}$, $C_{\#}^{\beta}$ depend only on c, c',c'', C_α, ε, K, N in hypotheses (H0)... (H6*).*

WKB Theorem on Low Eigenvalues. *Assume (H0*)... (H6*). Then there is a finite sequence $E_0, E_1, \dots, E_{k_{\max}}$ with the following properties.*

(a) *Let $w = \phi(E_*) + \frac{1}{48}\psi(E_*)$ with $E_* = min\{E_\infty, V(x_0) + c_{\#}\lambda^{-2\varepsilon} S\}$, and let $\overline{n}$ be the largest integer with $\pi(\overline{n} + 1/2) \le w$. If $\min_{k \in \mathbb{Z}} |w - \pi(k + 1/2)| > C_{\#}\lambda^{-2+4\varepsilon}$, then $k_{\max} = \overline{n}$. In any case, $|k_{\max} - \overline{n}| \le 1$.*

(b) *If* $0 \le k < k_{\max}$*, then* E_k *is an eigenvalue of* H*.*

(c) *Either* $E_{k_{\max}} = E_\infty$ *or else* $E_{k_{\max}}$ *is an eigenvalue of* H*.*

(d) *Every eigenvalue* E *of* H *satisfying* $E \le E_\infty$, $|E - V(x_0)| < c_\# \lambda^{-2\varepsilon} S$ *is one of the* E_k*.*

(e) *For* $0 \le k \le k_{\max}$ *we have* $V(x_0) < E_k < V(x_0) + 2c_\# \lambda^{-2\varepsilon} S$ *and* $|\phi(E_k) + \frac{1}{48}\psi(E_k) - \pi(k + 1/2)| \le C_\# \lambda^{-2+4\varepsilon}$.

The constants $c_\#$, $C_\#$ *depend only on* ε, K, N, c, c', c'', C_α *in hypotheses (H0*)... (H6*).* ∎

Lemma 2. *Assume (H0*)... (H6*). Let* $F(x)$ *be an eigenfunction of* H *whose eigenvalue* E *satisfies* $V(x_0) < E < V(x_0) + c_\# \lambda^{-2\varepsilon} S$ *and* $E \le E_\infty$*. Then*

$$\int_{I_{\mathrm{BVP}} \setminus \{|x - x_0| < \lambda^{-\varepsilon} B\}} |F(x)|^2 \, dx \le C_\# \lambda^{-N} \int_{\{|x - x_0| < \lambda^{-\varepsilon} B\}} |F(x)|^2 \, dx.$$

The constants $c_\#$, $C_\#$ *depend only on* ε, K, N, c, c', c'', C_α *in the hypotheses (H0*)... (H6*).*

WKB Theory with Weak Turning Points.

Set–up. We are given an energy E_0 and a potential $V(x)$ defined on a (possibly unbounded) interval I_{BVP}. The interval I_{BVP} is partitioned into subintervals $I_{\text{far left}}$, I_{left}, I_{center}, I_{rt}, $I_{\text{far rt}}$ with $I_{\text{far left}}$ to the left of I_{left}, I_{left} to the left of I_{center}, etc. Here, $I_{\text{far left}}$ and $I_{\text{far right}}$ may be empty. On I_{center} we are given positive weight functions $S(x)$, $B(x)$. Set $\lambda(x) = S^{1/2}(x) B(x)$ and $\Lambda = (\int_{I_{\text{center}}} \frac{dx}{\lambda(x) B(x)})^{-1}$. We make the following assumptions.

Hypotheses.

($\hat{\mathrm{H0}}$) I_{center} is non–empty, and for $x, y \in I_{\text{center}}$ with $|x - y| < cB(x)$ we have $c < B(y)/B(x) < C$ and $c < S(y)/S(x) < C$, and $|I_{\text{center}}| > cB(x)$.

($\hat{\mathrm{H1}}$) For $x \in I_{\text{center}}$ we have $|(\frac{d}{dx})^\alpha V(x)| \le C_\alpha S(x) B^{-\alpha}(x)$ and $cS(x) < E_0 - V(x) < CS(x)$.

(H$\hat{2}$) Λ is bounded below by a large number depending only on c, C, C_α in (H$\hat{0}$), (H$\hat{1}$).

(H$\hat{3}$) I_{left}, I_{rt} are non–empty. If $I_{\text{center}} = [x_{\text{left}}, x_{\text{rt}}]$, then we have $|I_{\text{left}}| \leq \underline{C}B(x_{\text{left}})$, $|I_{\text{rt}}| \leq \underline{C}B(x_{\text{rt}})$, $\lambda(x_{\text{left}}) \leq \underline{C}$, $\lambda(x_{\text{rt}}) \leq \underline{C}$.

(H$\hat{4}$) If $I_{\text{left}} = [x_{\text{far left}}, x_{\text{left}}]$, then we have $|V(x) - E_0| \leq \underline{C}|I_{\text{left}}|^{-1}(x - \hat{x}_{\text{far left}})^{-1}$ in I_{left}. Here $\hat{x}_{\text{far left}} \leq x_{\text{far left}}$ with strict inequality unless $I_{\text{far left}} = \emptyset$.

(H$\hat{5}$) For $x \in I_{\text{rt}}$ we have $|V(x) - E_0| \leq \underline{C}|I_{\text{rt}}|^{-2}$.

(H$\hat{6}$) For $x \in I_{\text{far left}}$ we have $V(x) - E_0 \geq \underline{c}|I_{\text{left}}|^{-2}$. $V(x)$ is C^∞ in the interior of $I_{\text{far left}}$.

(H$\hat{7}$) For $x \in I_{\text{far rt}}$ we have $V(x) - E_0 \geq \frac{-10^{-9}}{(x - x_{\text{rt}})^2}$.

Our goal is to understand the eigenfunctions and eigenvalues of $H = -\frac{d^2}{dx^2} + V(x)$ on $L^2(I_{\text{BVP}})$ with Dirichlet boundary conditions.

Denote by $C_\#$ a constant depending on c, C, C_α in (H$\hat{0}$), (H$\hat{1}$). Denote by C_*, c_* etc. constants depending only on c, C, C_α, $\underline{c}$, $\underline{C}$ in (H$\hat{0}$)…(H$\hat{7}$).

Note that I_{left} and I_{rt} don't play completely analogous rôles in our hypotheses.

Theorem 1. *Under the assumptions (H$\hat{0}$)…(H$\hat{7}$) we have*

$$\left|(\textit{Number of eigenvalues of } H < E_0) - \frac{1}{\pi}\int_{I_{\text{center}}} (E_0 - V(t))^{1/2}\, dt\right| \leq C_* \ .$$

Suppose E_0 is an eigenvalue of H, and suppose u is the corresponding eigenfunction, with u real–valued and having L^2–norm 1 on I_{BVP}. Then

$$|u(x)|^2 \leq C_* \Big(\int_{I_{\text{center}}} (E_0 - V(t))^{-1/2}\, dt\Big)^{-1} (E_0 - V(x))^{-1/2} \ \textit{for } x \in I_{\text{center}}$$

$$|u(x)|^2 \leq C_* \Big(\int_{I_{\text{center}}} (E_0 - V(t))^{-1/2}\, dt\Big)^{-1} (E_0 - V(x_{\text{left}}))^{-1/2} \exp\Big(\frac{-c_*(x_{\text{left}} - x)}{B(x_{\text{left}})}\Big)$$
$$\textit{for } x \in I_{\text{BVP}}, x \leq x_{\text{left}} = \min I_{\text{center}} \ .$$

$$|u(x)|^2 \leq C_* \Big(\int_{I_{\text{center}}} (E_0 - V(t))^{-1/2}\, dt\Big)^{-1} (E_0 - V(x_{\text{right}}))^{-1/2} \quad \textit{for } x \in I_{\text{right}} \ .$$

As an application of this theorem, we study the following situation.

Set–up. $V(x)$ is a potential defined on a (possibly unbounded) interval I_{BVP}. We are given a subinterval $I \subset I_{\mathrm{BVP}}$ and weight functions $S(x)$, $B(x) > 0$ defined on I. Set $\lambda(x) = S^{1/2}(x)B(x)$. We are given an energy E_0.

We make the following assumptions.

Hypotheses.

($\mathrm{H}\overline{0}$) For $x, y \in I$ with $|x-y| < cB(x)$ we have $c < \frac{B(y)}{B(x)} < C$ and $c < \frac{S(y)}{S(x)} < C$, and $|I| > cB(x)$.

($\mathrm{H}\overline{1}$) For $x \in I$ we have $|(\frac{d}{dx})^\alpha V(x)| \leq C_\alpha S(x) B^{-\alpha}(x)$.

($\mathrm{H}\overline{2}$) $\{x \in I_{\mathrm{BVP}} \mid V(x) < E_0\} = (x_{\mathrm{left}}, x_{\mathrm{right}}) \subset I$ with $\mathrm{dist}(x_{\mathrm{left}}, \partial I) > cB(x_{\mathrm{left}})$, $\mathrm{dist}(x_{\mathrm{right}}, \partial I) > cB(x_{\mathrm{right}})$.

($\mathrm{H}\overline{3}$) In $[x_{\mathrm{left}}, x_{\mathrm{left}} + c_1 B(x_{\mathrm{left}})]$ we have $-V'(x) \geq cS(x_{\mathrm{left}})/B(x_{\mathrm{left}})$, and in $[x_{\mathrm{right}} - c_1 B(x_{\mathrm{right}}), x_{\mathrm{right}}]$ we have $+V'(x) \geq cS(x_{\mathrm{right}})/B(x_{\mathrm{right}})$.

($\mathrm{H}\overline{4}$) In $[x_{\mathrm{left}} + c_1 B(x_{\mathrm{left}}), x_{\mathrm{right}} - c_1 B(x_{\mathrm{right}})]$ we have $cS(x) < E_0 - V(x) < CS(x)$.

($\mathrm{H}\overline{5}$) In $I_{\mathrm{BVP}}^{\mathrm{interior}} \cap (-\infty, x_{\mathrm{left}})$, $V(x)$ is decreasing and C^∞.

($\mathrm{H}\overline{6}$) $\Lambda = (\int_{x_{\mathrm{left}}}^{x_{\mathrm{right}}} \frac{dx}{\lambda(x)B(x)})^{-1}$ is bigger than a large positive number depending only on c, C, c_1, C_α in ($\mathrm{H}\overline{0}$)… ($\mathrm{H}\overline{4}$).

Let $H = -\frac{d^2}{dx^2} + V(x)$ on $L^2(I_{\mathrm{BVP}})$ with Dirichlet boundary conditions.

Theorem 2. *Under assumptions ($\mathrm{H}\overline{0}$)… ($\mathrm{H}\overline{6}$) above, we have*

$$|(\textit{Number of eigenvalues of } H < E_0) - \frac{1}{\pi}\int_{I_{\mathrm{BVP}}} (E_0 - V(t))_+^{1/2}\, dt| \leq C_*.$$

If E_0 is an eigenvalue of H, and if $u(x)$ (real–valued, with norm 1 in $L^2(I_{\mathrm{BVP}})$) is the corresponding eigenfunction, then

$$|u(x)|^2 \leq C_* \Big(\int_{I_{\mathrm{BVP}}} (E_0 - V(t))_+^{-1/2}\, dt\Big)^{-1} (E_0 - V(x))^{-1/2} \quad \textit{for } x_{\mathrm{left}} < x < x_{\mathrm{right}}$$

$$|u(x)|^2 \le C_* \Big(\int_{I_{\mathrm{BVP}}} (E_0 - V(t))_+^{-1/2}\, dt \Big)^{-1} \Big[S(x_{\mathrm{left}}) \lambda^{-2/3}(x_{\mathrm{left}}) \Big]^{-1/2} \cdot$$
$$\cdot \exp\Big(-c_* \lambda^{2/3}(x_{\mathrm{left}}) \frac{(x_{\mathrm{left}} - x)}{B(x_{\mathrm{left}})} \Big) \quad \text{for } x \in I_{\mathrm{BVP}} \cap (-\infty, x_{\mathrm{left}}] \; .$$

The constants c_*, C_* depend only on c, C, c_1, C_α, in ($H\bar{0}$)... ($H\bar{4}$).

A REFORMULATION OF THE EIGENVALUE THEOREM

In this section, we restate the WKB Eigenvalue Theorem in the form in which we will use it.

Reformulated Eigenvalue Theorem. *Assume the hypotheses of the WKB Eigenvalue Theorem, with* $E_\infty = 0$*. Let* $\phi(E)$ *and* $\psi(E)$ *be defined as in that theorem. Then the eigenvalues of* H *that lie in the interval*

(1) $$[E_{\ell o}, E_{hi}] \equiv (-\infty, 0] \cap \{E \mid |E - E_0| \le c^1_\# S_{\min}\}$$

may be written as $\{E_k \mid k_{\ell o} \le k \le k_{hi}\}$*, where the following properties are satisfied:*

(2) $$k_{\ell o} < k_{hi} \ .$$

(3) *For* $k_{\ell o} \le k \le k_{hi}$ *we have* $|\phi(E_k) + \frac{1}{48}\psi(E_k) - \pi(k + 1/2)| \le C_\# \Lambda^{-2}$.

(4) *The set* $\{k \in \mathbb{Z} \mid k_{\ell o} \le k \le k_{hi}\}$ *consists exactly of all those integers* k *that lie in the interval*

$$[a,b] \equiv [\frac{1}{\pi}\phi(E_{\ell o}) + \frac{1}{48\pi}\psi(E_{\ell o}) - \frac{1}{2} + \omega_{\ell o}, \frac{1}{\pi}\phi(E_{hi}) + \frac{1}{48\pi}\psi(E_{hi}) - \frac{1}{2} + \omega_{hi}]$$

with $|\omega_{\ell o}|, |\omega_{hi}| \le C_\# \Lambda^{-2}$.

(5) *If* $\min_{k \in \mathbb{Z}} |\frac{1}{\pi}\phi(E_{hi}) + \frac{1}{48\pi}\psi(E_{hi}) - \frac{1}{2} - k| \ge C_\# \Lambda^{-2}$,

then we can take $\omega_{hi} = 0$ *in (4)* .

Here, $c^1_\#$ *and* $C_\#$ *depend only on the constants* ε*,* K*,* N*, etc. appearing in the hypotheses of the WKB Eigenvalue Theorem.*

To derive the result from the WKB Eigenvalue Theorem is merely a tedious exercise. We provide details for the reader's convenience.

Proof of the Reformulated Eigenvalue Theorem. We use $c_\#$, $c'_\#$ $C_\#$ etc. for constants depending only on the constants ε, K, N etc. appearing in the hypotheses of the WKB Eigenvalue Theorem. From that theorem, we get a function $\Phi(E)$ on $[E_0 - c_\# S_{\min}, E_0 + c_\# S_{\min}]$ and a finite sequence $E_{k_{\min}}, E_{k_{\min}+1}, \dots, E_{k_{\max}} \le 0$ with the following properties.

(6) $\{k \in \mathbb{Z} \mid k_{\min} \le k \le k_{\max}\}$ is exactly the set of those integers k for which

$$|\Phi(E) - \pi k| < C_\# \Lambda^{-N''} \quad \text{for some}$$
$$E \in (-\infty, 0] \cap [E_0 - \frac{1}{4} c_\# S_{\min}, E_0 + \frac{1}{4} c_\# S_{\min}] \ .$$

(7) $$\text{If } k_{\min} \le k < k_{\max} \text{ , then } E_k \text{ is an eigenvalue of } H \ .$$

(8) $$E_{k_{\max}} \text{ is either } 0 \text{ or an eigenvalue of } H \ .$$

(9) For $k_{\min} \le k \le k_{\max}$ we have $E_k \in [E_0 - c_\# S_{\min}, E_0 + c_\# S_{\min}]$

and $|\Phi(E_k) - \pi k| < C_\# \Lambda^{-N''}$.

(10)

Every eigenvalue of H in the interval $(-\infty, 0] \cap [E_0 - \frac{1}{4} c_\# S_{\min}, E_0 + \frac{1}{4} c_\# S_{\min}]$

is one of the E_k $(k_{\min} \le k \le k_{\max})$.

Moreover, either

(11) $$\Phi(E) = -\frac{\pi}{2} + \phi(E) + \frac{1}{48}\psi(E) + \phi_{\text{error}}(E) \quad \text{with}$$

(12) $$\left|\left(\frac{d}{dE}\right)^\beta \phi_{\text{error}}(E)\right| \le C_\#^\beta \Lambda^{-1} \int_{x_{\text{left}}}^{x_{\text{rt}}} \frac{dx}{S^{\frac{1}{2}+\beta}(x) B^2(x)} \quad \text{for all } \beta \ge 0 \ ,$$
$$\text{and all } E \in [E_0 - c_\# S_{\min}, E_0 + c_\# S_{\min}]$$

or else

(13) $$\Phi(E) = +\frac{\pi}{2} + \phi(E) + \frac{1}{48}\psi(E) + \phi_{\text{error}}(E) \quad \text{with } \phi_{\text{error}} \text{ satisfying (12)} \ .$$

We can assume (11) holds instead of (13). For, if Φ satisfies (13), then in place of Φ, E_k, $k_{\min}$, $k_{\max}$ we use $\tilde{\Phi}(E) = \Phi(E) - \pi$, $\tilde{E}_k = E_{k+1}$, $\tilde{k}_{\min} = k_{\min} - 1$, $\tilde{k}_{\max} = k_{\max} - 1$, and we see that $\tilde{\Phi}$, $\tilde{E}_k$, $\tilde{k}_{\min}$, $\tilde{k}_{\max}$ satisfy (6)...(12).

Next we estimate the derivative of the phase Φ. Lemma 1 preceding the WKB Eigenvalue Theorem gives

$$\begin{aligned}(14)\quad \left|\frac{d}{dE}\psi(E)\right| &\leq C_\# \int_{x_{\text{left}}}^{x_{\text{rt}}} \frac{dx}{S^{3/2}(x)B^2(x)} \\ &\leq C_\# \big(\inf_{x\in[x_{\text{left}},x_{\text{rt}}]} S(x)B^2(x)\big)^{-1} \int_{x_{\text{left}}}^{x_{\text{rt}}} \frac{dx}{S^{1/2}(x)} \leq C_\# \Lambda^{-2} \int_{x_{\text{left}}}^{x_{\text{rt}}} \frac{dx}{S^{1/2}(x)} \quad \text{and}\end{aligned}$$

$$(15)\qquad c_\# \int_{x_{\text{left}}}^{x_{\text{rt}}} \frac{dx}{S^{1/2}(x)} < \frac{d}{dE}\phi(E) < C_\# \int_{x_{\text{left}}}^{x_{\text{rt}}} \frac{dx}{S^{1/2}(x)}$$

for $E \in [E_0 - c_\# S_{\min}, E_0 + c_\# S_{\min}]$. From (12) we get

$$(16)\qquad \left|\frac{d}{dE}\phi_{\text{error}}(E)\right| \leq C_\# \Lambda^{-1} \int_{x_{\text{left}}}^{x_{\text{rt}}} \frac{dx}{S^{3/2}(x)B^2(x)} \leq C_\# \Lambda^{-3} \int_{x_{\text{left}}}^{x_{\text{rt}}} \frac{dx}{S^{1/2}(x)}$$

for $E \in [E_0 - c_\# S_{\min}, E_0 + c_\# S_{\min}]$, where the last inequality follows as in (14). Putting (14), (15), (16) into (11), we find that

$$(17)\quad c_\# \int_{x_{\text{left}}}^{x_{\text{rt}}} \frac{dx}{S^{1/2}(x)} < \frac{d}{dE}\Phi(E) < C_\# \int_{x_{\text{left}}}^{x_{\text{rt}}} \frac{dx}{S^{1/2}(x)} \qquad \text{for } E \in [E_0 - c_\# S_{\min}, E_0 + c_\# S_{\min}]\ .$$

In particular, Φ is strictly increasing on $[E_0 - c_\# S_{\min}, E_0 + c_\# S_{\min}]$. Moreover if $J \subset (-\infty, 0] \cap [E_0 - \frac{c_\#}{4} S_{\min}, E_0 + \frac{c_\#}{4} S_{\min}]$ is an interval of length $\geq c'_\# S_{\min}$, then the image $\Phi(J)$ is an interval with length at least

$$c'_\# S_{\min} \cdot c_\# \int_{x_{\text{left}}}^{x_{\text{rt}}} \frac{dx}{S^{1/2}(x)} \geq c''_\# \int_{[x_{\text{left}},x_{\text{rt}}]\cap[\tilde{x}-c_\# B(\tilde{x}),\tilde{x}+c_\# B(\tilde{x})]} S_{\min} \frac{dx}{S^{1/2}(x)}$$

for any $\tilde{x} \in [x_{\text{left}}, x_{\text{rt}}]$. Picking $\tilde{x}$ so that $S(\tilde{x}) \sim S_{\min} \equiv \inf_{x\in[x_{\text{left}},x_{\text{rt}}]} S(x)$, we conclude that

$$\text{length } \Phi(J) \geq c_\# S^{1/2}(\tilde{x}) B(\tilde{x}) \geq c''_\# \Lambda\ .$$

Therefore, $\Phi(J)$ contains at least $c_{\#}\Lambda$ distinct intervals of the form

$$I_k = [\pi k - C_{\#}\Lambda^{-N''}, \pi k + C_{\#}\Lambda^{-N''}] \quad (k \in \mathbb{Z}) \ .$$

However, suppose $I_k \subset \Phi(J)$. Then

$$\pi k \in I_k \subset \Phi(J) \subset \Phi((-\infty, 0] \cap [E_0 - \frac{c_{\#}}{4} S_{\min}, E_0 + \frac{c_{\#}}{4} S_{\min}]) \ ,$$

so that $k_{\min} \leq k \leq k_{\max}$ by (6) and

$$\Phi(E_k) \in I_k \subset \Phi(J) \quad \text{by (9)} \ . \tag{18}$$

Since Φ is strictly increasing on $[E_0 - c_{\#}S_{\min}, E_0 + c_{\#}S_{\min}]$, and since $J \subset [E_0 - c_{\#}S_{\min}, E_0 + c_{\#}S_{\min}]$ while $E_k \in [E_0 - c_{\#}S_{\min}, E_0 + c_{\#}S_{\min}]$ by (9), it follows from (18) that E_k belongs to J. Consequently, J contains at least $c_{\#}\Lambda$ of the E_k ($k_{\min} \leq k \leq k_{\max}$). In view of (7), all but at most one of these E_k are eigenvalues of H. Thus we have proven the following fact.

(19)
If $J \subset (-\infty, 0] \cap [E_0 - \frac{1}{4} c_{\#} S_{\min}, E_0 + \frac{1}{4} c_{\#} S_{\min}]$ is an interval of length at least $c'_{\#} S_{\min}$, then at least $c''_{\#}\Lambda$ distinct eigenvalues of H lie in J.

Next we make a trivial observation that will be used repeatedly.

(20)
Suppose $k \in \mathbb{Z}, \xi \in \mathbb{R}$ and $0 < \tau < \frac{1}{10}$. If $k + 1 \geq \xi - \tau$ and $k \leq \xi + \tau$, then for some ω with $|\omega| \leq 2\tau$ we have $k =$ (greatest integer $\leq \xi + \omega$).

To see this, assume first that $k + 1 > \xi + \tau$. Then we have $k =$ (greatest integer $\leq \xi + \tau$). On the other hand, if $k + 1 \leq \xi + \tau$ then since $\xi - \tau \leq k + 1 \leq \xi + \tau$ with $0 < \tau < \frac{1}{10}$, we have $k =$ (greatest integer $\leq \xi - 2\tau$). Thus, (20) holds in either case.

Now define a set

$$\mathcal{K} = \{k \in \mathbb{Z} \mid k_{\min} \le k \le k_{\max} \text{ and } E_k \text{ is an eigenvalue of } H \text{ belonging to } [E_{\ell o}, E_{hi}]\} \ . \tag{21}$$

Note that $\mathcal{K}$ contains at least $c_\# \Lambda$ integers, by (1), (10) and (19). (Here we take $c_\#^1$ in (1) to be smaller than $\frac{1}{4}c_\#$ in (10).)

Also, note that

$$k_1 < k_2 < k_3 \ , \ k_1 \in \mathcal{K} \text{ and } k_3 \in \mathcal{K} \text{ imply } k_2 \in \mathcal{K} \ . \tag{22}$$

To see this, note that the E_k increase with k, as follows from (9) and the fact that Φ is strictly increasing on $[E_0 - c_\# S_{\min}, E_0 + c_\# S_{\min}]$. Suppose $k_1 < k_2 < k_3$ with $k_1, k_3 \in \mathcal{K}$. Then

$$k_{\min} \le k_1 < k_2 < k_3 \le k_{\max} \ , \text{ and} \tag{24}$$

$$E_{k_1}, E_{k_3} \in [E_{\ell o}, E_{hi}] \ . \tag{25}$$

From (25) and the fact that the E_k increase, we see that $E_{k_2} \in [E_{\ell o}, E_{hi}]$. From (24) and (7), we see that E_{k_2} is an eigenvalue of H. Hence $k_2 \in \mathcal{K}$, completing the proof of (22).

Since $\mathcal{K}$ is a non–empty finite set of integers satisfying (22), it follows that

$$\mathcal{K} = \{k \in \mathbb{Z} \mid k_{\ell o} \le k \le k_{hi}\} \tag{26}$$

for integers $k_{\ell o} \le k_{hi}$. Recalling that $\mathcal{K}$ contains at least $c_\# \Lambda$ integers, we conclude from (26) that

$$k_{hi} > k_{\ell o} + c_\# \Lambda \ . \tag{27}$$

Also, comparing (21) with (26), we get

$$k_{\min} \le k_{\ell o} \le k_{hi} \le k_{\max} \ . \tag{28}$$

From (10) and (1), we know that every eigenvalue of H that lies in $[E_{\ell o}, E_{hi}]$ is equal to $E_{\tilde{k}}$ for some $\tilde{k}$ ($k_{\min} \le \tilde{k} \le k_{\max}$). The definition (21) then shows that $\tilde{k} \in \mathcal{K}$. So the set of eigenvalues of H in $[E_{\ell o}, E_{hi}]$ is contained in $\{E_k \mid k \in \mathcal{K}\}$. The converse inclusion is immediate from (21), so the eigenvalues of H that lie in $[E_{\ell o}, E_{hi}]$ are precisely the E_k for $k \in \mathcal{K}$. In view of (26), this means that the eigenvalues of H that lie in $[E_{\ell o}, E_{hi}]$ are precisely the E_k for $k_{\ell o} \le k \le k_{hi}$, as in the statement of the Reformulated Eigenvalue Theorem. Moreover, (2) follows from (27); and (3) follows from (9), (11) and the estimate

$$(29) \qquad |\phi_{\text{error}}(E)| \le C_{\#}\Lambda^{-2} \text{ , valid for } E \in [E_0 - c_{\#}S_{\min}, E_0 + c_{\#}S_{\min}] \ .$$

Estimate (29) is merely the special case $\beta = 0$ of (12). So the only assertions of the Reformulated Eigenvalue Theorem that remain to be proved are (4) and (5).

To establish (4) and (5), the main step is to show that

$$(30) \qquad k_{hi} = (\text{ greatest integer } \le \frac{1}{\pi}\phi(E_{hi}) + \frac{1}{48\pi}\psi(E_{hi}) - \frac{1}{2} + \tilde{\omega})$$

for an $\tilde{\omega}$ with

$$(31) \qquad |\tilde{\omega}| \le C_{\#}\Lambda^{-2} \ .$$

To prove (30) and (31), we distinguish three cases.

Case A. $k_{hi} = k_{\max}$

Case B. $k_{hi} < k_{\max}$ and $E_{k_{hi}+1} \in [E_{\ell o}, E_{hi}]$ but $E_{k_{hi}+1}$ isn't an eigenvalue of H.

Case C. $k_{hi} < k_{\max}$ and $E_{k_{hi}+1} \notin [E_{\ell o}, E_{hi}]$.

Let us check that we are always in one of these three cases. If we're not in Case A, then (28) gives $k_{\min} \le k_{hi}+1 \le k_{\max}$, yet $k_{hi}+1 \notin \mathcal{K}$ by (26). Hence, (21) shows that $E_{k_{hi}+1} \notin [E_{\ell o}, E_{hi}]$ or else $E_{k_{hi}+1}$ isn't an eigenvalue of H. Consequently, we're either in Case B or Case C. This shows we are always in one of the three cases A, B, C.

Next we prove (30), (31) in case A. Thus we assume $k_{hi} = k_{\max}$. ¿From (6) we see that $|\Phi(E) - \pi(k_{\max}+1)| \geq C_{\#}\Lambda^{-N''}$ for all $E \in (-\infty, 0] \cap [E_0 - \frac{1}{4}c_{\#}S_{\min}, E_0 + \frac{1}{4}c_{\#}S_{\min}] \equiv [E_-, E_+]$. That is,

(32) $$\pi(k_{\max}+1) \text{ has distance at least } C_{\#}\Lambda^{-N''} \text{ from } [\Phi(E_-), \Phi(E_+)\,]\ .$$

(Recall that Φ is increasing on an interval containing $[E_-, E_+]$, so $[\Phi(E_-),\ \Phi(E_+)]$ is the image of $[E_-, E_+]$ under Φ.) On the other hand, another application of (6) shows similarly that

(33) $$\pi k_{\max} \text{ has distance less than } C_{\#}\Lambda^{-N''} \text{ from } [\Phi(E_-), \Phi(E_+)]\ .$$

The constant $C_{\#}$ is the same in (32) and (33). Hence (32) and (33) imply

(34) $$\pi(k_{\max}+1) \geq \Phi(E_+) + C_{\#}\Lambda^{-N''}\ .$$

Comparing (1) with the definition of $[E_-, E_+]$ and taking $c^1_{\#}$ small enough, we see that $E_{hi} \leq E_+$, and therefore $\Phi(E_{hi}) \leq \Phi(E_+)$. So (34) implies

(35) $$\pi(k_{hi}+1) \geq \Phi(E_{hi}) + C_{\#}\Lambda^{-N''}\ , \text{ since we're assuming } k_{\max} = k_{hi}\ .$$

On the other hand, (21) and (26) show that $E_{k_{hi}} \in [E_{\ell o}, E_{hi}]$.

In particular, $E_{k_{hi}} \leq E_{hi}$, so $\Phi(E_{k_{hi}}) \leq \Phi(E_{hi})$. Hence (9) implies

(36) $$\pi k_{hi} \leq \Phi(E_{hi}) + C_{\#}\Lambda^{-N''}\ .$$

In view of (11) and (29) estimates (35) and (36) imply

(37) $$k_{hi} + 1 \geq \frac{1}{\pi}\phi(E_{hi}) + \frac{1}{48\pi}\psi(E_{hi}) - \frac{1}{2} - C'_{\#}\Lambda^{-2} \text{ and}$$

(38) $$k_{hi} \leq \frac{1}{\pi}\phi(E_{hi}) + \frac{1}{48\pi}\psi(E_{hi}) - \frac{1}{2} + C'_{\#}\Lambda^{-2}\ .$$

The desired conclusions (30) and (31) follow at once from (37), (38) and (20). So we have proven (30) and (31) in Case A.

Next we verify (30) and (31) in Case B. Thus we assume $k_{hi} < k_{max}$ and

(39) $$E_{k_{hi}+1} \in [E_{\ell o}, E_{hi}] \ ,$$

but $E_{k_{hi}+1}$ isn't an eigenvalue of H. From (7) and (8) we see that $k_{hi} + 1 = k_{\max}$ and $E_{k_{hi}+1} = 0$. Then from (39) we get $0 \in [E_{\ell o}, E_{hi}] \subset (-\infty, 0]$, so that $E_{hi} = 0$. Since $E_{hi} = E_{k_{hi}+1} = 0$, (9) shows that $|\Phi(E_{hi}) - \pi(k_{hi} + 1)| \leq C_{\#}\Lambda^{-N''}$. In view of (11) and (29), this implies $|(\frac{1}{\pi}\phi(E_{hi}) + \frac{1}{48\pi}\psi(E_{hi}) - \frac{1}{2}) - (k_{hi} + 1)| \leq C'_{\#}\Lambda^{-2}$. Therefore, $k_{hi} = ($ greatest integer $\leq \frac{1}{\pi}\phi(E_{hi}) + \frac{1}{48\pi}\psi(E_{hi}) - \frac{1}{2} - 2C'_{\#}\Lambda^{-2})$, which immediately yields (30), (31). So we have proven (30), (31) in Case B.

Next we prove (30), (31) in Case C. Thus we assume $k_{hi} < k_{max}$ and

(40) $$E_{k_{hi}+1} \notin [E_{\ell o}, E_{hi}] \ .$$

From (21) and (26) we get

(41) $$E_{k_{hi}} \in [E_{\ell o}, E_{hi}] \ .$$

Since we noted that E_k increases with k, (40) and (41) imply $E_{k_{hi}+1} > E_{hi}$. Therefore $\Phi(E_{k_{hi}+1}) > \Phi(E_{hi})$ since Φ is strictly increasing on $[E_0 - c_{\#}S_{\min}, E_0 + c_{\#}S_{\min}]$. Applying (9), we obtain

(42) $$\pi(k_{hi} + 1) > \Phi(E_{hi}) - C_{\#}\Lambda^{-N''} \ .$$

On the other hand, (41) yields $\Phi(E_{k_{hi}}) \leq \Phi(E_{hi})$ since Φ is increasing, and therefore (9) implies

(43) $$\pi k_{hi} \leq \Phi(E_{hi}) + C_{\#}\Lambda^{-N''} \ .$$

From (42), (43) and (29), we conclude that

$$k_{hi} + 1 \geq \frac{1}{\pi}\phi(E_{hi}) + \frac{1}{48\pi}\psi(E_{hi}) - \frac{1}{2} - C'_{\#}\Lambda^{-2} \quad \text{and}$$

$$k_{hi} \leq \frac{1}{\pi}\phi(E_{hi}) + \frac{1}{48\pi}\psi(E_{hi}) - \frac{1}{2} + C'_{\#}\Lambda^{-2} \ .$$

These two inequalities and (20) show that (30), (31), hold in Case C. Thus, we have verified (30), (31) in all three cases A, B, C, so we know that (30), (31) hold always.

Note that under the additional assumption

$$\min_{k\in\mathbb{Z}} \Big| \big(\frac{1}{\pi}\phi(E_{hi}) + \frac{1}{48\pi}\psi(E_{hi}) - \frac{1}{2}\big) - k \Big| > C_{\#}\Lambda^{-2} \quad \text{with } C_{\#} \text{ as in (31)} ,$$

we obtain

$$k_{hi} = (\text{greatest integer } \leq \frac{1}{\pi}\phi(E_{hi}) + \frac{1}{48\pi}\psi(E_{hi}) - \frac{1}{2}) \quad \text{as a consequence of (30), (31)} .$$

Therefore, we have proven (30), (31) in the following stronger form:

(44) $$k_{hi} = (\text{greatest integer } \leq \frac{1}{\pi}\phi(E_{hi}) + \frac{1}{48\pi}\psi(E_{hi}) - \frac{1}{2} + \omega_{hi}) \text{ , with}$$

(45) $$|\omega_{hi}| \leq C'_{\#}\Lambda^{-2} \ , \quad \text{and}$$

(46) $$\omega_{hi} = 0 \ \text{ if } \ \min_{k\in\mathbb{Z}} \Big| \big(\frac{1}{\pi}\phi(E_{hi}) + \frac{1}{48\pi}\psi(E_{hi}) - \frac{1}{2}\big) - k \Big| \geq C'_{\#}\Lambda^{-2} \ .$$

We turn our attention to $k_{\ell o}$. Applying (20) to $-\xi$, $-k$, τ in place of ξ, k, τ, we get the following trivial observation.

(47) Suppose $k \in \mathbb{Z}, \xi \in \mathbb{R}$ and $0 < \tau \leq \frac{1}{10}$. If $k - 1 \leq \xi + \tau$ and $k \geq \xi - \tau$,

then $k = (\text{least integer } \geq \xi + \omega)$ for some ω with $|\omega| \leq 2\tau$.

We shall use (47) to prove

(48) $$k_{\ell o} = (\text{least integer } \geq \frac{1}{\pi}\phi(E_{\ell o}) + \frac{1}{48\pi}\psi(E_{\ell o}) - \frac{1}{2} + \omega_{\ell o}) \ ,$$

with

$$|\omega_{\ell o}| \leq C'_{\#}\Lambda^{-2} \ . \tag{49}$$

To prove (48), (49), we first note that

$$k_{\min} < k_{\ell o} \ . \tag{50}$$

In fact, applying (19) to $J = [E_0 - \frac{1}{4}c_{\#}S_{\min}, E_0 - 2c^1_{\#}S_{\min}]$ (and taking $c^1_{\#} < \frac{c_{\#}}{20}$ so that length $J > c'_{\#}S_{\min}$), we find that there are eigenvalues $\tilde{E}$ of H that belong to J. ¿From (10) we get $\tilde{E} = E_{\tilde{k}}$ for $k_{\min} \leq \tilde{k} \leq k_{\max}$. Thus, $E_{\tilde{k}} \in J$. On the other hand, $E_{k_{\ell o}} \in [E_{\ell o}, E_{hi}]$ by (21), (26). Since J lies to the left of $[E_{\ell o}, E_{hi}]$, it follows that $E_{\tilde{k}} < E_{k_{\ell o}}$. Since the E_k increase with k, we obtain $\tilde{k} < k_{\ell o}$, and therefore $k_{\min} \leq \tilde{k} < k_{\ell o}$, proving (50).

In view of (50), $k_{\min} \leq k_{\ell o} - 1 \leq k_{\max}$, so $E_{k_{\ell o}-1}$ is well–defined. Also, $k_{\ell o} - 1 < k_{\ell o} \leq k_{\max}$, so $E_{k_{\ell o}-1}$ is an eigenvalue of H, by (7). If $E_{k_{\ell o}-1}$ belonged to $[E_{\ell o}, E_{hi}]$, then (21) would give $k_{\ell o} - 1 \in \mathcal{K}$, contradicting (26). So $E_{k_{\ell o}-1}$ doesn't belong to $[E_{\ell o}, E_{hi}]$. On the other hand, $E_{k_{\ell o}}$ does belong to $[E_{\ell o}, E_{hi}]$, as we noted above. Since the E_k increase with k, it follows that $E_{k_{\ell o}-1} < E_{\ell o} \leq E_{k_{\ell o}}$. Hence $\Phi(E_{k_{\ell o}-1}) < \Phi(E_{\ell o}) \leq \Phi(E_{k_{\ell o}})$ since Φ is increasing on $[E_0 - c_{\#}S_{\min}, E_0 + c_{\#}S_{\min}]$. Applying (9), we find that

$$\pi(k_{\ell o} - 1) - C_{\#}\Lambda^{-N''} < \Phi(E_{\ell o}) \leq \pi k_{\ell o} + C_{\#}\Lambda^{-N''} \ .$$

Together with (11) and (29), this shows that

$$k_{\ell o} - 1 - C_{\#}\Lambda^{-2} < \frac{1}{\pi}\phi(E_{\ell o}) + \frac{1}{48\pi}\psi(E_{\ell o}) - \frac{1}{2} \leq k_{\ell o} + C_{\#}\Lambda^{-2} \ . \tag{51}$$

From (51) and (47), we deduce the desired inequalities (48), (49).

Now it is trivial to complete the proof of the Reformulated Eigenvalue Theorem by checking (4) and (5). In fact, (4) is immediate from (44), (48) and the estimates (45), (49). Finally, (5) is contained in (46). The proof is complete. ■

Remark. When we picked $c^1_\#$ in the proof of the Reformulated Eigenvalue Theorem, we needed only to make sure that $c^1_\#$ is small enough. Hence we can take $c^1_\# < \frac{1}{33}c_\#$, where $c_\#$ is the small constant appearing in the statements of the WKB Theorems (and Lemma 1 preceding them). This ensures that we have precise information on the phases $\phi(E)$, $\psi(E)$ for $E \in [E_0 - 33c^1_\# S_{\min}, E_0 + 33c^1_\# S_{\min}]$, and on eigenfunctions corresponding to eigenvalues in $[E_{\ell o}, E_{hi}]$.

The Reformulated Eigenvalue Theorem identifies conveniently those eigenvalues of $H = -\frac{d^2}{dx^2} + V(x)$ that lie in an interval $[E_0 - c^1_\# S_{\min}, E_0 + c^1_\# S_{\min}] \cap (-\infty, 0]$.

This makes it trivial to identify those eigenvalues of H that lie in a union of such intervals. Again we provide details for the reader's convenience.

Corollary to the Reformulated Eigenvalue Theorem. *Let* $[E_{\ell o}, E_{hi}] \subset (-\infty, 0]$. *Assume for each* $E_0 \in [E_{\ell o}, E_{hi}]$ *that the hypotheses of the WKB Eigenvalue Theorem are satisfied, with* $E_\infty = 0$, *and with constants* ε, K, N, C, c, c_1, c_2, C_α *independent of* E_0. *Let* $S_{\min}(E_0)$, $\Lambda(E_0)$ *denote the quantities called* $S_{\min}$, Λ *in the statement of the WKB Eigenvalue Theorem, corresponding to the given* E_0. *Define* $\Lambda_{\min} = \inf_{E_0 \in [E_{\ell o}, E_{hi}]} \Lambda(E_0)$. *Assume that* $\phi(E_{hi}) - \phi(E_{\ell o}) \geq 100$, *with* $\phi(E) \equiv \int_{I_{\mathrm{BVP}}} (E - V(x))_+^{1/2} dx$ *for* $E \in [E_{\ell o}, E_{hi}]$. *Then the eigenvalues of* H *that lie in* $[E_{\ell o}, E_{hi}]$ *may be written in the form* $(E_k)_{k_{\ell o} \leq k \leq k_{hi}}$, *so that the following properties hold.*

(52)
$$k_{\ell o} < k_{hi}$$

(53)
$$|\phi(E_k) - \pi(k + 1/2)| \leq C_\# \Lambda^{-1}_{\min} \quad \text{for} \quad k_{\ell o} \leq k \leq k_{hi} \ .$$

(54)
$$\{k \in \mathbb{Z} \mid k_{\ell o} \leq k \leq k_{hi}\} = \mathbb{Z} \cap [\frac{1}{\pi}\phi(E_{\ell o}) - \frac{1}{2} + \omega_{\ell o}, \frac{1}{\pi}\phi(E_{hi}) - \frac{1}{2} + \omega_{hi}] \ ,$$

with

$$|\omega_{\ell o}|, |\omega_{hi}| \le C_\# \Lambda_{\min}^{-1} \quad \text{and} \tag{55}$$

$$\textit{If} \quad \min_{k\in\mathbb{Z}} |\frac{1}{\pi}\phi(E_{hi}) - \frac{1}{2} - k| \ge C_\# \Lambda_{\min}^{-1} \ , \quad \textit{then} \quad \omega_{hi} = 0 \ . \tag{56}$$

Here $C_\#$ depends only on the constants ε, K, N, c, c_1, c_2, C, C_α in the hypotheses of the WKB Theorems.

Proof. Let $c_\#$, $C_\#$ etc. denote constants determined by ε, K, N, c, C, c_1, c_2, C_α. We begin with a few remarks on the phase ϕ. Recall that in the WKB Theorems we defined $\phi(E) = \int_I (E - V(x))_+^{1/2}\, dx$ for a certain subinterval $I \subset I_{\text{BVP}}$.

If $E_0 \in [E_{\ell o}, E_{hi}]$, then the hypotheses of the WKB Theorems hold with $E_\infty = 0$. It follows that $\{x \in I_{\text{BVP}} \mid V(x) < E\} \subset I$ for $E \in [E_0 - c_\#^1 S_{\min}(E_0),\ E_0 + c_\#^1 S_{\min}(E_0)] \cap (-\infty, 0] \equiv [E_{\ell o}(E_0), E_{hi}(E_0)]$. Hence

$$\phi(E) = \int\limits_{I_{\text{BVP}}} (E - V(x))_+^{1/2}\, dx \quad \text{for} \quad E_0 \in [E_{\ell o}, E_{hi}] \quad \text{and} \tag{57}$$
$$E \in [E_{\ell o}(E_0), E_{hi}(E_0)] \ .$$

We recall an important lower bound for $\phi'(E) = \frac{1}{2}\int_I (E - V(x))_+^{-1/2}\, dx$, when E is as in (57). In fact, we can find an interval of the form $J = [\tilde{x} - c_\# B(\tilde{x}), \tilde{x} + c_\# B(\tilde{x})]$ contained in $I \cap \{c_\# S(\tilde{x}) < E - V(x) < C_\# S(\tilde{x})\}$ and with $S(\tilde{x}) \sim S_{\min}(E_0)$. Restricting the region of integration to J in the formula for $\phi'(E)$, we get $\phi'(E) \ge c_\# (S(\tilde{x}))^{-1/2} B(\tilde{x}) = c_\# [(S(\tilde{x}))^{1/2} B(\tilde{x})](S(\tilde{x}))^{-1} = c_\# \lambda(\tilde{x})(S(\tilde{x}))^{-1} \ge c_\# (\Lambda(E_0))(S_{\min}(E_0))^{-1}$. This holds for $E \in [E_{\ell o}(E_0), E_{hi}(E_0)]$. Therefore,

$$\begin{aligned} \phi(E_0) &\ge \phi(E_{\ell o}(E_0)) + \Big(\frac{c_\# \Lambda(E_0)}{S_{\min}(E_0)}\Big) \cdot (E_0 - E_{\ell o}(E_0)) \\ &= \phi(E_{\ell o}(E_0)) + \frac{c_\# \Lambda(E_0)}{S_{\min}(E_0)} \cdot c_\#^1 S_{\min}(E_0) \ge \phi(E_{\ell o}(E_0)) + c'_\# \Lambda_{\min} \end{aligned} \tag{58}$$

for $E_0 \in [E_{\ell o}, E_{hi}]$. This concludes our small discussion of ϕ.

For each $E_0 \in [E_{\ell o}, E_{hi}]$, we apply the Reformulated Eigenvalue Theorem, to obtain the following results.

(59)

The eigenvalues of H that lie in $[E_{\ell o}(E_0), E_{hi}(E_0)]$ may be written as

$$\{\mathcal{E}(k, E_0) \colon k_{\ell o}(E_0) \le k \le k_{hi}(E_0)\}$$

(60) $$k_{\ell o}(E_0) < k_{hi}(E_0)$$

(61) $$|\phi(\mathcal{E}(k, E_0)) + \frac{1}{48}\psi(\mathcal{E}(k, E_0)) - \pi(k + 1/2)| \le C_{\#}(\Lambda(E_0))^{-2}$$

(62) $$\{k \in \mathbb{Z} \colon k_{\ell o}(E_0) \le k \le k_{hi}(E_0)\} = \mathbb{Z} \cap [a(E_0), b(E_0)] \ , \quad \text{where}$$

(63) $$a(E_0) = \frac{1}{\pi}\phi(E_{\ell o}(E_0)) + \frac{1}{48\pi}\psi(E_{\ell o}(E_0)) - \frac{1}{2} + \omega_{\ell o}(E_0) \ ,$$

(64) $$b(E_0) = \frac{1}{\pi}\phi(E_{hi}(E_0)) + \frac{1}{48\pi}\psi(E_{hi}(E_0)) - \frac{1}{2} + \omega_{hi}(E_0) \ , \quad \text{and}$$

(65) $$|\omega_{\ell o}(E_0)|, |\omega_{hi}(E_0)| \le C_{\#}(\Lambda(E_0))^{-2} \ .$$

Lemma 1 immediately preceding the WKB Eigenvalue Theorem gives $|\psi(E)| \le C_{\#}(\Lambda(E_0))^{-1}$ for $E \in [E_{\ell o}(E_0), E_{hi}(E_0)]$, $E_0 \in [E_{\ell o}, E_{hi}]$. Applying this to $E = E_{\ell o}(E_0)$, $E = E_{hi}(E_0)$ and $E = \mathcal{E}(k, E_0)$, and setting

$$\tilde{\omega}_{\ell o}(E_0) = \frac{1}{48\pi}\psi(E_{\ell o}(E_0)) + \omega_{\ell o}(E_0) \ , \quad \tilde{\omega}_{hi}(E_0) = \frac{1}{48\pi}\psi(E_{hi}(E_0)) + \omega_{hi}(E_0) \ ,$$

we obtain the following from (61)...(65).

(66) $$|\phi(\mathcal{E}(k, E_0)) - \pi(k + 1/2)| \le C_{\#}(\Lambda(E_0))^{-1} \quad \text{for} \quad k_{\ell o}(E_0) \le k \le k_{hi}(E_0) \ ;$$

$$\{k \in \mathbb{Z}: k_{\ell o}(E_0) \leq k \leq k_{hi}(E_0)\} = \mathbb{Z} \cap [a(E_0), b(E_0)] \; ; \tag{67}$$

$$a(E_0) = \frac{1}{\pi}\phi(E_{\ell o}(E_0)) - \frac{1}{2} + \tilde{\omega}_{\ell o}(E_0) \; ; \tag{68}$$

$$b(E_0) = \frac{1}{\pi}\phi(E_{hi}(E_0)) - \frac{1}{2} + \tilde{\omega}_{hi}(E_0) \; ; \tag{69}$$

$$|\tilde{\omega}_{\ell o}(E_0)|, |\tilde{\omega}_{hi}(E_0)| \leq C_\#(\Lambda(E_0))^{-1} \; . \tag{70}$$

Here (66)...(70) hold for all $E_0 \in [E_{\ell o}, E_{hi}]$.

We shall prove the following for a constant $C'_\#$ larger than $C_\#$ in (66)...(70).

Claim A:

Suppose $\overline{k^{-}} \in \mathbb{Z}$ satisfies

$$\frac{1}{\pi}\phi(E_{\ell o}) - \frac{1}{2} + C'_\#\Lambda_{\min}^{-1} \leq \overline{k^{-}} \leq \frac{1}{\pi}\phi(E_{hi}) - \frac{1}{2} - C'_\#\Lambda_{\min}^{-1} \; . \tag{71}$$

Then there is an eigenvalue $\overline{E}$ of H that lies in $[E_{\ell o}, E_{hi}]$ and satisfies

$$|\phi(\overline{E}) - \pi(\overline{k^{-}} + 1/2)| \leq C_\#\Lambda_{\min}^{-1} \; . \tag{72}$$

To see this, note that $\frac{1}{\pi}\phi(E_{\ell o}) - \frac{1}{2} \leq \overline{k^{-}} + C'_\#\Lambda_{\min}^{-1} \leq \frac{1}{\pi}\phi(E_{hi}) - \frac{1}{2}$, so we can find an $E_0 \in [E_{\ell o}, E_{hi}]$ with $\frac{1}{\pi}\phi(E_0) - \frac{1}{2} = \overline{k^{-}} + C'_\#\Lambda_{\min}^{-1}$. We will check that $a(E_0) \leq \overline{k^{-}} \leq b(E_0)$.

In fact, $\overline{k^{-}} + C'_\#\Lambda_{\min}^{-1} = \frac{1}{\pi}\phi(E_0) - \frac{1}{2} \leq \frac{1}{\pi}\phi(E_{hi}(E_0)) - \frac{1}{2} = b(E_0) - \tilde{\omega}_{hi}(E_0) \leq b(E_0) + C_\#\Lambda_{\min}^{-1}$ by (69), (70). Hence $\overline{k^{-}} \leq b(E_0)$. On the other hand, (58) gives $\frac{1}{\pi}\phi(E_{\ell o}(E_0)) - \frac{1}{2} \leq \frac{1}{\pi}\phi(E_0) - \frac{1}{2} - c_\#\Lambda_{\min} = \overline{k^{-}} + C'_\#\Lambda_{\min}^{-1} - c_\#\Lambda_{\min} \leq \overline{k^{-}} - \frac{1}{2}c_\#\Lambda_{\min}$. This, together with (68), (70), gives $a(E_0) \leq \overline{k^{-}}$. Thus $a(E_0) \leq \overline{k^{-}} \leq b(E_0)$, as asserted. Hence $k_{\ell o}(E_0) \leq \overline{k^{-}} \leq k_{hi}(E_0)$ by (67), so that $\overline{E} = \mathcal{E}(\overline{k^{-}}, E_0)$ is an eigenvalue of H lying in $[E_{\ell o}(E_0), E_{hi}(E_0)]$ by (59). We have

$|\phi(\overline{E}) - \pi(k^{--} + 1/2)| \le C_\# \Lambda_{\min}^{-1}$ by (66), which proves (72). It remains to check that $\overline{E} \in [E_{\ell o}, E_{hi}]$. Since $\overline{E} \in [E_{\ell o}(E_0), E_{hi}(E_0)]$, we have $\phi(\overline{E}) = \int_{I_{\rm BVP}} (\overline{E} - V(x))_+^{1/2}\, dx$ by (57). Equation (57) yields also $\phi(E_{\ell o}) = \int_{I_{\rm BVP}} (E_{\ell o} - V(x))_+^{1/2}\, dx$ and $\phi(E_{hi}) = \int_{I_{\rm BVP}} (E_{hi} - V(x))_+^{1/2}\, dx$.

¿From (71) and (72) we get $\phi(E_{\ell o}) < \phi(\overline{E}) < \phi(E_{hi})$. Thus,

$$\int_{I_{\rm BVP}} (E_{\ell o} - V(x))_+^{1/2}\, dx < \int_{I_{\rm BVP}} (\overline{E} - V(x))_+^{1/2}\, dx < \int_{I_{\rm BVP}} (E_{hi} - V(x))_+^{1/2}\, dx \ ,$$

which implies $\overline{E} \in [E_{\ell o}, E_{hi}]$, completing the proof of *CLAIM A*.

The next step is to prove the following.

CLAIM B: If E, E' are distinct eigenvalues of H belonging to $[E_{\ell o}, E_{hi}]$, then $|\phi(E) - \phi(E')| > \frac{1}{100}$.

To see this suppose instead $|\phi(E) - \phi(E')| \le \frac{1}{100}$. We may assume $E' < E$. If E', E both belonged to $[E_{\ell o}(E), E_{hi}(E)]$, then we would get a contradiction from (59), (66). Since $E \in [E_{\ell o}(E), E_{hi}(E)]$, we must have $E' \notin [E_{\ell o}(E), E_{hi}(E)]$. Since also $E' < E \in [E_{\ell o}(E), E_{hi}(E)]$, we must have

$$E' < E_{\ell o}(E) \le E \ . \tag{73}$$

From (73) we see that E', $E_{\ell o}(E)$, E all belong to $[E_{\ell o}, E_{hi}]$, so that (73) implies

$$\phi(E') \le \phi(E_{\ell o}(E)) \ .$$

Therefore, $\phi(E') \le \phi(E) - c_\# \Lambda_{\min}$ by (58), which contradicts our original assumption $|\phi(E) - \phi(E')| \le \frac{1}{100}$. This completes the proof of *CLAIM B*.

Next we prove

CLAIM C: Let $E \in [E_{\ell o}, E_{hi}]$ be an eigenvalue of H. Then

$$|\phi(E) - \pi(k + 1/2)| \le C_\# \Lambda_{\min}^{-1} \tag{74}$$

for an integer k which satisfies

$$\frac{1}{\pi}\phi(E_{\ell o}) - \frac{1}{2} - C_\# \Lambda_{\min}^{-1} \le k \le \frac{1}{\pi}\phi(E_{hi}) - \frac{1}{2} + C_\# \Lambda_{\min}^{-1} \ . \tag{75}$$

In fact, since $E \in [E_{\ell o}(E), E_{hi}(E)]$, (59) gives $E = \mathcal{E}(k, E)$ for an integer k with $k_{\ell o}(E) \le k \le k_{hi}(E)$. Then from (66) we get (74). To prove (75), we recall that ϕ is increasing on $[E_{\ell o}, E_{hi}]$, so that $\frac{1}{\pi}\phi(E_{\ell o}) - \frac{1}{2} \le \frac{1}{\pi}\phi(E) - \frac{1}{2} \le \frac{1}{\pi}\phi(E_{hi}) - \frac{1}{2}$. These inequalities and (74) imply (75).

Let $\mathcal{K}$ be the set of all integers k for which there is an eigenvalue $E \in [E_{\ell o}, E_{hi}]$ satisfying

(76) $$|\phi(E) - \pi(k + 1/2)| \le C_{\#}\Lambda_{\min}^{-1} .$$

Claim B shows that for each $k \in \mathcal{K}$ there is only one eigenvalue $E \in [E_{\ell o}, E_{hi}]$ satisfying (76); we call that eigenvalue E_k. Thus, E_k is well-defined and belongs to $[E_{\ell o}, E_{hi}]$ for $k \in \mathcal{K}$. Moreover, E_k is an eigenvalue of H and satisfies

(77) $$|\phi(E_k) - \pi(k + 1/2)| \le C_{\#}\Lambda_{\min}^{-1} \quad \text{for} \quad k \in \mathcal{K} \ ; \quad \text{and}$$

(78) If $k \in \mathcal{K}$ and E is an eigenvalue of H belonging to $[E_{\ell o}, E_{hi}]$ and satisfying $|\phi(E) - \pi(k + 1/2)| \le C_{\#}\Lambda_{\min}^{-1}$, then $E = E_k$.

CLAIM C, (78) and the definition of $\mathcal{K}$ show that the eigenvalues of H that belong to $[E_{\ell o}, E_{hi}]$ are precisely the $(E_k)_{k \in \mathcal{K}}$.

Moreover, *CLAIMS A* and *C* show that

(79) $$\mathcal{K} \supset \mathbb{Z} \cap [\frac{1}{\pi}\phi(E_{\ell o}) - \frac{1}{2} + C'_{\#}\Lambda_{\min}^{-1}, \frac{1}{\pi}\phi(E_{hi}) - \frac{1}{2} - C'_{\#}\Lambda_{\min}^{-1}] \text{ and}$$

(80) $$\mathcal{K} \subset \mathbb{Z} \cap [\frac{1}{\pi}\phi(E_{\ell o}) - \frac{1}{2} - C_{\#}\Lambda_{\min}^{-1}, \frac{1}{\pi}\phi(E_{hi}) - \frac{1}{2} + C_{\#}\Lambda_{\min}^{-1}] .$$

Define $\tilde{\omega}_{\ell o}$ and $\tilde{\omega}_{hi}$ as follows.

(81) $\tilde{\omega}_{\ell o} = -C_{\#}\Lambda_{\min}^{-1}$ if there is some

$$k \in \mathcal{K} \cap [\frac{1}{\pi}\phi(E_{\ell o}) - \frac{1}{2} - C_{\#}\Lambda_{\min}^{-1}, \frac{1}{\pi}\phi(E_{\ell o}) - \frac{1}{2} + C'_{\#}\Lambda_{\min}^{-1}] ,$$

$$\tilde{\omega}_{\ell o} = +C'_{\#}\Lambda_{\min}^{-1} \quad \text{otherwise} \ . \tag{82}$$

(83) $\tilde{\omega}_{hi} = +C_{\#}\Lambda_{\min}^{-1}$ if there is some

$$k \in \mathcal{K} \ \cap [\frac{1}{\pi}\phi(E_{hi}) - \frac{1}{2} - C'_{\#}\Lambda_{\min}^{-1}, \frac{1}{\pi}\phi(E_{hi}) - \frac{1}{2} + C_{\#}\Lambda_{\min}^{-1}] \ ,$$

$$\tilde{\omega}_{hi} = -C'_{\#}\Lambda_{\min}^{-1} \quad \text{otherwise} \ . \tag{84}$$

Evidently,

$$|\tilde{\omega}_{\ell o}|, |\tilde{\omega}_{hi}| \leq C'_{\#}\Lambda_{\min}^{-1} \ . \tag{85}$$

Let us check that

$$\mathcal{K} = \mathbb{Z} \cap [\frac{1}{\pi}\phi(E_{\ell o}) - \frac{1}{2} + \tilde{\omega}_{\ell o}, \frac{1}{\pi}\phi(E_{hi}) - \frac{1}{2} + \tilde{\omega}_{hi}] \ . \tag{86}$$

Suppose $\bar{k}^{--} \in \mathcal{K}$. From (80) we get $\bar{k}^{--} \leq \frac{1}{\pi}\phi(E_{hi}) - \frac{1}{2} + C_{\#}\Lambda_{\min}^{-1}$. If $\bar{k}^{--} \geq \frac{1}{\pi}\phi(E_{hi}) - \frac{1}{2} - C'_{\#}\Lambda_{\min}^{-1}$, then the definition (83) gives $\tilde{\omega}_{hi} = C_{\#}\Lambda_{\min}^{-1}$ so that

$$\bar{k}^{--} \leq \frac{1}{\pi}\phi(E_{hi}) - \frac{1}{2} + \tilde{\omega}_{hi} \ . \tag{87}$$

On the other hand, if $\bar{k}^{--} \leq \frac{1}{\pi}\phi(E_{hi}) - \frac{1}{2} - C'_{\#}\Lambda_{\min}^{-1}$, then (87) follows from the definition (83), (84) of $\tilde{\omega}_{hi}$. Hence (87) holds in all cases.

Similarly, (80) yields $\bar{k}^{--} \geq \frac{1}{\pi}\phi(E_{\ell o}) - \frac{1}{2} - C_{\#}\Lambda_{\min}^{-1}$. If $\bar{k}^{--} \leq \frac{1}{\pi}\phi(E_{\ell o}) - \frac{1}{2} + C'_{\#}\Lambda_{\min}^{-1}$, then (81) gives $\tilde{\omega}_{\ell o} = -C_{\#}\Lambda_{\min}^{-1}$, so that

$$\bar{k}^{--} \geq \frac{1}{\pi}\phi(E_{\ell o}) - \frac{1}{2} + \tilde{\omega}_{\ell o} \ . \tag{88}$$

On the other hand, if $\bar{k}^{--} \geq \frac{1}{\pi}\phi(E_{\ell o}) - \frac{1}{2} + C'_{\#}\Lambda_{\min}^{-1}$, then (88) follows from the definition (81), (82) of $\tilde{\omega}_{\ell o}$. Hence (88) holds in all cases.

¿From (87), (88) we see that

$$\mathcal{K} \subset \mathbb{Z} \cap [\frac{1}{\pi}\phi(E_{\ell o}) - \frac{1}{2} + \tilde{\omega}_{\ell o}, \frac{1}{\pi}\phi(E_{hi}) - \frac{1}{2} + \tilde{\omega}_{hi}] \ , \tag{89}$$

which is half of (86) .

Now suppose $k^{--} \in \mathbb{Z} \cap [\frac{1}{\pi}\phi(E_{\ell o}) - \frac{1}{2} + \tilde{\omega}_{\ell o}, \frac{1}{\pi}\phi(E_{hi}) - \frac{1}{2} + \tilde{\omega}_{hi}]$. Then either

$$\text{(90)} \qquad \frac{1}{\pi}\phi(E_{\ell o}) - \frac{1}{2} + \tilde{\omega}_{\ell o} \leq k^{--} < \frac{1}{\pi}\phi(E_{\ell o}) - \frac{1}{2} + C'_{\#}\Lambda_{\min}^{-1}, \text{ or}$$

$$\text{(91)} \qquad \frac{1}{\pi}\phi(E_{\ell o}) - \frac{1}{2} + C'_{\#}\Lambda_{\min}^{-1} \leq k^{--} \leq \frac{1}{\pi}\phi(E_{hi}) - \frac{1}{2} - C'_{\#}\Lambda_{\min}^{-1}, \text{ or}$$

$$\text{(92)} \qquad \frac{1}{\pi}\phi(E_{hi}) - \frac{1}{2} - C'_{\#}\Lambda_{\min}^{-1} < k^{--} \leq \frac{1}{\pi}\phi(E_{hi}) - \frac{1}{2} + \tilde{\omega}_{hi} \ .$$

If (90) holds, then $\tilde{\omega}_{\ell o} \neq C'_{\#}\Lambda_{\min}^{-1}$, so (81), (82) show that $\tilde{\omega}_{\ell o} = -C_{\#}\Lambda_{\min}^{-1}$ and that there is some $k \in \mathcal{K}$ satisfying

$$\text{(93)} \qquad \frac{1}{\pi}\phi(E_{\ell o}) - \frac{1}{2} - C_{\#}\Lambda_{\min}^{-1} \leq k \leq \frac{1}{\pi}\phi(E_{\ell o}) - \frac{1}{2} + C'_{\#}\Lambda_{\min}^{-1} \ .$$

From (90), (93) we get $|k - k^{--}| \leq (C_{\#} + C'_{\#})\Lambda_{\min}^{-1} << 1$. Since k and k^{--} are both integers, it follows that $k^{--} = k$. Hence k^{--} belongs to $\mathcal{K}$.

If (91) holds, then k^{--} belongs to $\mathcal{K}$, by (79).

If (92) holds, then $\tilde{\omega}_{hi} \neq -C'_{\#}\Lambda_{\min}^{-1}$, so (83), (84) show that $\tilde{\omega}_{hi} = C_{\#}\Lambda_{\min}^{-1}$, and there is some $k \in \mathcal{K}$ satisfying

$$\frac{1}{\pi}\phi(E_{hi}) - \frac{1}{2} - C'_{\#}\Lambda_{\min}^{-1} \leq k \leq \frac{1}{\pi}\phi(E_{hi}) - \frac{1}{2} + C_{\#}\Lambda_{\min}^{-1} \ .$$

This and (92) yield $|k^{--} - k| \leq (C_{\#} + C'_{\#})\Lambda_{\min}^{-1} << 1$. Since k and k^{--} are both integers, it follows that $k^{--} = k$. Hence $k^{--} \in \mathcal{K}$. Thus, in all three cases (90), (91), (92) we have $k^{--} \in \mathcal{K}$. So we have proven

$$\mathbb{Z} \cap [\frac{1}{\pi}\phi(E_{\ell o}) - \frac{1}{2} + \tilde{\omega}_{\ell o}, \frac{1}{\pi}\phi(E_{hi}) - \frac{1}{2} + \tilde{\omega}_{hi}] \subset \mathcal{K} \ .$$

This together with (89) completes the proof of (86).

Now define $\omega_{\ell o}$, ω_{hi} by setting

$$\omega_{\ell o} = \tilde{\omega}_{\ell o}, \ \omega_{hi} = 0 \text{ if } \min_{k \in \mathbb{Z}} |\frac{1}{\pi}\phi(E_{hi}) - \frac{1}{2} - k| \geq 2C'_{\#}\Lambda_{\min}^{-1} \ ,$$

$$\omega_{hi} = \tilde{\omega}_{hi} \text{ otherwise} \ .$$

Evidently,

$$(94)\quad \mathbb{Z}\cap\ [\frac{1}{\pi}\phi(E_{\ell o})-\frac{1}{2}+\omega_{\ell o},\frac{1}{\pi}\phi(E_{hi})-\frac{1}{2}+\omega_{hi}]$$
$$=\mathbb{Z}\cap\ [\frac{1}{\pi}\phi(E_{\ell o})-\frac{1}{2}+\tilde{\omega}_{\ell o},\frac{1}{\pi}\phi(E_{hi})-\frac{1}{2}+\tilde{\omega}_{hi}]=\mathcal{K}\quad \text{by (86)}\ .$$

Also,

$$(95)\qquad |\omega_{\ell o}|,|\omega_{hi}|\le C'_{\#}\Lambda_{\min}^{-1}\ ,\ \text{by (85); and}$$

$$(96)\qquad \omega_{hi}=0\quad \text{if}\quad \min_{k\in\mathbb{Z}}|\frac{1}{\pi}\phi(E_{hi})-\frac{1}{2}-k|\ge 2C'_{\#}\Lambda_{\min}^{-1}\ .$$

¿From (94) and the hypothesis $\phi(E_{hi})-\phi(E_{\ell o})\ge 100$, we see that $\mathcal{K}$ has the form

$$(97)\qquad \mathcal{K}=\{k\in\mathbb{Z}\mid k_{\ell o}\le k\le k_{hi}\}\ ,\quad \text{with}$$

$$(98)\qquad k_{\ell o}<k_{hi}\ .$$

Since we saw already that the eigenvalues of H belonging to $[E_{\ell o},E_{hi}]$ are precisely the $(E_k)_{k\in\mathcal{K}}$, we now have from (97) that these eigenvalues are $(E_k)_{k_{\ell o}\le k\le k_{hi}}$. The conclusions of the present Corollary are now obvious. In fact, (52) is the same as (98); (53) follows from (77) and (97); (54) follows from (94) and (97); and (55), (56) with an enlarged constant $C_{\#}$ are contained in (95), (96). The proof of the Corollary is complete.

■

APPROXIMATING SUMS BY INTEGRALS

In this section we study the error that arises when we approximate a Riemann sum by an integral. Our formulas involve the following elementary functions.

$$\chi_+(x) = k - x - \frac{1}{2} \quad \text{for} \quad k \quad \text{the smallest integer } \geq x\ ;$$
$$\chi_-(x) = x - k - \frac{1}{2} \quad \text{for} \quad k \quad \text{the largest integer } \leq x\ ;$$
$$\tilde{\chi}(x) = |x - k - \frac{1}{2}|^2 - \frac{1}{12} \quad \text{for an integer} \quad k \quad \text{that minimizes } |x - k - \frac{1}{2}|\ .$$

Lemma 1. *Assume* $|(\frac{d}{dx})^m f(x)| \leq C\, R^{-m}$ *for* $0 \leq m \leq 4$ *with* $R \geq 1$*. If* $a \leq b$*, then*

$$\sum_{k \in \mathbb{Z} \cap [a,b]} f(k) = \int_a^b f(t)\, dt - f(b)\chi_-(b) - f(a)\chi_+(a) + \frac{1}{2} f'(b)\tilde{\chi}(b)$$
$$- \frac{1}{2} f'(a)\tilde{\chi}(a) + \textit{Error},$$

with $|\text{Error}| \leq C' R^{-2} + C' R^{-4} |b - a|$ *and* C' *determined entirely by* C*.*

Proof. First assume $\mathbb{Z} \cap [a,b] \neq \emptyset$. Thus for suitable $m_0, m_1 \in \mathbb{Z}$ we have $m_0 - 1 < a \leq m_0 \leq m_1 \leq b < m_1 + 1$. Then $\chi_+(a) = m_0 - a - 1/2$, $\chi_-(b) = b - m_1 - 1/2$, $\tilde{\chi}(a) = |a - m_0 + \frac{1}{2}|^2 - \frac{1}{12}$, $\tilde{\chi}(b) = |b - m_1 - \frac{1}{2}|^2 - \frac{1}{12}$. For $m_0 \leq k \leq m_1$ we have the following:

$$\int_{k-1/2}^{k+1/2} f(t)\, dt = f(k) + \frac{1}{24} f''(k) + \text{Error}_k, \quad |\text{Error}_k| \leq C' R^{-4}\,,$$
$$\int_{k-1/2}^{k+1/2} f''(t)\, dt = f''(k) + \text{Error}''_k, \quad |\text{Error}''_k| \leq C' R^{-4}\,.$$

These formulas follow by Taylor–expanding f to order 3 with remainder, and f'' to order 1 with remainder. Summing over k from m_0 to m_1, we get the following equations:

$$\int\limits_{m_0-1/2}^{m_1+1/2} f(t)dt = \sum_{m_0\le k\le m_1} f(k) + \frac{1}{24}\sum_{m_0\le k\le m_1} f''(k) + \text{ Error},$$

$$f'(m_1+\frac{1}{2}) - f'(m_0-1/2) = \int\limits_{m_0-1/2}^{m_1+1/2} f''(t)dt = \sum_{m_0\le k\le m_1} f''(k) + \text{ Error}''$$

with $|\text{Error}|$, $|\text{Error}''| \le C'R^{-4}(b-a+1)$. Therefore

$$\sum_{m_0\le k\le m_1} f(k) = \int\limits_{m_0-1/2}^{m_1+1/2} f(t)dt - \frac{1}{24}\Big(f'(m_1+\frac{1}{2}) - f'(m_0-\frac{1}{2})\Big) + \text{ Error},$$

with $|\text{Error}| \le C'R^{-4}(b-a+1)$.

We simplify slightly, by noting that $|f'(m_1+1/2)-f'(b)|$, $|f'(m_0-1/2)-f'(a)|$ are dominated by $C'R^{-2}$. Thus,

$$\text{(1)} \qquad \sum_{m_0\le k\le m_1} f(k) = \int\limits_{m_0-1/2}^{m_1+1/2} f(t)dt - \frac{1}{24}(f'(b)-f'(a)) + \text{ Error},$$

$$|\text{Error}| \le C'R^{-2} + C'R^{-4}(b-a)\ .$$

Next, Taylor–expanding $f(t)$ about $t=b$ to order 1 with remainder, we get

$$\int\limits_{m_1+1/2}^{b} f(t)dt = f(b)\cdot(b-m_1-1/2) - \frac{1}{2}f'(b)\cdot(b-m_1-1/2)^2 + \text{ Error}\ ,$$

$$|\text{Error}| \le C'R^{-2}\ .$$

Similarly,

$$\int\limits_{a}^{m_0-1/2} f(t)dt = f(a)\cdot(m_0-a-1/2) + \frac{1}{2}f'(a)\cdot(m_0-a-1/2)^2 + \text{ Error},$$

$$|\text{Error}| \le C'R^{-2}\ .$$

Combining these equations with (1), we get

$$\sum_{m_0\le k\le m_1} f(k) = \int\limits_{a}^{b} f(t)dt - f(b)\cdot(b-m_1-1/2) - f(a)\cdot(m_0-1/2-a) + \frac{1}{2}f'(b)$$

$$\cdot\{(b-m_1-1/2)^2 - \frac{1}{12}\} - \frac{1}{2}f'(a)\cdot\{(m_0-a-1/2)^2 - \frac{1}{12}\} + \text{ Error}\ ,$$

$$|\text{Error}| \le C'R^{-2} + C'R^{-4}(b-a)\ .$$

That is the conclusion of Lemma 1. It remains to check the case $[a,b]\cap\mathbb{Z}=\emptyset$. Thus $m<a\le b<m+1$ for an integer m. We then have $\chi_+(a)=m+\frac{1}{2}-a$, $\chi_-(b)=b-m-\frac{1}{2}$, $\tilde{\chi}(a)=(a-m-\frac{1}{2})^2-\frac{1}{12}$, $\tilde{\chi}(b)=(b-m-\frac{1}{2})^2-\frac{1}{12}$, so our lemma asserts that

$$(2)\quad 0=\int_a^b f(t)dt-f(b)(b-m-\frac{1}{2})-f(a)(m+\frac{1}{2}-a)$$
$$+\frac{1}{2}f'(b)\cdot\{\big(b-m-\frac{1}{2}\big)^2-\frac{1}{12}\}-\frac{1}{2}f'(a)\cdot\{(a-m-\frac{1}{2})^2-\frac{1}{12}\}+\text{ Error },$$
$$|\text{Error}|\le C'R^{-2}\ .$$

Taylor–expanding $f(t)$ about $t=a$ yields

$$\int_a^{m+\frac{1}{2}} f(t)\,dt=f(a)\cdot(m+\frac{1}{2}-a)+\frac{1}{2}f'(a)\cdot(m+\frac{1}{2}-a)^2+\text{ Error },$$
$$|\text{Error}|\le C'R^{-2}\ .$$

Taylor–expanding $f(t)$ about $t=b$ yields

$$\int_{m+1/2}^{b} f(t)\,dt=f(b)\cdot(b-m-\frac{1}{2})-\frac{1}{2}f'(b)\cdot(m+\frac{1}{2}-b)^2+\text{ Error },$$
$$|\text{Error}|\le C'R^{-2}\,.$$

Adding these and noting that $|f'(b)-f'(a)|\le C'R^{-2}$, we obtain (2). So the lemma holds also when $\mathbb{Z}\cap[a,b]=\emptyset$. ■

Lemma 2. *Assume* $|(\frac{d}{dx})^m f(x)|\le C_m R^{-m}(m\ge 0)$ *with* $R\ge 1$. *Assume also that* f *is supported in an interval of length* CR. *Then*

$$\sum_{k\in\mathbb{Z}} f(k)=\int_{-\infty}^{\infty} f(t)\,dt+\text{ Error},\quad \textit{and}\quad |\text{Error}|\le C_N' R^{-N}\quad \textit{with}\quad N\quad \textit{as large}$$

as we please. The constant C_N' *depends only on* N, *on* C, *and on finitely many of the* C_m.

Proof. We may assume that supp f is contained in an interval of length CR centered at 0. Since $x \mapsto f(xR)$ is a Schwartz function, the Fourier transform $\hat{f}(\xi)$ satisfies the estimate $|\hat{f}(\xi)| \le C'_N R(1+R|\xi|)^{-N}$. Therefore,

$$\Big| \sum_{k\in\mathbb{Z}\setminus\{0\}} \hat{f}(k)\Big| \le C'_N R \sum_{k\in\mathbb{Z}\setminus\{0\}} (R|k|)^{-N} \le C'_N R^{1-N} ,$$

so the lemma follows at once from the Poisson summation formula. ■

The interested reader can easily supply a more direct proof of Lemma 2. The main result of this section is as follows.

Lemma on Riemann Sums. *Let $f(t)$, $\sigma(t)$, $\tau(t)$ be defined on a non–empty interval $[a,b]$. Suppose $\sigma(t) > 0$, $\tau(t) \ge 1$ in $[a,b]$; and assume that whenever $t_1, t_2 \in [a,b]$ with $|t_1 - t_2| < c\tau(t_1)$, we have $c < \frac{\tau(t_2)}{\tau(t_1)} < C$ and $c < \frac{\sigma(t_2)}{\sigma(t_1)} < C$. Finally assume $|(\frac{d}{dt})^m f(t)| \le C_m \sigma(t)\tau^{-m}(t)$ for $t \in [a,b]$. Then $\sum\limits_{k\in\mathbb{Z}\cap[a,b]} f(k) = \int\limits_a^b f(t)dt - f(b)\chi_-(b) - f(a)\chi_+(a) + \frac{1}{2}f'(b)\tilde{\chi}(b) - \frac{1}{2}f'(a)\tilde{\chi}(a) +$ Error with $|\text{Error}| \le C'\sigma(a)\tau^{-2}(a) + C'\sigma(b)\tau^{-2}(b) + C'_N \int_a^b \sigma(t)\tau^{-N}(t)dt$. Here, C' depends only on c, C, C_m; and C'_N depends only on c, C, C_m, N. If $f(t) = 0$ to infinite order at $t = a$, then we have the sharper estimate $|\text{Error}| \le C'\sigma(b)\tau^{-2}(b) + C'_N \int_a^b \sigma(t)\tau^{-N}(t)dt$, with C', C'_N as before. Similarly, if $f(t) = 0$ to infinite order at $t = b$, then $|\text{Error}| \le C'\sigma(a)\tau^{-2}(a) + C'_N \int_a^b \sigma(t)\tau^{-N}(t)dt$. If $f(t) = 0$ to infinite order at both $t = a$ and $t = b$, then $|\text{Error}| \le C'_N \int\limits_a^b \sigma(t)\tau^{-N}(t)dt$.*

Proof. We distinguish several cases.

Case 1. $b - a \ge c\tau(a)$.

Then we can find a partition of unity $\theta_0(x)\ldots\theta_M(x)$ with the following properties

- ($\cdot$) Each $\theta_\nu(x)$ is supported in an interval I_ν of length $\sim \tau(x_\nu)$, $x_\nu \in I_\nu \cap [a,b]$.

(·) If $\nu = 0$ then $x_\nu = a$; if $\nu = M$ then $x_\nu = b$; if $0 < \nu < M$ then $I_\nu \subset [a,b]$.

(·) If $\nu = 0$, then $\theta_\nu = 1$ in a neighborhood of a; and if $\nu = M$, then $\theta_\nu = 1$ in a neighborhood of b.

(·) Each $\theta_\nu(x)$ satisfies $|(\frac{d}{dx})^m \theta_\nu(x)| \le C'_m \tau^{-m}(x_\nu)$.

(·) $\theta_0(x) + \theta_1(x) + \ldots + \theta_M(x) = 1$ on $[a,b]$.

(·) No point belongs to more than C' of the intervals I_ν.

Using the θ_ν we write $f = \sum\limits_{\nu=0}^{M} \theta_\nu f \equiv \sum\limits_{\nu=0}^{M} f_\nu$.

For $0 < \nu < M$, we have $|(\frac{d}{dx})^m f_\nu| \le C''_m \sigma(x_\nu)\tau^{-m}(x_\nu)$ and diam supp $f_\nu \le C\tau(x_\nu), \mathrm{supp} f_\nu \subset [a,b]$. For $\nu = 0$ or M, we can extend f_ν to a C_0^∞ function satisfying again $|(\frac{d}{dx})^m f_\nu| \le C''_m \sigma(x_\nu)\tau^{-m}(x_\nu)$ and diam supp $f_\nu \le C\tau(x_\nu)$, but we don't have supp $f_\nu \subset [a,b]$. However, if $\nu = 0$ and $f(t)$ vanishes to infinite order at a, then we can take $f_\nu \equiv 0$ outside $[a,b]$ and thus supp $f_\nu \subset [a,b]$. Similarly for b. Now we compute $\sum\limits_{k\in\mathbb{Z}\cap[a,b]} f(k) = \sum\limits_{\nu=0}^{M}\big(\sum\limits_{k\in\mathbb{Z}\cap[a,b]} f_\nu(k)\big)$ by applying lemmas 1 and 2. If supp $f_\nu \subset [a,b]$, then $\sum\limits_{k\in\mathbb{Z}\cap[a,b]} f_\nu(k) = \sum\limits_{k\in\mathbb{Z}} f_\nu(k)$, so lemma 2 applies. If supp $f_\nu \not\subset [a,b]$, then we must apply lemma 1. Since $\sum\limits_{\nu=0}^{M} \sigma(x_\nu)\tau^{-(N-1)}(x_\nu) \sim \sum\limits_{\nu=0}^{M} \int\limits_{I_\nu} \sigma(t)\tau^{-N}(t)dt \sim \int\limits_a^b \frac{\sigma(t)}{\tau^N(t)}\,dt$, $f_0(a) = f(a)$, $f'_0(a) = f'(a)$, $f_M(b) = f(b)$, $f'_M(b) = f'(b)$, the result of our computation is the conclusion of the present lemma. This settles the case $b - a \ge c\tau(a)$.

Case 2. $b - a < c\tau(a)$, and $f(t)$ vanishes to only finite order at a or b. Then the conclusion of the present lemma follows from lemma 1.

Case 3. $c(\tau(a))^{1/2} < b - a < c\tau(a)$, and $f(t)$ vanishes to infinite order at a and b. Replacing $\tau(t)$ by $\tilde{\tau} = \tau^{1/2}(t)$, we find that σ, $\tilde{\tau}$, f satisfy the hypotheses of the present lemma, and we find ourselves in Case 1. So our desired conclusion follows by the reasoning of Case 1.

Case 4. $c \le b - a \le c(\tau(a))^{1/2}$ and $f(t)$ vanishes to infinite order at a and b. Then in $[a,b]$ we have $|f(t)| \le \max_{[a,b]} |(\frac{d}{dt'})^{2N} f(t')| \cdot (b-a)^{2N}$ by Taylor's theorem, hence $|f(t)| \le C_N \frac{\sigma(t)}{\tau^{2N}(t)}(b-a)^{2N} \le C_N \frac{\sigma(t)}{\tau^N(t)} \le C'_N \frac{\sigma(a)}{\tau^N(a)}$ for $t \in [a,b]$. Consequently,

$$\Big| \sum_{k\in\mathbb{Z}\cap[a,b]} f(k)\Big| \le C'_N \frac{\sigma(a)}{\tau^N(a)} \cdot (b-a+1)$$

$$\le C''_N \frac{\sigma(a)}{\tau^N(a)} \cdot (b-a) \le C''_N \le C''_N \int_a^b \frac{\sigma(t)}{\tau^N(t)}\, dt$$

and

$$\Big| \int_a^b f(t)dt \Big| \le C'_N \frac{\sigma(a)}{\tau^N(a)} \cdot (b-a) \le C''_N \int_a^b \frac{\sigma(t)}{\tau^N(t)}\, dt\ .$$

From these estimates, the conclusion of the lemma is obvious.

Case 5. $b - a \le c$ and $f(t)$ vanishes to infinite order at a and b. Then Taylor's theorem tells us that $|f(\bar{t})| \le \max_{[a,b]} |(\frac{d}{dx})^{2N} f(x)| \cdot (b-a)^{2N} \le C'_N \frac{\sigma(a)}{\tau^{2N}(a)}(b-a)^{2N} \le C'_N \frac{\sigma(a)}{\tau^N(a)}(b-a) \le C''_N \int_a^b \frac{\sigma(t)}{\tau^N(t)}\, dt$ for $\bar{t} \in [a,b]$. Therefore

$$\Big| \int_a^b f(t)\ dt \Big| \le C''_N \int_a^b \frac{\sigma(t)}{\tau^N(t)}\, dt \qquad \text{since } b-a<1\ ,$$

and $|\sum_{k\in\mathbb{Z}\cap[a,b]} f(k)| \le C''_N \int_a^b \frac{\sigma(t)}{\tau^N(t)}\, dt$ since the sum has at most one term.

¿From these estimates our desired conclusion is obvious. ■

Trivial Remarks. There are obvious variants of the Lemma on Riemann sums with the interval $[a,b]$ replaced by an open or half–open interval. These results follow from the case of a closed interval by a simple limiting argument. For instance, applying the above lemma to $[a+\delta, b-\delta]$ and letting $\delta \to 0+$, we obtain

$$\sum_{k\in\mathbb{Z}\cap(a,b)} f(k) = \int_a^b f(t)dt - f(b)\chi_-(b-) - f(a)\chi_+(a+) + \frac{1}{2} f'(b)\tilde{\chi}(b)$$

$$- \frac{1}{2} f'(a)\tilde{\chi}(a) + \text{ Error }\ ,$$

with $\chi_+(a+) \equiv \lim_{\delta\to 0+} \chi_+(a+\delta)$, $\chi_-(b-) \equiv \lim_{\delta\to 0+} \chi_-(b-\delta)$, and $|\text{Error}|$ estimated as before. Note that $\tilde{\chi}$ is a continuous function, so we needn't write $\tilde{\chi}(b-)$ or $\tilde{\chi}(a+)$.

Note also that $\frac{1}{2}f'(b)\tilde{\chi}(b)$ is dominated by $\frac{\sigma(b)}{\tau(b)}$, and similarly for $\frac{1}{2}f'(a)\tilde{\chi}(a)$. Hence in the lemma on Riemann Sums, we may omit the terms $\frac{1}{2}f'(b)\tilde{\chi}(b)$ and $\frac{1}{2}f'(a)\tilde{\chi}(a)$, provided we weaken our estimates on the error to

$$|\text{Error}| \leq C'\frac{\sigma(a)}{\tau(a)} + C'\frac{\sigma(b)}{\tau(b)} + C'_N \int_a^b \frac{\sigma(t)}{\tau^N(t)}\, dt \ .$$

Again we may omit $\frac{\sigma(a)}{\tau(a)}$ here if f vanishes to infinite order at a, and similarly for b.

THE MICROLOCALIZED DENSITY IN THE AIREY REGION I

In this section we adopt the notation and hypotheses of the WKB Theorems, with $E_\infty = 0$. Our goal is to understand the microlocalized density $\rho(x, g)$ when $g(x, E)$ is supported in the region where eigenfunctions are closely approximated in terms of the Airey function. More precisely, we make the following assumptions on $g(x, E)$.

ASSUMPTION 1. supp $g(x, E) \subset \{|E - E_0| < \hat{c}\, S_{\min},\ |x - x_{\text{left}}(E)| < \delta x\}$.

ASSUMPTION 2. $|\partial_E^\beta g(x, E)| \leq \hat{C}_\beta (\delta E)^{-\beta}$ for all x, E.

ASSUMPTION 3. δx and δE are positive numbers with $\lambda_{\text{left}}^{-2/3+\varepsilon} B_{\text{left}} < \delta x < \frac{1}{20} \lambda_{\text{left}}^{-\varepsilon} B_{\text{left}}$ and $\delta E = \min\{S_{\min}, \frac{S_{\text{left}}}{B_{\text{left}}}(\delta x)\}$.

ASSUMPTION 4. The constant $\hat{c}$ in *ASSUMPTION* 1 is bounded above by a certain small positive number, determined by the constants ε, K, N, c, C, c_1, c_2, C_α in the hypotheses of the WKB Theorems.

ASSUMPTION 5. Λ is bounded below by a certain large positive number, determined by $\hat{c}$ and the $\hat{C}_\beta$ in *ASSUMPTIONS* 1 and 2, and by ε, K, N, c, C, c_1, c_2, C_α in the hypotheses of the WKB Theorems.

Here, *ASSUMPTION* 5 strengthens the hypothesis on the hugeness of Λ in the WKB Theorems. We denote by $c_\#$, $C_\#$, $C_\#^\alpha$ etc. constants that depend only on ε, K, N, c, C, c_1, c_2, C_α in the hypotheses of the WKB Theorems. Constants denoted by c_*, C_*, C_*^α etc. are allowed to depend on ε, K, N, c, C, c_1, c_2, C_α and also on $\hat{c}$, $\hat{C}_\beta$ in *ASSUMPTIONS* 1 and 2.

The microlocalized density $\rho(x, g)$ is defined as a weighted sum of squares of eigenfunctions. The WKB Eigenfunction Theorem lets us approximate the eigenfunctions by explicit formulas involving the Airey function. This is possible by virtue of our assumption on the support of $g(x, E)$. Hence $\rho(x, g)$ is given approximately as a weighted sum of squares of Airey functions. The Lemma on Riemann sums then lets us approximate this sum by an integral involving the square of the Airey function. In this section we carry out the details, and show that $\rho(x, g)$ is closely approximated by an integral involving the square of the Airey function. In the next section, we will relate that integral to the semiclassical density $\rho_{sc}(x, g)$.

We begin by applying the WKB Eigenvalue Theorem to understand the eigenvalues in the interval

$$(1) \qquad [E_{\ell o}, E_{hi}] = (-\infty, 0] \cap \{E \colon |E - E_0| \le c_\#^1 S_{\min}\}.$$

These eigenvalues may be written as $E_{k_{\ell o}}, E_{k_{\ell o}+1} \dots, E_{k_{hi}}$, with the following properties.

$$(2) \qquad \text{For } k_{\ell 0} \le k \le k_{hi} \text{ we have } |\phi(E_k) + \frac{1}{48}\psi(E_k) - \pi(k + 1/2)| \le C_\# \Lambda^{-2} \ .$$

(3) The integers $k_{\ell o} \le k \le k_{hi}$ are precisely those integers k that lie in the interval

$$[a, b] = [\frac{1}{\pi}\phi(E_{\ell o}) + \frac{1}{48\pi}\psi(E_{\ell o}) - \frac{1}{2} + \omega_{\ell o} \ , \frac{1}{\pi}\phi(E_{hi}) + \frac{1}{48\pi}\psi(E_{hi}) - \frac{1}{2} + \omega_{hi}] \text{ with } |\omega_{\ell o}|, |\omega_{hi}| \le C_\# \Lambda^{-2} \ .$$

$$(4) \qquad \text{If } \min_{k \in \mathbb{Z}} |\phi(E_{hi}) + \frac{1}{48}\psi(E_{hi}) - \pi(k + 1/2)| \ge C_\# \Lambda^{-2} \ ,$$

then we can take $\omega_{hi} = 0$.

These properties are the conclusions of the reformulated WKB Eigenvalue Theorem. Next we recall the basic estimates for the phase $\phi(E)$, $\psi(E)$, from the section on the WKB Theorems. They are as follows

$$(5) \qquad \left|\left(\frac{d}{dE}\right)^m \phi(E)\right| \le C_\#^m \int_{x_{\text{left}}}^{x_{\text{rt}}} S^{\frac{1}{2}-m}(x)\, dx \quad \text{and}$$

$$\left|\left(\frac{d}{dE}\right)^m \psi(E)\right| \le C_\#^m \int_{x_{\text{left}}}^{x_{\text{rt}}} S^{-\frac{1}{2}-m}(x) B^{-2}(x)\, dx \quad (m \ge 0)$$

$$\text{for} \quad |E - E_0| \le 33\ c_\#^1 S_{\min} \ .$$

Also,

$$\frac{d\phi}{dE} \geq c_{\#} \int_{x_{\text{left}}}^{x_{\text{rt}}} S^{-1/2}(x)dx \quad \text{for} \quad |E - E_0| \leq 33\ c_{\#}^1 S_{\min}\ . \tag{6}$$

(See the remark just after the proof of the Reformulated Eigenvalue Theorem.)

Since $S(x) \geq S_{\min}$ and $S(x)B^2(x) = \lambda^2(x) \geq c_{\#}\Lambda^2$ in $(x_{\text{left}}, x_{\text{rt}})$, it follows that

$$\Big|\big(\frac{d}{dE}\big)^m \phi\Big| \leq C_{\#}^m\ \Gamma S_{\min}^{-m} \ \ (m \geq 1) \quad \text{for} \quad |E - E_0| \leq 33\ c_{\#}^1 S_{\min} \tag{7}$$

$$\Big|\big(\frac{d}{dE}\big)^m \psi\Big| \leq C_{\#}^m \Lambda^{-2}\Gamma S_{\min}^{-m} \ \ (m \geq 1) \quad \text{for} \quad |E - E_0| \leq 33\ c_{\#}^1 S_{\min} \tag{8}$$

$$\frac{d\phi}{dE} \geq c_{\#}\Gamma S_{\min}^{-1} \quad \text{for} \quad |E - E_0| \leq 33\ c_{\#}^1 S_{\min}\ , \quad \text{with} \tag{9}$$

$$\Gamma = S_{\min} \int_{x_{\text{left}}}^{x_{\text{rt}}} S^{-1/2}(x)dx\ . \tag{10}$$

We note two useful lower bounds for Γ. Restricting the region of integration to $[x_{\text{left}}, x_{\text{left}} + c_{\#}B_{\text{left}}]$ in (10), we get

$$\Gamma S_{\min}^{-1} \geq c_{\#} S_{\text{left}}^{-1/2} B_{\text{left}}\ . \tag{11}$$

Also, we can find $x_{\min} \in (x_{\text{left}}, x_{\text{rt}})$ with $S(x_{\min}) \sim S_{\min}$ and $\text{dist}(x_{\min}, x_{\text{left}})$, $\text{dist}(x_{\min}, x_{\text{rt}}) > c_{\#}B(x_{\min})$. Restricting the region of integration in (10) to $[x_{\min} - c_{\#}B(x_{\min}), x_{\min} + c_{\#}B(x_{\min})]$, we get

(12)

$$\Gamma \geq c_{\#}S_{\min} \cdot S^{-1/2}(x_{\min})B(x_{\min}) \geq c'_{\#}S^{1/2}(x_{\min})B(x_{\min}) = c'_{\#}\lambda(x_{\min}) \geq c_{\#}\Lambda\ .$$

We use (7), (8), (9) to solve the equation $\phi(E) + \frac{1}{48}\psi(E) = \pi(t + 1/2)$. In fact, those estimates show that

$$\Gamma^{-1}\Big[\Big\{\frac{1}{\pi}\phi(E) + \frac{1}{48\pi}\psi(E) - \frac{1}{2}\Big\} - \Big\{\frac{1}{\pi}\phi(E_0) + \frac{1}{48\pi}\psi(E_0) - \frac{1}{2}\Big\}\Big] = f_1\big(\frac{E - E_0}{S_{\min}}\big)\ ,$$

with a–priori estimates on the C^∞ seminorms of f_1 and an a–priori lower bound for $(f_1)'$. Here f_1 is defined on $[-33\ c^1_\#, +33\ c^1_\#]$. Hence on the image $f_1([-33c^1_\#, +33c^1_\#])$, the inverse function f_1^{-1} is well–defined, and we have a–priori bounds on its C^∞ seminorms. In other words, we have the following results.

Let $\mathcal{J} = \{\frac{1}{\pi}\phi(E) + \frac{1}{48\pi}\psi(E) - \frac{1}{2} \mid |E - E_0| \leq 33\ c^1_\# S_{\min}\}$. For $t \in \mathcal{J}$, the equation $\frac{1}{\pi}\phi(E) + \frac{1}{48\pi}\psi(E) - \frac{1}{2} = t$ has a unique solution $E = E(t)$ with $|E - E_0| \leq 33\ c^1_\# S_{\min}$, and we have

$$\left|\left(\frac{d}{dt}\right)^m E(t)\right| \leq C^m_\# S_{\min}\Gamma^{-m} \quad (m \geq 1) \quad \text{for} \quad t \in \mathcal{J} \tag{13}$$

$$\frac{d}{dt}E(t) \geq c_\# S_{\min}\Gamma^{-1} \quad \text{for} \quad t \in \mathcal{J}\ . \tag{14}$$

¿From (8), (9) we see that $\mathcal{J}$ is an interval and that

$$\begin{aligned}
\min\ \mathcal{J} &= \left\{\frac{1}{\pi}\phi(E_0 - 33\ c^1_\# S_{\min}) + \frac{1}{48\pi}\psi(E_0 - 33\ c^1_\# S_{\min}) - \frac{1}{2}\right\} \\
&\leq \left\{\frac{1}{\pi}\phi(E_0 - c^1_\# S_{\min}) + \frac{1}{48\pi}\psi(E_0 - c^1_\# S_{\min}) - \frac{1}{2}\right\} - c_\#\Gamma \\
&\qquad = a - \omega_{\ell o} - c_\#\Gamma\ .
\end{aligned}$$

Here we use the definitions of a and $E_{\ell o}$. By (12) and $|\omega_{\ell o}| \leq C_\#\Lambda^{-2}$, we obtain

$$\min \mathcal{J} \leq a - c'_\#\Gamma\ . \tag{15}$$

Similarly, (9) implies

$$\begin{aligned}
\max \mathcal{J} &= \left\{\frac{1}{\pi}\phi(E_0 + 33\ c^1_\# S_{\min}) + \frac{1}{48\pi}\psi(E_0 + 33\ c^1_\# S_{\min}) - \frac{1}{2}\right\} \\
&\geq \left\{\frac{1}{\pi}\phi(E_{hi}) + \frac{1}{48\pi}\psi(E_{hi}) - \frac{1}{2}\right\} + c_\#\Gamma = b - \omega_{hi} + c_\#\Gamma\ ,
\end{aligned}$$

so that

$$\max \mathcal{J} \geq b + c'_\#\Gamma\ . \tag{16}$$

This tells us in particular that

$$[a,b] \subset \mathcal{J} \ . \tag{17}$$

For $k_{\ell o} \leq k \leq k_{hi}$ we have $k \in [a,b]$ by (3), so $E(k)$ is well–defined and lies in $\{|E_0 - E(k)| \leq 33\ c^1_{\#} S_{\min}\}$. Also $|E_k - E_0| \leq c^1_{\#} S_{\min}$ by definition of the E_k. Hence

$$\begin{aligned} |E_k - E(k)| &\leq \max_{t\in\mathcal{J}} \left|\frac{dE(t)}{dt}\right| \cdot \Big|\{\frac{1}{\pi}\phi(E_k) + \frac{1}{48\pi}\psi(E_k) - \frac{1}{2}\} \\ &\qquad\qquad - \{\frac{1}{\pi}\phi(E(k)) + \frac{1}{48\pi}\psi(E(k)) - \frac{1}{2}\}\Big| \\ &= \max_{t\in\mathcal{J}} \left|\frac{dE(t)}{dt}\right| \cdot \left|\frac{1}{\pi}\phi(E_k) + \frac{1}{48\pi}\psi(E_k) - \frac{1}{2} - k\right| \leq C_{\#} S_{\min}\Gamma^{-1}\Lambda^{-2} \ , \end{aligned} \tag{18}$$

by estimates (2) and (13).

Similarly, define $E_{\max} = E(b)$ and $E_{\min} = E(a)$, and we have

$$\begin{aligned} |E(b) - E_{hi}| &\leq \max_{t\in\mathcal{J}} \left|\frac{dE(t)}{dt}\right| \cdot \left|b - \{\frac{1}{\pi}\phi(E_{hi}) + \frac{1}{48\pi}\psi(E_{hi}) - \frac{1}{2}\}\right| \\ &\leq C_{\#} S_{\min}\Gamma^{-1}|\omega_{hi}| \leq C_{\#} S_{\min}\Gamma^{-1}\Lambda^{-2} \ . \end{aligned} \tag{19}$$

The same reasoning gives also

$$|E(a) - E_{\ell o}| \leq C_{\#} S_{\min}\Gamma^{-1}\Lambda^{-2} \ . \tag{20}$$

We know that $g(x,E) = 0$ for $|E - E_0| > \hat{c} S_{\min}$. In particular, if $E \leq E_{\min} = E(a) \leq E_{\ell o} + C_{\#} S_{\min}\Gamma^{-1}\Lambda^{-2}$ (by (20)) $= (E_0 - c^1_{\#} S_{\min}) + C_{\#} S_{\min}\Gamma^{-1}\Lambda^{-2}$ (by definition of $E_{\ell o}$), then evidently $E \leq E_0 - \hat{c} S_{\min}$ if Λ and Γ are big enough and $\hat{c}$ is small enough. *ASSUMPTIONS* 4 and 5 and estimate (12) show that Λ and Γ are big enough and $\hat{c}$ is small enough, so we get

$$g(x,E) = 0 \quad \text{if} \quad E \leq E_{\min} \ . \tag{21}$$

Similarly, suppose $E \geq E_{\max}$ and $|E_0| \geq 2\hat{c} S_{\min}$. Then $E_{hi} = \min(0, E_0 + c^1_{\#} S_{\min}) \geq E_0 + 2\hat{c}\ S_{\min}$, so

$$\begin{aligned} E \geq E_{\max} = E(b) &\geq E_{hi} - C_{\#} S_{\min}\Gamma^{-1}\Lambda^{-2} \quad \text{((by (19))} \\ &\geq E_0 + 2\hat{c}\ S_{\min} - C_{\#} S_{\min}\Gamma^{-1}\Lambda^{-2} \geq E_0 + \hat{c}\ S_{\min} \quad \text{by } \textit{ASSUMPTION } 5 \ . \end{aligned}$$

Hence we get

$$g(x,E)=0 \quad \text{if} \quad E \geq E_{\max} \quad \text{and} \quad |E_0| \geq 2\hat{c}S_{\min} \ . \tag{22}$$

If $|E_0| < 2\hat{c}\ S_{\min}$, then perhaps $g(x, E_{\max})$ will not vanish.

Next we apply the WKB Eigenfunction Theorem to approximate closely the (normalized) eigenfunction $u_k(x)$ associated with E_k $(k_{\ell o} \leq k \leq k_{hi})$. (See also the remark just after the proof of the Reformulated Eigenvalue Theorem.) We obtain

$$\int_{|x-x_{\text{left}}(E_k)|<\lambda_{\text{left}}^{-\varepsilon}B_{\text{left}}} |u_k(x) - b_{\text{left}}F(x,E_k)|^2\, dx \leq \Lambda^{-N''} \qquad \text{for} \quad k_{\ell o} \leq k \leq k_{hi}\ , \tag{23}$$

with

$$F(x,E) = \lambda_{\text{left}}^{-1/3}\Big(\frac{\partial Y_{\text{left}}(x,E)}{\partial x}\Big)^{-1/2}\ A(\lambda_{\text{left}}^{2/3} Y_{\text{left}}(x,E))\ . \tag{24}$$

The WKB Normalization Theorem gives

$$\Big| b_{\text{left}}^2 \cdot \Big(\frac{1}{2} \int_{E_k - V(x)>0} (E_k - V(x))^{-1/2}\ dx\Big) - 1\Big| \leq \Lambda^{4\varepsilon-2}\ , \quad \text{i.e.}$$

$$|b_{\text{left}}^2 \phi'(E_k) - 1| \leq \Lambda^{4\varepsilon-2}\ . \tag{25}$$

In (23) and (25) we want to replace E_k by $E(k)$. To do this for (25), note that

$$\begin{aligned}
|\phi'(E_k) - \phi'(E(k))| &\leq \max_{|E-E_0|\leq 33\ c_{\#}^1 S_{\min}} \Big|\frac{d^2\phi}{dE^2}\Big| \cdot |E_k - E(k)| \\
&\leq C_\#\Gamma S_{\min}^{-2} \cdot C_\# S_{\min}\Gamma^{-1}\Lambda^{-2} \quad \text{(by (7) and (18))} \\
&= C_\#(\Gamma S_{\min}^{-1}) \cdot \Gamma^{-1}\Lambda^{-2} \leq C'_\#\phi'(E_k)\cdot\Gamma^{-1}\Lambda^{-2} \\
&\qquad \text{(by (9))} \quad \leq C''_\#\phi'(E_k)\cdot\Lambda^{-3} \quad \text{(by (12))}\ .
\end{aligned}$$

This and (25) show that

$$|b_{\text{left}}^2\phi'(E(k)) - 1| \leq C_\#\Lambda^{4\varepsilon-2}\ , \quad \text{as desired}\ . \tag{26}$$

The argument for changing E_k to $E(k)$ in (23) is more involved. First of all, we want an interval $I_k \subset \{|x - x_{\rm left}(E_k)| < \lambda_{\rm left}^{-\varepsilon} B_{\rm left}\}$ that contains the support of $x \mapsto g(x,E)$ for all E between E_k and $E(k)$. We take

$$I_k = \{|x - x_{\rm left}(E_k)| < 2(\delta x)\} \quad \text{and check that it works.} \tag{27}$$

Certainly $I_k \subset \{|x - x_{\rm left}(E_k)| < \lambda_{\rm left}^{-\varepsilon} B_{\rm left}\}$ by *ASSUMPTION* 3. For E between E_k and $E(k)$ we have

$$\begin{aligned}
|x_{\rm left}(E) - x_{\rm left}(E_k)| &\le \frac{C_\# B_{\rm left}}{S_{\rm left}} |E - E_k| \le \frac{C_\# B_{\rm left}}{S_{\rm left}} |E(k) - E_k| \\
&\le \frac{C_\# B_{\rm left}}{S_{\rm left}} \cdot S_{\min} \Gamma^{-1} \Lambda^{-2} \le \frac{C_\# B_{\rm left}}{S_{\rm left}} S_{\rm left}^{1/2} B_{\rm left}^{-1} \Lambda^{-2} \quad \text{(by (11))} \\
&= \frac{C_\# \Lambda^{-2} B_{\rm left}}{S_{\rm left}^{1/2} B_{\rm left}} = \frac{C_\# \Lambda^{-2} B_{\rm left}}{\lambda_{\rm left}} < \lambda_{\rm left}^{-\frac{2}{3}+\varepsilon} B_{\rm left} \le (\delta x) \quad \text{by } \textit{ASSUMPTION } 3\ .
\end{aligned} \tag{28}$$

For fixed E between E_k and $E(k)$, $x \mapsto g(x,E)$ is supported in $\{|x - x_{\rm left}(E)| < (\delta x)\} \subset I_k$ by *ASSUMPTION* 1 and the previous chain of inequalities. So I_k does what we said it does.

Next for $x \in I_k$ and E between E_k and $E(k)$ we estimate

$$\begin{aligned}
\left|\frac{\partial F(x,E)}{\partial E}\right| &\le C_\# \lambda_{\rm left}^{-1/3} \left|\frac{\partial Y_{\rm left}(x,E)}{\partial x}\right|^{-3/2} \left|\frac{\partial^2 Y_{\rm left}(x,E)}{\partial x\, \partial E}\right| \, |A(\lambda_{\rm left}^{2/3} Y_{\rm left}(x,E))| \\
&\quad + C_\# \lambda_{\rm left}^{-1/3} \left|\frac{\partial Y_{\rm left}(x,E)}{\partial x}\right|^{-1/2} \lambda_{\rm left}^{2/3} \left|\frac{\partial Y_{\rm left}(x,E)}{\partial E}\right| \, |A'(\lambda_{\rm left}^{2/3} Y_{\rm left}(x,E))|\ .
\end{aligned}$$

We know that $|\frac{\partial Y_{\rm left}}{\partial x}| \sim B_{\rm left}^{-1}$, $|\frac{\partial Y_{\rm left}}{\partial E}| \sim S_{\rm left}^{-1}$, $|\frac{\partial^2 Y_{\rm left}}{\partial x\, \partial E}| \le C_\# B_{\rm left}^{-1} S_{\rm left}^{-1}$, by the properties of $Y_{\rm left}$ enumerated in the WKB Eigenfunction Theorem. Also, $|A(\xi)| \le C(1 + |\xi|)^{-1/4}$ and $|A'(\xi)| \le C(1+|\xi|)^{+1/4}$ for a universal constant C. Hence,

$$\begin{aligned}
\left|\frac{\partial F(x,E)}{\partial E}\right| &\le C_\# \lambda_{\rm left}^{-1/3} B_{\rm left}^{1/2} S_{\rm left}^{-1} \, (1 + \lambda_{\rm left}^{2/3} |Y|)^{-1/4} \\
&\quad + C_\# \lambda_{\rm left}^{+1/3} B_{\rm left}^{1/2} S_{\rm left}^{-1} \, (1 + \lambda_{\rm left}^{2/3} |Y|)^{+1/4}\ .
\end{aligned}$$

The second term on the right clearly dominates the first term, so

$$\left|\frac{\partial F(x,E)}{\partial E}\right| \le C_\# \lambda_{\rm left}^{1/3} B_{\rm left}^{1/2} S_{\rm left}^{-1} (1 + \lambda_{\rm left}^{2/3} |\, Y_{\rm left}(x,E)|)^{+\frac{1}{4}}$$

$$\text{for } \ x \in I_k,\ E \ \text{ between } \ E_k \ \text{ and } \ E(k)\ .$$

Using again the fact that $\frac{\partial Y}{\partial x} \sim B_{\text{left}}^{-1}$, we conclude that

$$(29) \qquad \int\limits_{I_k} \left|\frac{\partial F(x,E)}{\partial E}\right|^2 dx \le C_\# B_{\text{left}}^2 S_{\text{left}}^{-2} \int\limits_{I_k} (1+\lambda_{\text{left}}^{2/3}|\ Y_{\text{left}}(x,E)|)^{+\ 1/2}$$

$$\cdot \lambda_{\text{left}}^{2/3} \frac{\partial Y_{\text{left}}(x,E)}{\partial x}\ dx = C_\# B_{\text{left}}^2 S_{\text{left}}^{-2} \int\limits_{\hat{I}_k} (1+|\xi|)^{+\frac{1}{2}}\ d\xi\ ,$$

where $\hat{I}_k$ is the image of I_k under $x \mapsto \lambda_{\text{left}}^{2/3}\ Y_{\text{left}}(x,E)$.

Estimate (28) shows that $x_{\text{left}}(E) \in I_k$, hence $\lambda_{\text{left}}^{2/3} Y_{\text{left}}(x_{\text{left}}(E),E) \in \hat{I}_k$. On the other hand, $|Y_{\text{left}}(x_{\text{left}}(E),E)| = |Y_{\text{left}}(x_{\text{left}}(E),E) - Y_0^{\text{left}}(x_{\text{left}}(E),E)| \le C_\# \lambda_{\text{left}}^{-2}$. Hence $\hat{I}_k$ contains a point ξ with $|\xi| \le C_\# \lambda_{\text{left}}^{\frac{2}{3}-2} << 1$. The length of $\hat{I}_k$ is $\sim \lambda_{\text{left}}^{2/3} B_{\text{left}}^{-1} |I_k| \sim \lambda_{\text{left}}^{2/3} B_{\text{left}}^{-1}(\delta x)$, since $\frac{\partial Y_{\text{left}}}{\partial x} \sim B_{\text{left}}^{-1}$. For an interval $\hat{I}_k$ containing ξ with absolute value < 1 and having length $\sim \lambda_{\text{left}}^{2/3} B_{\text{left}}^{-1}(\delta x) > 1$ by *ASSUMPTION* 3, we have

$$\int\limits_{\hat{I}_k} (1+|\xi|)^{1/2}\ d\xi \sim |\hat{I}_k|^{3/2} \sim \lambda_{\text{left}} \left(\frac{\delta x}{B_{\text{left}}}\right)^{3/2}.$$

Putting this into (29), we get

$$(30) \qquad \int\limits_{I_k} \left|\frac{\partial F(x,E)}{\partial E}\right|^2 dx \le C_\# B_{\text{left}}^2 S_{\text{left}}^{-2} \lambda_{\text{left}} \left(\frac{\delta x}{B_{\text{left}}}\right)^{3/2}$$

$$\text{for } E \text{ between } E_k \text{ and } E(k)\ .$$

Since $(\int_{I_k} |F(x,E_k) - F(x,E(k))|^2 dx)^{1/2} \le \int_{\min(E_k,E(k))}^{\max(E_k,E(k))} (\int_{I_k} |\frac{\partial F(x,E)}{\partial E}|^2 dx)^{1/2} dE$, we conclude that

$$\int\limits_{I_k} |F(x,E_k) - F(x,E(k))|^2 dx \le C_\# B_{\text{left}}^2 S_{\text{left}}^{-2} \lambda_{\text{left}} \left(\frac{\delta x}{B_{\text{left}}}\right)^{3/2} |E_k - E(k)|^2$$

$$(31) \qquad \le C_\# B_{\text{left}}^2 S_{\text{left}}^{-2} \lambda_{\text{left}} \left(\frac{\delta x}{B_{\text{left}}}\right)^{3/2} (S_{\min}\Gamma^{-1}\Lambda^{-2})^2\ .$$

Also (7), (9), (25) show that $|b_{\text{left}}^2| \sim (S_{\min}\Gamma^{-1})$. This and (31) show that

$$\int\limits_{I_k} |b_{\text{left}} F(x,E_k) - b_{\text{left}} F(x,E(k))|^2 dx$$

$$\le C_\# (S_{\min}\Gamma^{-1})^3 \Lambda^{-4} \lambda_{\text{left}} \left(\frac{\delta x}{B_{\text{left}}}\right)^{3/2} B_{\text{left}}^2 S_{\text{left}}^{-2}\ .$$

Combining this with (23) and recalling that $I_k \subset \{|x - x_{\text{left}}(E_k)| < \lambda_{\text{left}}^{-\varepsilon} B_{\text{left}}\}$, we get

(32)

$$\int_{I_k} |u_k(x) - b_{\text{left}} F(x, E(k))|^2 dx \le C_\#(S_{\min}\Gamma^{-1})^3 \Lambda^{-4} \lambda_{\text{left}} \Big(\frac{\delta x}{B_{\text{left}}}\Big)^{3/2} B_{\text{left}}^2 S_{\text{left}}^{-2} + \Lambda^{-N''} .$$

Estimate (32) is the desired analogue of (23) with E_k replaced by $E(k)$. It doesn't matter that the region of integration has been cut down to I_k, since the support of $x \mapsto g(x, E)$ is contained in I_k for all E between E_k and $E(k)$. We next use (26) to replace b_{left} by $\pm[\phi'(E(k))]^{-1/2}$ in (32). After possibly replacing $u_k(x)$ by $-u_k(x)$, we may suppose $b_{\text{left}} > 0$. Estimate (26) may then be written as $|b_{\text{left}} - \big(\phi'(E(k))\big)^{-1/2}| \le C_\# \Lambda^{4\varepsilon - 2} |\phi'(E(k))|^{-1/2}$, which implies

(33) $$\int_{I_k} |b_{\text{left}} F(x, E(k)) - [\phi'(E(k))]^{-1/2} F(x, E(k))|^2 dx \le$$

$$C_\# \Lambda^{8\varepsilon - 4} [\phi'(E(k))]^{-1} \int_{I_k} |F(x, E(k))|^2 dx \le C_\# \Lambda^{8\varepsilon - 4} S_{\min} \Gamma^{-1} \int_{I_k} |F(x, E(k))|^2 dx$$

by (9) .

To estimate the right–hand side of (33), we again use the estimates $\frac{\partial Y_{\text{left}}}{\partial x} \sim B_{\text{left}}^{-1}$, $|A(\xi)| \le C(1 + |\xi|)^{-1/4}$ to conclude that

$$\int_{I_k} |F(x, E(k))|^2 dx = \int_{I_k} \lambda_{\text{left}}^{-2/3} \Big(\frac{\partial Y_{\text{left}}}{\partial x}\Big)^{-1} A^2(\lambda_{\text{left}}^{2/3} Y_{\text{left}}(x, E(k)))\, dx$$

$$< C_\# \int_{I_k} \lambda_{\text{left}}^{-4/3} \Big(\frac{\partial Y_{\text{left}}}{\partial x}\Big)^{-2} \big(1 + \lambda_{\text{left}}^{2/3}\, |Y_{\text{left}}(x, E(k))|\big)^{-1/2} \cdot \lambda_{\text{left}}^{2/3} \frac{\partial Y_{\text{left}}}{\partial x}\, dx$$

$$\le C_\# \lambda_{\text{left}}^{-4/3} B_{\text{left}}^2 \int_{I_k} \big(1 + \lambda_{\text{left}}^{2/3}\, |Y_{\text{left}}(x, E(k))|\big)^{-1/2} \cdot \lambda_{\text{left}}^{2/3} \frac{\partial Y_{\text{left}}(x, E(k))}{\partial x}\, dx$$

(34)

$$= C_\# \lambda_{\text{left}}^{-4/3} B_{\text{left}}^2 \int_{\hat{I}_k} (1 + |\xi|)^{-1/2} d\xi \quad \text{with} \quad \hat{I}_k \quad \text{as in (29)} .$$

Recall from the paragraph following (29) that $\hat{I}_k$ is an interval of length $\sim \lambda_{\text{left}}^{+2/3}(\frac{\delta x}{B_{\text{left}}}) > 1$, containing a point ξ with $|\xi| < 1$. Therefore, $\int_{\hat{I}_k} (1 + |\xi|)^{-1/2} d\xi \sim$

$|\hat{I}_k|^{1/2} \sim \lambda_{\text{left}}^{+1/3}(\frac{\delta x}{B_{\text{left}}})^{1/2}$, so that (34) implies

$$\int_{I_k} |F(x,E(k))|^2 \, dx \le C_\# \lambda_{\text{left}}^{-1} B_{\text{left}}^2 \Big(\frac{\delta x}{B_{\text{left}}}\Big)^{1/2} . \tag{35}$$

From (33) and (35), we see that

$$\int_{I_k} |b_{\text{left}} F(x,E(k)) - [\phi'(E(k))]^{-1/2} F(x,E(k))|^2 \, dx \le C_\# \Lambda^{8\varepsilon-4} S_{\min} \Gamma^{-1} \lambda_{\text{left}}^{-1} B_{\text{left}}^2 \Big(\frac{\delta x}{B_{\text{left}}}\Big)^{1/2} . \tag{36}$$

This and (32) imply

$$\begin{aligned} \int_{I_k} |u_k(x) - [\phi'(E(k))]^{-1/2} F(x,E(k))|^2 \, dx \\ \le C_\# \Lambda^{8\varepsilon-4} (S_{\min}\Gamma^{-1}) \lambda_{\text{left}}^{-1} B_{\text{left}}^2 \Big(\frac{\delta x}{B_{\text{left}}}\Big)^{1/2} \\ + C_\# B_{\text{left}}^2 S_{\text{left}}^{-2} \lambda_{\text{left}} \Big(\frac{\delta x}{B_{\text{left}}}\Big)^{3/2} \Lambda^{-4} (S_{\min}\Gamma^{-1})^3 + \Lambda^{-N''} . \end{aligned} \tag{37}$$

Fortunately, the right–hand side of (37) simplifies. In fact, (11) shows that

$$\begin{aligned} C_\# B_{\text{left}}^2 S_{\text{left}}^{-2} \lambda_{\text{left}} \Big(\frac{\delta x}{B_{\text{left}}}\Big)^{3/2} \Lambda^{-4} (S_{\min}\Gamma^{-1})^3 \\ \le C_\# B_{\text{left}}^2 S_{\text{left}}^{-2} \lambda_{\text{left}} \Big(\frac{\delta x}{B_{\text{left}}}\Big)^{3/2} \Lambda^{-4} (S_{\min}\Gamma^{-1}) (S_{\text{left}}^{1/2} B_{\text{left}}^{-1})^2 \\ = C_\# B_{\text{left}}^2 \lambda_{\text{left}} \Big(\frac{\delta x}{B_{\text{left}}}\Big)^{3/2} \Lambda^{-4} (S_{\min}\Gamma^{-1}) \cdot (S_{\text{left}}^{-1} B_{\text{left}}^{-2}) \\ = C_\# B_{\text{left}}^2 \lambda_{\text{left}}^{-1} \Big(\frac{\delta x}{B_{\text{left}}}\Big)^{3/2} \Lambda^{-4} (S_{\min}\Gamma^{-1}) \\ \le C_\# \Lambda^{8\varepsilon-4} (S_{\min}\Gamma^{-1}) \lambda_{\text{left}}^{-1} B_{\text{left}}^2 \Big(\frac{\delta x}{B_{\text{left}}}\Big)^{1/2} . \end{aligned}$$

This means that the second term on the right of (37) is dominated by the first term. So (37) simplifies to

$$\int_{I_k} |u_k(x) - [\phi'(E(k))]^{-1/2} F(x,E(k))|^2 \, dx \le C_\# \Lambda^{8\varepsilon-4} (S_{\min}\Gamma^{-1}) \lambda_{\text{left}}^{-1} B_{\text{left}}^2 \Big(\frac{\delta x}{B_{\text{left}}}\Big)^{1/2} + \Lambda^{-N''} . \tag{38}$$

Next we use the inequality

$$(39)\quad \int_{I_k} |u^2(x) - v^2(x)|\ dx \le \Big(\int_{I_k} |u(x) - v(x)|^2\ dx\Big)^{1/2} \Big(\int_{I_k} |u(x) + v(x)|^2\ dx\Big)^{1/2}$$

$$\le \Big(\int_{I_k} |u(x) - v(x)|^2\ dx\Big)^{1/2} \Big[2\Big(\int_{I_k} |v(x)|^2\ dx\Big)^{1/2} + \Big(\int_{I_k} |u(x) - v(x)|^2\ dx\Big)^{1/2}\Big]\ ,$$

as follows from $\|u + v\| = \|2v - (v - u)\| \le 2\|v\| + \|u - v\|$. We take $u = u_k(x)$ and $v(x) = [\phi'(E(k))]^{-1/2} F(x, E(k))$. By (35) and (9) we have

$$(40)\quad \int_{I_k} |v(x)|^2\ dx \le C_\# S_{\rm min} \Gamma^{-1} \int_{I_k} |F(x, E(k))|^2\ dx$$

$$\le C_\# S_{\rm min} \Gamma^{-1} \lambda_{\rm left}^{-1} B_{\rm left}^2 \Big(\frac{\delta x}{B_{\rm left}}\Big)^{1/2}\ .$$

Hence (38) and (39) show that

$$(41)\quad \int_{I_k} |u_k^2(x) - [\phi'(E(k))]^{-1} F^2(x, E(k))|\ dx$$

$$\le \Big[C_\# \Lambda^{8\varepsilon - 4} (S_{\rm min} \Gamma^{-1}) \lambda_{\rm left}^{-1} B_{\rm left}^2 \Big(\frac{\delta x}{B_{\rm left}}\Big)^{1/2} + \Lambda^{-N''}\Big]^{1/2}$$

$$\cdot \Big[C_\# (S_{\rm min} \Gamma^{-1}) \lambda_{\rm left}^{-1} B_{\rm left}^2 \Big(\frac{\delta x}{B_{\rm left}}\Big)^{1/2} + \Lambda^{-N''}\Big]^{1/2}$$

$$\le C_\# \Lambda^{4\varepsilon - 2} (S_{\rm min} \Gamma^{-1}) \lambda_{\rm left}^{-1} B_{\rm left}^2 \Big(\frac{\delta x}{B_{\rm left}}\Big)^{1/2}$$

$$+ C_\# \Lambda^{-\frac{N''}{2}} \Big[(S_{\rm min} \Gamma^{-1}) \lambda_{\rm left}^{-1} B_{\rm left}^2 \Big(\frac{\delta x}{B_{\rm left}}\Big)^{1/2}\Big]^{1/2} + \Lambda^{-N''}$$

$$\le C_\# \Lambda^{4\varepsilon - 2} (S_{\rm min} \Gamma^{-1}) \lambda_{\rm left}^{-1} B_{\rm left}^2 \Big(\frac{\delta x}{B_{\rm left}}\Big)^{1/2} + C_\# \Lambda^{10 - N''}$$

(we see the last inequality by using $\Lambda^{-\frac{N''}{2}} X^{1/2} \le \Lambda^{10 - N''} + \Lambda^{-2} X$) .

Since I_k contains the supports of both $g(x, E_k)$ and $g(x, E(k))$, we have

$$(42)\quad \int_{I_{\rm BVP}} |g(x, E_k) u_k^2(x) - g(x, E(k)) [\phi'(E(k))]^{-1} F^2(x, E(k))|\ dx$$

$$\le \Big[\max_{x \in I_{\rm BVP}} |g(x, E_k)|\Big] \int_{I_k} |u_k^2(x) - [\phi'(E(k))]^{-1} F^2(x, E(k))|\ dx$$

$$+ \Big[\max_{x \in I_{\rm BVP}} |g(x, E_k) - g(x, E(k))|\Big] \int_{I_k} \Big|[\phi'(E(k))]^{-1} F^2(x, E(k))\Big|\ dx\ .$$

We know that $\max_{x\in I_{\rm BVP}} |g(x,E_k)| \le C_*$ and

$$\Big[\max_{x\in I_{\rm BVP}} |g(x,E_k)-g(x,E(k))|\Big] \le C_*(\delta E)^{-1}|E_k-E(k)|$$
$$\le C_*(\delta E)^{-1}S_{\min}\Gamma^{-1}\Lambda^{-2}$$

(by (18)) $\le C_*[S_{\rm left}^{-1}(\frac{B_{\rm left}}{\delta x}) + S_{\min}^{-1}]S_{\min}\Gamma^{-1}\Lambda^{-2}$. (See *ASSUMPTION* 3.) The integrals on the right of (42) are controlled by (40) and (41). Substituting these estimates into (42), we get the following.

$$\text{(43)}\quad \int_{I_{\rm BVP}} |g(x,E_k)u_k^2(x) - g(x,E(k))[\phi'(E(k))]^{-1}F^2(x,E(k))|\,dx$$
$$\le C_*\Lambda^{4\varepsilon-2}(S_{\min}\Gamma^{-1})\lambda_{\rm left}^{-1}B_{\rm left}^2\Big(\frac{\delta x}{B_{\rm left}}\Big)^{1/2} + C_*\Lambda^{10-N''}$$
$$+\Big\{C_*\Big[S_{\rm left}^{-1}\Big(\frac{B_{\rm left}}{\delta x}\Big)+S_{\min}^{-1}\Big](S_{\min}\Gamma^{-1})\Lambda^{-2}$$
$$\cdot(S_{\min}\Gamma^{-1})\lambda_{\rm left}^{-1}B_{\rm left}^2\Big(\frac{\delta x}{B_{\rm left}}\Big)^{1/2}\Big\}\ .$$

The right–hand side simplifies. We shall check that the term in curly brackets on the right is dominated by the first term on the right. This amounts to checking that $\Big[S_{\rm left}^{-1}\Big(\frac{B_{\rm left}}{\delta x}\Big)+S_{\min}^{-1}\Big](S_{\min}\Gamma^{-1}) \le \Lambda^{4\varepsilon}$, i.e.

$$\text{(44)}\qquad S_{\rm left}^{-1}(B_{\rm left}/\delta x)\ (S_{\min}\Gamma^{-1}) + \Gamma^{-1} \le \Lambda^{4\varepsilon}\ .$$

We have $\Gamma >> 1$ by (12), so (44) will follow from

$$\text{(45)}\qquad S_{\rm left}^{-1}\Big(\frac{B_{\rm left}}{\delta x}\Big)(S_{\min}\Gamma^{-1}) \le \frac{1}{2}\Lambda^{4\varepsilon}\ .$$

Since $\frac{B_{\rm left}}{\delta x} \le \lambda_{\rm left}^{2/3}$ by *ASSUMPTION* 3, estimate (11) shows that

$$S_{\rm left}^{-1}\Big(\frac{B_{\rm left}}{\delta x}\Big)(S_{\min}\Gamma^{-1}) \le C_\# S_{\rm left}^{-1}\Big(\frac{B_{\rm left}}{\delta x}\Big)(S_{\rm left}^{1/2}B_{\rm left}^{-1})$$
$$\le C_\#\lambda_{\rm left}^{2/3}(S_{\rm left}^{-1/2}B_{\rm left}^{-1}) = C_\#\lambda_{\rm left}^{-1/3} << 1\ .$$

So (45) holds, and the term in curly brackets in (43) is dominated by the other terms on the right–hand side. Thus (43) may be rewritten as

$$\text{(46)}\quad \int_{I_{\rm BVP}} |g(x,E_k)u_k^2(x) - g(x,E(k))[\phi'(E(k))]^{-1}F^2(x,E(k))|\,dx$$
$$\le C_*\Lambda^{4\varepsilon-2}(S_{\min}\Gamma^{-1})\lambda_{\rm left}^{-1}B_{\rm left}^2\Big(\frac{\delta x}{B_{\rm left}}\Big)^{1/2} + C_*\Lambda^{10-N''}\ .$$

We have proven (46) for $k_{\ell o} \le k \le k_{hi}$.

Next recall that $g(x,E) \equiv 0$ outside $\{|E-E_0| \le c^1_{\#} S_{\min}\}$, and that $E_{k_{\ell o}} \dots E_{k_{hi}}$ are precisely the eigenvalues of $-\frac{d^2}{dx^2} + V(x)$ belonging to $\{|E-E_0| \le c^1_{\#} S_{\min}\}$ $\cap\, (-\infty, 0]$. The definition $\rho(x,g) = \sum\limits_{E \text{ eigenvalue } \le 0} g(x,E) u^2(x,E)$ (with $u(x,E) =$ the real normalized eigenfunction corresponding to E) therefore shows that $\rho(x,g) = \sum\limits_{k=k_{\ell o}}^{k_{hi}} g(x,E_k) u_k^2(x)$. So (46) implies:

$$(47) \qquad \int\limits_{I_{\mathrm{BVP}}} \Big|\rho(x,g) - \sum_{k=k_{\ell o}}^{k_{hi}} g(x,E(k))[\phi'(E(k))]^{-1} F^2(x,E(k))\Big|\, dx$$
$$\le [C_* \Lambda^{4\varepsilon-2} (S_{\min}\Gamma^{-1}) \lambda_{\mathrm{left}}^{-1} B_{\mathrm{left}}^2 \Big(\frac{\delta x}{B_{\mathrm{left}}}\Big)^{1/2} + C_* \Lambda^{10-N''}] \cdot (k_{hi} - k_{\ell o} + 1) \ .$$

It is easy to estimate $(k_{hi} - k_{\ell o} + 1)$, since $\{k_{hi} \le k \le k_{\ell o}\} = \mathbb{Z} \cap [a,b]$ by (3), and thus $k_{hi} - k_{\ell o} + 1 \le b - a + 10 \le \frac{1}{\pi}\phi(E_{hi}) - \frac{1}{\pi}\phi(E_{\ell o}) + 20$. (To see the last estimate, recall that $b = \frac{1}{\pi}\phi(E_{hi}) + \frac{1}{48\pi}\psi(E_{hi}) - \frac{1}{2} + \omega_{hi}$, $a = \frac{1}{\pi}\phi(E_{\ell o}) + \frac{1}{48\pi}\psi(E_{\ell o}) - \frac{1}{2} + \omega_{\ell o}$, with $|\psi(E_{hi})|$, $|\psi(E_{\ell o})|$, $|\omega_{hi}|$, $|\omega_{\ell o}| < 1$.) From (7) we get $\phi(E_{hi}) - \phi(E_{\ell o}) \le C_{\#}\Gamma S_{\min}^{-1} |E_{hi} - E_{\ell o}| \le C_{\#}\Gamma$ by definition (1) of E_{hi}, $E_{\ell o}$. Hence

$$(48) \qquad k_{hi} - k_{\ell o} + 1 \le C_{\#}\Gamma + 20 \le C_{\#}\Gamma \qquad \text{by (12)} \ .$$

Putting this into (47) and using (3), we get

$$(49) \qquad \int\limits_{I_{\mathrm{BVP}}} \Big|\rho(x,g) - \sum_{k \in \mathbb{Z}\cap[a,b]} g(x,E(k))[\phi'(E(k))]^{-1} F^2(x,E(k))\Big|\, dx$$
$$\le C_* \Lambda^{4\varepsilon-2} S_{\min} \lambda_{\mathrm{left}}^{-1} B_{\mathrm{left}}^2 \Big(\frac{\delta x}{B_{\mathrm{left}}}\Big)^{1/2} + C_* \Lambda^{10-N''} \Gamma \ .$$

We want to approximate the sum in (49) by using the Lemma on Riemann sums. This requires that we bound the derivatives of $t \mapsto g(x,E(t))[\phi'(E(t))]^{-1} F^2(x,E(t))$. To produce the bounds, we write

$$(50) \qquad g(x,E(t))[\phi'(E(t))]^{-1} F^2(x,E(t)) = f_{\mathrm{Airey}}(x,t) f_{\mathrm{other}}(x,t) \ , \qquad \text{with}$$

$$f_{\text{Airey}}(x,t) = A^2\big(\lambda_{\text{left}}^{2/3} Y_{\text{left}}(x,E(t))\big) \qquad \text{and} \tag{51}$$

$$f_{\text{other}}(x,t) = g(x,E(t))\lambda_{\text{left}}^{-2/3}\Big(\frac{\partial Y_{\text{left}}(x,E(t))}{\partial x}\Big)^{-1}[\phi'(E(t))]^{-1} \ . \tag{52}$$

(See (24).) To handle f_{Airey}, we note that

$$\Big|\big(\frac{d}{d\xi}\big)^m A^2(\xi)\Big| \le C_m(1+|\xi|)^{-\frac{1}{2}+\frac{m}{2}} \quad (m \ge 0) \quad \text{for a universal constant } C_m \ . \tag{53}$$

The estimates for $Y_{\text{left}}(x,E)$ in the WKB Eigenfunction Theorem include

$$\Big|\big(\frac{\partial}{\partial E}\big)^\beta Y_{\text{left}}(x,E)\Big| \le C_\#^\beta S_{\text{left}}^{-\beta} \quad \text{for } (x,E) \in \text{ supp } g \ . \tag{54}$$

The derivative $(\frac{\partial}{\partial E})^\beta A^2(\lambda_{\text{left}}^{2/3} Y_{\text{left}}(x,E))$ is a sum of terms of the form

$$\Big[\big(\frac{d}{d\xi}\big)^m A^2(\xi)\ \big|_{\xi=\lambda_{\text{left}}^{2/3}Y_{\text{left}}(x,E)}\Big] \cdot \prod_{\nu=1}^{m}\Big[\big(\frac{\partial}{\partial E}\big)^{\beta_\nu}\{\lambda_{\text{left}}^{2/3} Y_{\text{left}}(x,E)\}\Big] \tag{55}$$

with $\beta_\nu \ge 1$ and $\beta_1 + \ldots + \beta_m = \beta$. In particular, $0 \le m \le \beta$. By (53) and (54), the term (55) is dominated by

$$C_\#^\beta(1+\lambda_{\text{left}}^{2/3}|Y_{\text{left}}(x,E)|)^{-\frac{1}{2}+\frac{m}{2}}\lambda_{\text{left}}^{\frac{2}{3}m} S_{\text{left}}^{-\beta} \ ;$$

and since $0 \le m \le \beta$, this is in turn dominated by

$$C_\#^\beta(1+\lambda_{\text{left}}^{2/3}|Y_{\text{left}}(x,E)|)^{-\frac{1}{2}}\Big[(\lambda_{\text{left}}^{2/3} + \lambda_{\text{left}}|Y_{\text{left}}(x,E)|^{1/2})S_{\text{left}}^{-1}\Big]^\beta \ .$$

Hence,

$$\begin{aligned}\Big|\big(\frac{\partial}{\partial E}\big)^\beta A^2(\lambda_{\text{left}}^{2/3} Y_{\text{left}}(x,E))\Big| &\le C_\#^\beta(1+\lambda_{\text{left}}^{2/3}|Y_{\text{left}}(x,E)|)^{-1/2} \\ &\cdot\Big[\frac{\lambda_{\text{left}}^{2/3} + \lambda_{\text{left}}|Y_{\text{left}}(x,E)|^{1/2}}{S_{\text{left}}}\Big]^\beta \text{for} \quad (x,E) \in \text{ supp } g \ .\end{aligned} \tag{56}$$

The derivative $(\frac{d}{dt})^m f_{\text{Airey}}(x,t)$ is a sum of terms of the form

$$\Big[\big(\frac{\partial}{\partial E}\big)^\beta A^2(\lambda_{\text{left}}^{2/3} Y_{\text{left}}(x,E))\ \big|_{E=E(t)}\Big] \ \cdot \prod_{\nu=1}^{\beta}\Big[\big(\frac{d}{dt}\big)^{m_\nu} E(t)\Big] \tag{57}$$

with $m_\nu \geq 1$ and $m_1 + \ldots + m_\beta = m$. In particular, $0 \leq \beta \leq m$. For $t \in \mathcal{J}$ and $(x, E(t)) \in$ supp g we have (56) and (13), so that the term (57) is dominated by

$$(58) \quad C_\#^m \big(1 + \lambda_{\text{left}}^{2/3} |Y_{\text{left}}(x, E(t))|\big)^{-1/2} \Big[\frac{\lambda_{\text{left}}^{2/3} + \lambda_{\text{left}} |Y_{\text{left}}(x, E(t))|^{1/2}}{S_{\text{left}}}\Big]^\beta S_{\min}^\beta \Gamma^{-m} .$$

We rewrite (58) in the form

$$(59) \qquad C_\#^m \big(1 + \lambda_{\text{left}}^{2/3} |Y_{\text{left}}(x, E(t))|\big)^{-1/2} \Big[\frac{(\lambda_{\text{left}}^{2/3} + \lambda_{\text{left}} |Y_{\text{left}}(x, E(t))|^{1/2})}{S_{\text{left}}} \cdot (S_{\min}\Gamma^{-1})\Big]^\beta \; \Gamma^{\beta - m} .$$

Hence $|(\frac{d}{dt})^m f_{\text{Airey}}(x,t)|$ is dominated by a sum of terms (59) for $t \in \mathcal{J}$, $(x, E(t)) \in$ supp g, with $0 \leq \beta \leq m$. Estimate (11) shows that the quantity in square brackets in (59) is dominated by

$$\frac{(\lambda_{\text{left}}^{2/3} + \lambda_{\text{left}} |Y_{\text{left}}(x, E(t))|^{1/2})}{S_{\text{left}}} \; (S_{\text{left}}^{1/2} B_{\text{left}}^{-1}) = (\lambda_{\text{left}}^{-1/3} + |Y_{\text{left}}(x, E(t))|^{1/2}) \leq C_\# \big(\frac{\delta x}{B_{\text{left}}}\big)^{1/2} \quad \text{for} \quad (x, E(t)) \in \text{ supp } g ,$$

since then $|x - x_{\text{left}}(E(t))| < (\delta x)$ so

$$|Y_{\text{left}}(x, E(t))| \leq |Y_0^{\text{left}}(x, E(t))| + C_\# \lambda_{\text{left}}^{-2} \leq C_\# \big(\frac{\delta x}{B_{\text{left}}}\big) + C_\# \lambda_{\text{left}}^{-2}$$

by the WKB Eigenfunction Theorem. Hence, (59) is dominated by

$$C_\#^m (1 + \lambda_{\text{left}}^{2/3} |Y_{\text{left}}(x, E(t))|)^{-1/2} \big(\frac{\delta x}{B_{\text{left}}}\big)^{\frac{1}{2}\beta} \Gamma^{\beta - m} \quad (0 \leq \beta \leq m) ,$$

so

$$(60) \qquad \Big|\big(\frac{d}{dt}\big)^m f_{\text{Airey}}(x,t)\Big| \leq C_\#^m (1 + \lambda_{\text{left}}^{2/3} |Y_{\text{left}}(x, E(t))|)^{-1/2} \cdot [\min\{\Gamma, (\frac{B_{\text{left}}}{\delta x})^{1/2}\}]^{-m} , \quad \text{for} \quad t \in \mathcal{J}, (x, E(t)) \in \text{ supp } g .$$

Next we study $f_{\text{other}}(x,t)$. The WKB Eigenfunction Theorem gives

$$\Big|\Big(\frac{\partial}{\partial E}\Big)^{\beta}\frac{\partial Y_{\text{left}}(x,E)}{\partial x}\Big| \le C_{\#}^{\beta}B_{\text{left}}^{-1}S_{\text{left}}^{-\beta}$$
$$\text{and}\quad \Big|\frac{\partial Y_{\text{left}}(x,E)}{\partial x}\Big| > c_{\#}B_{\text{left}}^{-1}$$

in supp g, and therefore

$$\Big|\Big(\frac{\partial}{\partial E}\Big)^{\beta}\Big(\frac{\partial Y_{\text{left}}(x,E)}{\partial x}\Big)^{-1}\Big| \le C_{\#}^{\beta}B_{\text{left}}S_{\text{left}}^{-\beta}\quad \text{in}\quad \text{supp } g\ . \tag{61}$$

ASSUMPTION 2 and the inequalities $(\delta E) \le S_{\min} \le S_{\text{left}}$, together with (61), imply

(62)
$$\Big|\Big(\frac{\partial}{\partial E}\Big)^{\beta}\Big\{g(x,E)\Big(\frac{\partial Y_{\text{left}}(x,E)}{\partial x}\Big)^{-1}\Big\}\Big| \le C_{*}^{\beta}B_{\text{left}}(\delta E)^{-\beta}\quad \text{for all } x\ , E\ .$$

From (7) and (9) we see that

$$\Big|\Big(\frac{d}{dE}\Big)^{\beta}\Big[\frac{d\phi}{dE}\Big]^{-1}\Big| \le C_{\#}^{\beta}(S_{\min}\Gamma^{-1})S_{\min}^{-\beta}\quad \text{for}\quad |E-E_0| \le 33\ c_{\#}^{1}S_{\min}\ .$$

Combining this with (62) and recalling that $g(x,E)=0$ for $|E-E_0| > c_{\#}^{1}S_{\min}$, we get

$$\Big|\Big(\frac{\partial}{\partial E}\Big)^{\beta}\Big\{\lambda_{\text{left}}^{-2/3}g(x,E)\Big(\frac{\partial Y_{\text{left}}(x,E)}{\partial x}\Big)^{-1}[\phi'(E)]^{-1}\Big\}\Big|$$
$$\le C_{*}^{\beta}\lambda_{\text{left}}^{-2/3}B_{\text{left}}(S_{\min}\Gamma^{-1})(\delta E)^{-\beta}\ ,\ \text{all } x, E\ . \tag{63}$$

Here again we use $\delta E \le S_{\min}$.

The derivative $(\frac{d}{dt})^m f_{\text{other}}(x,t)$ is a sum of terms

$$\Big[\Big(\frac{\partial}{\partial E}\Big)^{\beta}\Big\{\lambda_{\text{left}}^{-2/3}g(x,E)\Big(\frac{\partial Y_{\text{left}}(x,E)}{\partial x}\Big)^{-1}$$
$$\cdot[\phi'(E)]^{-1}\Big\}\Big|_{E=E(t)}\Big]\cdot\prod_{\nu=1}^{\beta}\Big[\Big(\frac{d}{dt}\Big)^{m_\nu}E(t)\Big] \tag{64}$$

with $m_\nu \ge 1$ and $m_1+\ldots+m_\beta = m$. (See (52).) In particular, $0 \le \beta \le m$. For $t \in \mathcal{J}$, estimates (63) and (13) show that the term (64) is dominated by

$$C_{*}^{m}\lambda_{\text{left}}^{-2/3}B_{\text{left}}(S_{\min}\Gamma^{-1})(\delta E)^{-\beta}\cdot S_{\min}^{\beta}\Gamma^{-m}\ . \tag{65}$$

Since $0 \le \beta \le m$ and $\delta E \le S_{\min}$, the term (65) is in turn dominated by

$$C_*^m \lambda_{\text{left}}^{-2/3} B_{\text{left}} (S_{\min}\Gamma^{-1}) \big[(\delta E)^{-1} S_{\min} \Gamma^{-1} \big]^m ,$$

and therefore

$$(66) \qquad \Big| \big(\frac{d}{dt}\big)^m f_{\text{other}}(x,t) \Big| \le C_*^m \lambda_{\text{left}}^{-2/3} B_{\text{left}} (S_{\min}\Gamma^{-1}) [(\delta E)^{-1} S_{\min}\Gamma^{-1}]^m \quad \text{for} \quad t \in \mathcal{J} \ .$$

ASSUMPTION 3 shows that the expression in square brackets in (66) has the order of magnitude

$$\begin{aligned} &(S_{\min}^{-1} + S_{\text{left}}^{-1} B_{\text{left}} (\delta x)^{-1}) S_{\min} \Gamma^{-1} \\ &= \Gamma^{-1} + S_{\text{left}}^{-1} (B_{\text{left}}/(\delta x))(S_{\min}\Gamma^{-1}) \le \Gamma^{-1} + \big(\frac{B_{\text{left}}}{\delta x}\big) S_{\text{left}}^{-1} (S_{\text{left}}^{1/2} B_{\text{left}}^{-1}) \end{aligned}$$

(by (11)) $= \Gamma^{-1} + (\frac{B_{\text{left}}}{\delta x}) \lambda_{\text{left}}^{-1}$. Therefore, (66) implies

$$(67) \qquad \Big| \big(\frac{d}{dt}\big)^m f_{\text{other}}(x,t) \Big| \le C_*^m \lambda_{\text{left}}^{-2/3} B_{\text{left}} (S_{\min}\Gamma^{-1}) \big[\min\{\Gamma, \big(\frac{\delta x}{B_{\text{left}}}\big) \lambda_{\text{left}}\} \big]^{-m} \quad \text{for} \quad t \in \mathcal{J} \ .$$

From (60) and (67), we learn that

$$(68) \qquad \begin{aligned} &\Big| \big(\frac{d}{dt}\big)^m \{ f_{\text{Airey}}(x,t) f_{\text{other}}(x,t) \} \Big| \\ &\le C_*^m \lambda_{\text{left}}^{-2/3} B_{\text{left}} (S_{\min}\Gamma^{-1}) \cdot (1 + \lambda_{\text{left}}^{2/3} | Y_{\text{left}}(x, E(t)) |)^{-1/2} \\ &\cdot \big[\min\{ \Gamma, \lambda_{\text{left}} \big(\frac{\delta x}{B_{\text{left}}}\big), \big(\frac{B_{\text{left}}}{\delta x}\big)^{1/2} \} \big]^{-m} \ , \\ &\text{for} \quad t \in \mathcal{J}, (x, E(t)) \in \ \text{supp}\ g \ . \end{aligned}$$

The right–hand side of (68) simplifies. First of all, the WKB Eigenfunction Theorem gives

$$|Y_{\text{left}}(x,E) - Y_0^{\text{left}}(x,E)| \le C_\# \lambda_{\text{left}}^{-2} \ , \ \ |Y_0^{\text{left}}(x,E)| \sim \frac{|x - x_{\text{left}}(E)|}{B_{\text{left}}}$$

in (supp g), and therefore

$$1+\lambda_{\text{left}}^{2/3}|Y_{\text{left}}(x,E(t))| \sim 1+\lambda_{\text{left}}^{2/3}B_{\text{left}}^{-1}|x-x_{\text{left}}(E(t))| \quad \text{for} \quad (x,E(t)) \in \text{ supp } g\ .$$

Also, *ASSUMPTION* 3 gives $\frac{\delta x}{B_{\text{left}}} > \lambda^{-2/3}$, so that $(\frac{B_{\text{left}}}{\delta x})^{1/2} \le \lambda_{\text{left}}^{+1/3} \le \lambda_{\text{left}}(\frac{\delta x}{B_{\text{left}}})$. Hence, $\min\{\Gamma, \lambda_{\text{left}}(\frac{\delta x}{B_{\text{left}}}), (\frac{B_{\text{left}}}{\delta x})^{1/2}\} = \min\{\Gamma, (\frac{B_{\text{left}}}{\delta x})^{1/2}\}$, so that (68) yields

$$(69) \qquad \left|\left(\frac{d}{dt}\right)^m \{f_{\text{Airey}}(x,t)f_{\text{other}}(x,t)\}\right| \le C_*^m \sigma_x(t)\tau^{-m} \quad \text{for} \quad t \in \mathcal{J}\ , x \in I_{\text{BVP}}$$

with

$$(70) \qquad \tau = \min\{\Gamma, (\frac{B_{\text{left}}}{\delta x})^{1/2}\} \quad \text{and}$$

$$(71) \qquad \sigma_x(t) = \lambda_{\text{left}}^{-2/3}B_{\text{left}}(S_{\min}\Gamma^{-1}) \cdot (1+\lambda_{\text{left}}^{2/3}B_{\text{left}}^{-1}\ |x-x_{\text{left}}(E(t))|)^{-1/2}\ .$$

Estimate (69) is the main hypothesis in the Lemma on Riemann sums. The other hypotheses are $\tau \ge 1$ and $c_* < \sigma_x(t_2)/\sigma_x(t_1) < C_*$ for $t_1, t_2 \in \mathcal{J}$, $|t_1-t_2| < c_*\tau$. Since $\frac{B_{\text{left}}}{\delta x} > \lambda_{\text{left}}^{\varepsilon}$ and $\Gamma \ge c_\# \Lambda$, (70) shows that $\tau \ge 1$. That $c_* < \sigma_x(t_2)/\sigma_x(t_1) < C_*$ amounts to saying that $|x_{\text{left}}(E(t_2)) - x_{\text{left}}(E(t_1))| \le C_*\lambda_{\text{left}}^{-2/3}B_{\text{left}}$ for $|t_1-t_2| < c_*\tau$; which in turn amounts to $|E(t_2)-E(t_1)| \le C_*\lambda_{\text{left}}^{-2/3}S_{\text{left}}$ for $t_1,t_2 \in \mathcal{J}$, $|t_1-t_2| < c_*\tau$. We know from (13) that $|E(t_2)-E(t_1)| \le C_\# S_{\min}\Gamma^{-1}|t_1-t_2| \le C_\# S_{\min}\Gamma^{-1}\tau \le C_\#(S_{\text{left}}^{1/2}B_{\text{left}}^{-1})(\frac{B_{\text{left}}}{\delta x})^{1/2}$ (by (11) and (70)) $\le C_\# S_{\text{left}}(S_{\text{left}}^{-1/2}B_{\text{left}}^{-1})\lambda_{\text{left}}^{1/3}$ (since $\frac{\delta x}{B_{\text{left}}} > \lambda_{\text{left}}^{-2/3}$) $= C_\# S_{\text{left}} \cdot \lambda_{\text{left}}^{-2/3}$, which is the desired hypothesis for the Lemma on Riemann sums. So we have verified all the hypotheses of the Lemma on Riemann sums. To know which variant of the lemma to apply, we must still check whether the integrand $\{f_{\text{Airey}}(x,t)f_{\text{other}}(x,t)\}$ vanishes to infinite order, both at $t=a$ and at $t=b$. From (21) we see that $g(x,E(t))=0$ for $t \le a$, so the integrand vanishes to infinite order at $t=a$, in all cases. Similarly, (22) shows that the integrand vanishes to infinite order at $t=b$, provided $|E_0| \ge 2\hat{c}S_{\min}$. If $|E_0| < 2\hat{c}S_{\min}$, then we still know that supp $g(x,E) \subset \{|x - x_{\text{left}}(E)| < \delta x\}$, and therefore $g(x,E(t))$ vanishes to infinite order at $t = b$ if $|x - x_{\text{left}}(E_{\max})| > (\delta x)$. To summarize,

(71bis) $f_{\text{Airey}}(x,t) f_{\text{other}}(x,t)$ vanishes to infinite order at $t = a$ always ,

and at $t = b$ unless $|E_0| < 2\hat{c}\, S_{\min}$

and $|x - x_{\text{left}}(E_{\max})| \le (\delta x)$.

Therefore, the lemma on Riemann sums yields the following conclusions.

(72)
$$\sum_{k \in \mathbb{Z} \cap [a,b]} f_{\text{Airey}}(x,k) \cdot f_{\text{other}}(x,k)$$
$$= \int_a^b f_{\text{Airey}}(x,t) f_{\text{other}}(x,t) dt - f_{\text{Airey}}(x,b) f_{\text{other}}(x,b) \chi_-(b) + \text{Error}_1(x)$$
$$\text{if} \quad |E_0| \le 2\hat{c}\, S_{\min} \quad \text{and} \quad |x - x_{\text{left}}(E_{\max})| \le \delta x \ , \quad \text{where}$$

(73)
$$|\text{Error}_1(x)| \le C_* \sigma_x(b) \tau^{-1} + C_*^{\overline{N}} \int_a^b \sigma_x(t) \tau^{-\overline{N}}\, dt \qquad (\overline{N} \text{ will be picked later}) \ .$$

Also

(74)
$$\sum_{k \in \mathbb{Z} \cap [a,b]} f_{\text{Airey}}(x,k) \cdot f_{\text{other}}(x,k) = \int_a^b f_{\text{Airey}}(x,t) \cdot f_{\text{other}}(x,t) dt + \text{Error}_2(x)$$
$$\text{if} \quad |E_0| \ge 2\hat{c}\, S_{\min} \quad \text{or} \quad |x - x_{\text{left}}(E_{\max})| > (\delta x) \ , \text{where}$$

(75)
$$|\text{Error}_2(x)| \le C_*^{\overline{N}} \int_a^b \sigma_x(t) \tau^{-\overline{N}}\, dt \ .$$

We estimate the L^1–norms of $\text{Error}_1(x)$ and $\text{Error}_2(x)$. ¿From the definitions of

$\sigma_x(t)$ and τ we have

$$\int\limits_{|x-x_{\text{left}}(E_{\max})|<\delta x} \sigma_x(b)\tau^{-1}dx$$
$$\leq C_*(\Gamma^{-1}+(\frac{\delta x}{B_{\text{left}}})^{1/2}) \int\limits_{|x-x_{\text{left}}(E_{\max})|<\delta x} \sigma_x(b)dx$$
$$\leq C_*(\Gamma^{-1}+(\frac{\delta x}{B_{\text{left}}})^{1/2}) \int\limits_{|x-x_{\text{left}}(E_{\max})|<\delta x} \lambda_{\text{left}}^{-2/3}B_{\text{left}}(S_{\min}\Gamma^{-1})$$
$$\cdot(1+\lambda_{\text{left}}^{2/3}B_{\text{left}}^{-1}|x-x_{\text{left}}(E_{\max})|)^{-1/2}\ dx$$
$$\leq C_*(\Gamma^{-1}+(\frac{\delta x}{B_{\text{left}}})^{1/2})\lambda_{\text{left}}^{-2/3}B_{\text{left}}(S_{\min}\Gamma^{-1})\cdot\lambda_{\text{left}}^{-1/3}B_{\text{left}}^{1/2}(\delta x)^{1/2}$$
$$= C_*(\Gamma^{-1}+(\frac{\delta x}{B_{\text{left}}})^{1/2})\lambda_{\text{left}}^{-1}B_{\text{left}}^{2}(S_{\min}\Gamma^{-1})(\frac{\delta x}{B_{\text{left}}})^{1/2}$$
$$= C_*(\Gamma^{-2}+(\frac{\delta x}{B_{\text{left}}})^{1/2}\Gamma^{-1})\lambda_{\text{left}}^{-1}B_{\text{left}}^{2}S_{\min}(\frac{\delta x}{B_{\text{left}}})^{1/2}$$
$$\leq C_*(\Lambda^{-2}+(\frac{\delta x}{B_{\text{left}}})^{1/2}\Lambda^{-1})\lambda_{\text{left}}^{-1}B_{\text{left}}^{2}S_{\min}(\frac{\delta x}{B_{\text{left}}})^{1/2} \quad \text{by (12)}\ . \tag{76}$$

Similarly,

(77)

$$\int\limits_{|x-x_{\text{left}}(E_0)|<c_\# B_{\text{left}}} \int_a^b \sigma_x(t)\tau^{-\overline{N}}dt\ dx \leq \int_a^b\Big\{\int\limits_{|x-x_{\text{left}}(E_0)|<c_\# B_{\text{left}}} \sigma_x(t)\ dx\Big\}dt\cdot\tau^{-\overline{N}}$$
$$\leq C_*\tau^{-\overline{N}}(b-a)\cdot\max_{t\in(a,b)}\Big\{\int\limits_{|x-x_{\text{left}}(E_0)|<c_\# B_{\text{left}}} \frac{\lambda_{\text{left}}^{-2/3}B_{\text{left}}(S_{\min}\Gamma^{-1})\ dx}{(1+\lambda_{\text{left}}^{2/3}B_{\text{left}}^{-1}|x-x_{\text{left}}(E(t))|)^{1/2}}\Big\}$$
$$\leq C_*\tau^{-\overline{N}}(b-a)\cdot\frac{\lambda_{\text{left}}^{-2/3}B_{\text{left}}^{2}(S_{\min}\Gamma^{-1})}{\lambda_{\text{left}}^{1/3}} \leq C_*\lambda_{\text{left}}^{-1}B_{\text{left}}^{2}S_{\min}\tau^{-\overline{N}}$$

(since we see from the proof of (48) that $b-a\leq C_\#\Gamma)\leq C_*(\Gamma^{-\overline{N}}+(\frac{\delta x}{B_{\text{left}}})^{\frac{\overline{N}}{2}})\lambda_{\text{left}}^{-1}B_{\text{left}}^{2}S_{\min}$ (by (70)) $\leq C_*(\Lambda^{-N}+(\frac{\delta x}{B_{\text{left}}})^{N})\lambda_{\text{left}}^{-1}B_{\text{left}}^{2}S_{\min}$. From (73), (75), (76), (77) we get

$$\int\limits_{|x-x_{\text{left}}(E_0)|<c_\# B_{\text{left}}} |\text{Error}_2(x)|\ dx \leq C_*(\Lambda^{-N}+(\frac{\delta x}{B_{\text{left}}})^{N})\lambda_{\text{left}}^{-1}B_{\text{left}}^{2}S_{\min}\ , \quad \text{and} \tag{78}$$

(79)

$$\int\limits_{|x-x_{\text{left}}(E_{\max})|<(\delta x)} |\text{Error}_1(x)|dx \leq C_*(\Lambda^{-2}+(\frac{\delta x}{B_{\text{left}}})^{1/2}\Lambda^{-1})\lambda_{\text{left}}^{-1}B_{\text{left}}^{2}S_{\min}(\frac{\delta x}{B_{\text{left}}})^{1/2}$$
$$+C_*(\Lambda^{-N}+(\frac{\delta x}{B_{\text{left}}})^{N})\lambda_{\text{left}}^{-1}B_{\text{left}}^{2}S_{\min}\ .$$

Note that $(\frac{\delta x}{B_{\text{left}}})^N \leq \lambda_{\text{left}}^{-\varepsilon N} \leq \Lambda^{-N'}$ and $\Lambda^{-N} \leq \Lambda^{-N'}$. Equation (74) may be rewritten as

$$\begin{aligned}(80) \qquad \sum_{k \in \mathbb{Z} \cap [a,b]} f_{\text{Airey}}(x,k) \cdot f_{\text{other}}(x,k) = \int_a^b f_{\text{Airey}}(x,t) f_{\text{other}}(x,t)\ dt \\ - f_{\text{Airey}}(x,b) f_{\text{other}}(x,b) \chi_-(b) + \text{Error}_2(x) \\ \text{for} \quad |E_0| \geq 2\hat{c} S_{\min} \quad \text{or} \quad |x - x_{\text{left}}(E_{\max})| > \delta x\ ,\end{aligned}$$

since we know in that case that $f_{\text{Airey}}(x,t) f_{\text{other}}(x,t)$ vanishes to infinite order at $t = b$. From (72) and (80), we have

$$\begin{aligned}(81) \qquad \sum_{k \in \mathbb{Z} \cap [a,b]} f_{\text{Airey}}(x,k) f_{\text{other}}(x,k) = \int_a^b f_{\text{Airey}}(x,t) f_{\text{other}}(x,t)\ dt \\ - f_{\text{Airey}}(x,b) f_{\text{other}}(x,b) \chi_-(b) + \text{Error}_3(x)\ ,\end{aligned}$$

with $\text{Error}_3(x) = \text{Error}_1(x)$ for $|E_0| \leq 2\hat{c} S_{\min}$, $|x - x_{\text{left}}(E_{\max})| \leq \delta x$, and $\text{Error}_3(x) =$ $\text{Error}_2(x)$ otherwise.

Equations (78) and (79) yield

$$\begin{aligned}(82) \qquad \int_{I_{\text{BVP}}} |\text{Error}_3(x)|\ dx \leq C_*(\Lambda^{-2} + (\frac{\delta x}{B_{\text{left}}})^{1/2} \Lambda^{-1}) \lambda_{\text{left}}^{-1} B_{\text{left}}^2 S_{\min} (\frac{\delta x}{B_{\text{left}}})^{1/2} \\ + C_* \Lambda^{-N'} \lambda_{\text{left}}^{-1} B_{\text{left}}^2 S_{\min} \quad \text{if} \quad |E_0| \leq 2\hat{c} S_{\min}\ ,\end{aligned}$$

and

$$(83) \qquad \int_{I_{\text{BVP}}} |\text{Error}_3(x)| dx \leq C_* \Lambda^{-N'} \lambda_{\text{left}}^{-1} B_{\text{left}}^2 S_{\min}, \quad \text{if} \quad |E_0| > 2\hat{c} S_{\min}\ .$$

Here we used the fact that $\text{Error}_3(x)$ is supported in $\{|x - x_{\text{left}}(E_0)| < c_\# B_{\text{left}}\}$, which follows from (52), (81) and *ASSUMPTIONS* 1 and 3.

To make further progress, we need to estimate $\int_{I_{\text{BVP}}} |f_{\text{Airey}}(x,t) f_{\text{other}}(x,t)| dx$ for fixed $t \in \mathcal{J}$. Using (69) and the fact that the integrand is supported in $\{|x -$

$x_{\text{left}}(E(t))| < \delta x\}$, we get

$$\int\limits_{I_{\text{BVP}}} |f_{\text{Airey}}(x,t) f_{\text{other}}(x,t)|\ dx \le C_* \int\limits_{|x - x_{\text{left}}(E(t))| < \delta x} \sigma_x(t)\ dx$$

$$\le C_* \int\limits_{|x - x_{\text{left}}(E(t))| < \delta x} \lambda_{\text{left}}^{-2/3} B_{\text{left}}(S_{\min}\Gamma^{-1})$$

$$\cdot (1 + \lambda_{\text{left}}^{2/3} B_{\text{left}}^{-1} |x - x_{\text{left}}(E(t))|)^{-1/2}\ dx$$

$$\le C_* \lambda_{\text{left}}^{-2/3} B_{\text{left}}(S_{\min}\Gamma^{-1}) \lambda_{\text{left}}^{-1/3} B_{\text{left}}^{1/2} (\delta x)^{1/2}$$

(84)
$$= C_* \lambda_{\text{left}}^{-1} B_{\text{left}}^2 (S_{\min}\Gamma^{-1}) (\frac{\delta x}{B_{\text{left}}})^{1/2} \quad \text{for} \quad t \in \mathcal{J} \ .$$

Our first use of (84) is to change the region of integration in (81) from $[a,b]$ to

(85) $$[a_0, b_0] = [\frac{1}{\pi}\phi(E_{\ell o}) + \frac{1}{48\pi}\psi(E_{\ell o}) - \frac{1}{2}\ ,\ \frac{1}{\pi}\phi(E_{hi}) + \frac{1}{48\pi}\psi(E_{hi}) - \frac{1}{2}]\ .$$

From (3) and (85) we get

(86) $$|a - a_0|\ ,\ |b - b_0| \le C_\# \Lambda^{-2}\ ,$$

and therefore by (84) we have

(87)
$$\int\limits_{I_{\text{BVP}}} \left| \int_{b_0}^{b} f_{\text{Airey}}(x,t) f_{\text{other}}(x,t)\ dt \right| dx \le |b_0 - b| \cdot C_* \lambda_{\text{left}}^{-1} B_{\text{left}}^2 (S_{\min}\Gamma^{-1}) (\frac{\delta x}{B_{\text{left}}})^{1/2}$$

$$\le C_* \Lambda^{-2} \lambda_{\text{left}}^{-1} B_{\text{left}}^2 (S_{\min}\Gamma^{-1}) (\frac{\delta x}{B_{\text{left}}})^{1/2} \le C_* \Lambda^{-3} \lambda_{\text{left}}^{-1} B_{\text{left}}^2 S_{\min} (\frac{\delta x}{B_{\text{left}}})^{1/2}$$

by (12) .

Similarly

(88) $$\int\limits_{I_{\text{BVP}}} \left| \int_{a_0}^{a} f_{\text{Airey}}(x,t) f_{\text{other}}(x,t)\ dt \right| dx \le C_* \Lambda^{-3} \lambda_{\text{left}}^{-1} B_{\text{left}}^2 S_{\min} (\frac{\delta x}{B_{\text{left}}})^{1/2}\ .$$

(In fact, the left–hand side is zero, but never mind.)

Equations (81)…(83) and (87), (88) imply

(89)
$$\sum_{k\in\mathbb{Z}\cap[a,b]} f_{\text{Airey}}(x,k)f_{\text{other}}(x,k) = \int_{a_0}^{b_0} f_{\text{Airey}}(x,t)f_{\text{other}}(x,t)\ dt$$
$$-f_{\text{Airey}}(x,b)f_{\text{other}}(x,b)\chi_{-}(b) + \ \text{Error}_4(x)$$

with

(90)
$$\int_{I_{\text{BVP}}} |\text{Error}_4(x)|\ dx \le C_*(\Lambda^{-2} + (\frac{\delta x}{B_{\text{left}}})^{1/2}\Lambda^{-1})\lambda_{\text{left}}^{-1}B_{\text{left}}^2 S_{\min}(\frac{\delta x}{B_{\text{left}}})^{1/2}$$
$$+C_*\Lambda^{-N'}\lambda_{\text{left}}^{-1}B_{\text{left}}^2 S_{\min} \quad \text{if} \quad |E_0| \le 2\hat{c}S_{\min}\ , \quad \text{and}$$

(91)
$$\int_{I_{\text{BVP}}} |\text{Error}_4(x)|\ dx \le C_*\Lambda^{-3}\lambda_{\text{left}}^{-1}B_{\text{left}}^2 S_{\min}(\frac{\delta x}{B_{\text{left}}})^{1/2}$$
$$+C_*\Lambda^{-N'}\lambda_{\text{left}}^{-1}B_{\text{left}}^2 S_{\min} \quad \text{if} \quad |E_0| > 2\hat{c}S_{\min}\ .$$

In the integral in (89), we want to change variable from t to $E = E(t)$. This introduces a Jacobian factor $dt = [\frac{1}{\pi}\phi'(E) + \frac{1}{48\pi}\psi'(E)]\ dE$, in place of the desired factor $\frac{1}{\pi}\phi'(E)$. To control the resulting error terms, we again use (84) and argue as follows. For $t \in \mathcal{J}$, estimates (8), (9) give $|\frac{\psi'(E)}{\phi'(E)}| \le C_\#\Lambda^{-2}$ with $E = E(t)$, hence $|[1 + \frac{1}{48}\frac{\psi'(E)}{\phi'(E)}]^{-1} - 1| \le C_\#\Lambda^{-2}$. Therefore,

(92)
$$\int_{I_{\text{BVP}}} \Big| \int_{a_0}^{b_0} f_{\text{Airey}}(x,t)f_{\text{other}}(x,t)\ dt$$
$$- \int_{a_0}^{b_0} f_{\text{Airey}}(x,t)f_{\text{other}}(x,t)\big[1 + \frac{1}{48}\frac{\psi'(E(t))}{\phi'(E(t))}\big]^{-1}dt\Big|\ dx$$
$$\le C_\#\Lambda^{-2}\int_{a_0}^{b_0}\int_{I_{\text{BVP}}} |f_{\text{Airey}}f_{\text{other}}(x,t)|\ dxdt$$
$$\le C_*\Lambda^{-2}(b_0 - a_0)\lambda_{\text{left}}^{-1}B_{\text{left}}^2(S_{\min}\Gamma^{-1})(\frac{\delta x}{B_{\text{left}}})^{1/2} \quad \text{(by (84))}$$
$$\le C_*\Lambda^{-2}\lambda_{\text{left}}^{-1}B_{\text{left}}^2 S_{\min}(\frac{\delta x}{B_{\text{left}}})^{1/2}\ , \text{by (86) and the fact that } b - a \le C_\#\Gamma\ .$$

On the other hand, changing variable from t to $E = E(t)$ gives

(93)
$$\int_{a_0}^{b_0} f_{\text{other}}(x,t)f_{\text{Airey}}(x,t)\big[1 + \frac{1}{48}\frac{\psi'(E(t))}{\phi'(E(t))}\big]^{-1}\ dt$$
$$= \frac{1}{\pi}\int_{E_{\ell 0}}^{E_{hi}} g(x,E)\lambda_{\text{left}}^{-2/3}\Big(\frac{\partial Y_{\text{left}}(x,E)}{\partial x}\Big)^{-1} A^2(\lambda_{\text{left}}^{2/3}Y_{\text{left}}(x,E))\ dE\ .$$

Here we used (51), (52) and $dt = \frac{1}{\pi}(\phi'(E) + \frac{1}{48}\psi'(E))\ dE$ for $E = E(t)$ to write down the integrand on the right, and we used (85) for the limits of integration.

Next we show that the region of integration on the right of (93) can be changed from $[E_{\ell o}, E_{hi}]$ to $(-\infty, 0]$ without affecting the value of the integral. Since $E_{\ell o} = E_0 - c^1_{\#} S_{\min}$, *ASSUMPTION* 1 shows that $g(x,E) = 0$ for $E < E_{\ell o}$. So changing the region of integration from $[E_{\ell o}, E_{hi}]$ to $(-\infty, E_{hi}]$ leaves the integral unchanged on the right–hand side of (93). Similarly, if $|E_0| > 2\hat{c}S_{\min}$, then $E_{hi} = \min\{0, E_0 + c^1_{\#} S_{\min}\} > E_0 + \hat{c}S_{\min}$, so $E_{hi} \leq 0$, and $g(x,E) = 0$ for $E_{hi} \leq E \leq 0$.

Thus the upper limit of integration in (93) may be changed from E_{hi} to 0 without changing the integral, provided $|E_0| > 2\hat{c}S_{\min}$. On the other hand, if $|E_0| \leq 2\hat{c}S_{\min}$, then already $E_{hi} = 0$ so there is nothing to prove. Hence, (93) may be rewritten as

$$(94)\quad \int_{a_0}^{b_0} f_{\text{other}}(x,t) f_{\text{Airey}}(x,t) \big[1 + \frac{1}{48}\frac{\psi'(E(t))}{\phi'(E(t))}\big]^{-1}\ dt$$
$$= \frac{1}{\pi}\int_{-\infty}^{0} g(x,E)\lambda_{\text{left}}^{-2/3}\Big(\frac{\partial Y_{\text{left}}(x,E)}{\partial x}\Big)^{-1} A^2(\lambda_{\text{left}}^{2/3} Y_{\text{left}}(x,E))\ dE\ .$$

From (92) and (94) we get

$$(95)\quad \int_{a_0}^{b_0} f_{\text{other}}(x,t) f_{\text{Airey}}(x,t)\ dt =$$
$$\frac{1}{\pi}\int_{-\infty}^{0} g(x,E)\lambda_{\text{left}}^{-2/3}\Big(\frac{\partial Y_{\text{left}}(x,E)}{\partial x}\Big)^{-1} A^2(\lambda_{\text{left}}^{2/3} Y_{\text{left}}(x,E))\ dE + \text{ Error}_5(x)$$

with

$$(96)\qquad \int_{I_{\text{BVP}}} |\text{Error}_5(x)| dx \leq C_* \Lambda^{-2} \lambda_{\text{left}}^{-1} B_{\text{left}}^2 S_{\min} \Big(\frac{\delta x}{B_{\text{left}}}\Big)^{1/2}\ .$$

Equations (89)...(91) and (95), (96) imply

$$(97)\quad \sum_{k \in \mathbb{Z}\cap[a,b]} f_{\text{Airey}}(x,k) f_{\text{other}}(x,k) =$$
$$\frac{1}{\pi}\int_{-\infty}^{0} g(x,E)\lambda_{\text{left}}^{-2/3}\Big(\frac{\partial Y_{\text{left}}(x,E)}{\partial x}\Big)^{-1} A^2(\lambda_{\text{left}}^{2/3} Y_{\text{left}}(x,E))\ dE$$
$$- f_{\text{Airey}}(x,b) f_{\text{other}}(x,b)\chi_-(b) + \text{ Error}_6(x)$$

with

(98)
$$\int_{I_{\rm BVP}} |\mathrm{Error}_6(x)|\ dx \le C_*(\Lambda^{-2} + (\frac{\delta x}{B_{\rm left}})^{1/2}\Lambda^{-1})\lambda_{\rm left}^{-1}B_{\rm left}^2 S_{\rm min}(\frac{\delta x}{B_{\rm left}})^{1/2}$$
$$+C_*\Lambda^{-N'}\lambda_{\rm left}^{-1}B_{\rm left}^2 S_{\rm min} \quad \text{if} \quad |E_0| \le 2\hat{c}S_{\rm min}\ , \quad \text{and}$$

(99)
$$\int_{I_{\rm BVP}} |\mathrm{Error}_6(x)|\ dx \le C_*\Lambda^{-2}\lambda_{\rm left}^{-1}B_{\rm left}^2 S_{\rm min}(\frac{\delta x}{B_{\rm left}})^{1/2}$$
$$+C_*\Lambda^{-N'}\lambda_{\rm left}^{-1}B_{\rm left}^2 S_{\rm min} \quad \text{if} \quad |E_0| > 2\hat{c}S_{\rm min}\ .$$

In equation (97) we want to change $f_{\rm Airey}(x,b)f_{\rm other}(x,b)\chi_-(b)$ to $f_{\rm Airey}(x,b_0)f_{\rm other}(x,b_0)$ $\cdot\,\chi_-(b)$. To do so, we estimate for fixed $t \in \mathcal{J}$:

$$\int_{I_{\rm BVP}} \Big|\frac{\partial}{\partial t}\{f_{\rm Airey}(x,t)f_{\rm other}(x,t)\}\Big|\ dx \le C_* \int_{|x-x_{\rm\ left}(E(t))|<\delta x} \sigma_x(t)\tau^{-1}\ dx$$
$$\le C_*(\Lambda^{-2} + (\frac{\delta x}{B_{\rm left}})^{1/2}\Lambda^{-1})\lambda_{\rm left}^{-1}B_{\rm left}^2 S_{\rm min}(\frac{\delta x}{B_{\rm left}})^{1/2}$$

by the proof of (76) with $E(t)$ in place of $E_{\rm max}$.

Since $|b-b_0| < C_\#\Lambda^{-2}$ by (86), this implies

(100)
$$\int_{I_{\rm BVP}} \Big|f_{\rm Airey}(x,b)f_{\rm other}(x,b) - f_{\rm Airey}(x,b_0)f_{\rm other}(x,b_0)\Big|\ dx$$
$$\le C_*\Big(\Lambda^{-4} + (\frac{\delta x}{B_{\rm left}})^{1/2}\Lambda^{-3}\Big)\lambda_{\rm left}^{-1}B_{\rm left}^2 S_{\rm min}(\frac{\delta x}{B_{\rm left}})^{1/2}\ .$$

We use (100) if $|E_0| \le 2\hat{c}S_{\rm min}$. If instead $|E_0| > 2\hat{c}S_{\rm min}$, then $f_{\rm other}(x,b)f_{\rm Airey}(x,b)$ $\equiv 0$ by (71 bis). Also, $|E_0| > 2\hat{c}S_{\rm min}$ implies $E(b_0) = E_{hi} = \min(0, E_0 + c_\#^1 S_{\rm min})$ $> E_0 + \hat{c}S_{\rm min}$, so that $g(x, E(b_0)) \equiv 0$ by *ASSUMPTION* 1, and hence $f_{\rm other}(x,b_0) \equiv 0$. Thus

(101)
$$f_{\rm Airey}(x,b)f_{\rm other}(x,b) = f_{\rm Airey}(x,b_0)f_{\rm other}(x,b_0) = 0$$
$$\text{for all } x \in I_{\rm BVP},\ \text{if } |E_0| > 2\hat{c}S_{\rm min}\ .$$

Putting (100) and (101) into (97)...(99) and noting that $E(b_0) = E_{hi} = 0$ if $|E_0| \le 2\hat{c}S_{\min}$, we derive the following conclusion

(102)
$$\sum_{k\in\mathbb{Z}\cap[a,b]} f_{\text{Airey}}(x,k) f_{\text{other}}(x,k)$$
$$= \frac{1}{\pi}\int_{-\infty}^{0} g(x,E)\lambda_{\text{left}}^{-2/3}\Big(\frac{\partial Y_{\text{left}}(x,E)}{\partial x}\Big)^{-1} A^2(\lambda_{\text{left}}^{2/3} Y_{\text{left}}(x,E))\; dE$$
$$- g(x,0)\lambda_{\text{left}}^{-2/3}\Big(\frac{\partial Y_{\text{left}}(x,0)}{\partial x}\Big)^{-1} A^2(\lambda_{\text{left}}^{2/3} Y_{\text{left}}(x,0))[\phi'(0)]^{-1}\chi_-(b)$$
$$+ \text{ Error}_7(x) \text{ , with}$$

(103)
$$\int_{I_{\text{BVP}}} |\text{Error}_7(x)|\; dx \le C_*\Big(\Lambda^{-2} + \big(\frac{\delta x}{B_{\text{left}}}\big)^{1/2}\Lambda^{-1}\Big)\lambda_{\text{left}}^{-1} B_{\text{left}}^2 S_{\min}\big(\frac{\delta x}{B_{\text{left}}}\big)^{1/2}$$
$$+ C_*\Lambda^{-N'}\lambda_{\text{left}}^{-1} B_{\text{left}}^2 S_{\min} \quad \text{if} \quad |E_0| \le 2\hat{c}S_{\min} \text{ ,} \quad \text{and}$$

(104)
$$\int_{I_{\text{BVP}}} |\text{Error}_7(x)|\; dx \le C_*\Lambda^{-2}\lambda_{\text{left}}^{-1} B_{\text{left}}^2 S_{\min}\big(\frac{\delta x}{B_{\text{left}}}\big)^{1/2}$$
$$+ C_*\Lambda^{-N'}\lambda_{\text{left}}^{-1} B_{\text{left}}^2 S_{\min} \quad \text{if} \quad |E_0| > 2\hat{c}S_{\min} \text{ .}$$

Here we used also the fact that $g(x,0) \equiv 0$ if $|E_0| > 2\hat{c}S_{\min}$. In particular, the expression $g(x,0)\lambda_{\text{left}}^{-2/3}\big(\frac{\partial Y_{\text{left}}(x,0)}{\partial x}\big)^{-1} A^2(\lambda_{\text{left}}^{2/3} Y_{\text{left}}(x,0))[\phi'(0)]^{-1}\chi_-(b)$ is defined to be zero whenever $g(x,0) \equiv 0$. We make this remark because eg. $Y_{\text{left}}(x,0)$ needn't be well-defined when $|E_0| > c_{\#}S_{\min}$. In (102), we want to replace $\chi_-(b)$ by $\chi_-(\frac{1}{\pi}\phi(0) - \frac{1}{2})$. If $|E_0| \le 2\hat{c}S_{\min}$, then $E(b_0) = E_{hi} = 0$ as we just noted, and therefore $b_0 = \frac{1}{\pi}\phi(0) + \frac{1}{48\pi}\psi(0) - \frac{1}{2}$ by (85), so the estimates for $\psi(E)$ in the WKB Theorems imply $|b_0 - (\frac{1}{\pi}\phi(0) - \frac{1}{2})| < C_{\#}\Lambda^{-1}$. By (86) we have also $|b - (\frac{1}{\pi}\phi(0) - \frac{1}{2})| \le C_{\#}\Lambda^{-1}$. This implies

(105)
$$\big|\chi_-(b) - \chi_-(\frac{1}{\pi}\phi(0) - \frac{1}{2})\big| \le C_{\#}\Lambda^{-1}$$
$$\text{if} \quad \min_{k\in\mathbb{Z}} |\phi(0) - \pi(k+1/2)| > \overline{C}_{\#}\Lambda^{-1} \text{ and } |E_0| \le 2\hat{c}S_{\min} \text{ .}$$

If $\min_{k\in\mathbb{Z}} |\phi(0) - \pi(k+1/2)| \le \overline{C}_{\#}\Lambda^{-1}$, then we know only that

(105 bis)
$$|\chi_-(b) - \chi_-(\frac{1}{\pi}\phi(0) - \frac{1}{2})| \le C_{\#} \text{ .}$$

Under the assumptions of (105), we have

$$(106)\quad \int_{I_{\rm BVP}} \Big|[\chi_-(b)-\chi_-(\frac{1}{\pi}\phi(0)-\frac{1}{2})]g(x,0)\lambda_{\rm left}^{-2/3}(\frac{\partial Y_{\rm left}(x,0)}{\partial x})^{-1} \cdot A^2(\lambda_{\rm left}^{2/3}Y_{\rm left}(x,0))[\phi'(0)]^{-1}\Big|\ dx$$

$$= |\chi_-(b)-\chi_-(\frac{1}{\pi}\phi(0)-\frac{1}{2})| \int_{I_{\rm BVP}} |f_{\rm Airey}(x,b_0)f_{\rm other}(x,b_0)|\ dx$$

$$\le C_*\Lambda^{-1}\lambda_{\rm left}^{-1}B_{\rm left}^2(S_{\rm min}\Gamma^{-1})(\frac{\delta x}{B_{\rm left}})^{1/2} \quad \text{(by (84) and (105))}$$

$$\le C_*\Lambda^{-2}\lambda_{\rm left}^{-1}B_{\rm left}^2 S_{\rm min}(\frac{\delta x}{B_{\rm left}})^{1/2} \quad \text{by (12) .}$$

Estimate (106) holds when $|E_0| \le 2\hat{c}S_{\rm min}$ and $\min_{k\in\mathbb{Z}}|\phi(0)-\pi(k+1/2)| > \overline{C}_{\#}\Lambda^{-1}$. If instead $|E_0| \le 2\hat{c}S_{\rm min}$ and $\min_{k\in\mathbb{Z}}|\phi(0)-\pi(k+1/2)| \le \overline{C}_{\#}\Lambda^{-1}$, then instead of (105) we can use merely (105 bis), so the proof of (106) gives only

(107)

$$\int_{I_{\rm BVP}} \Big|[\chi_-(b)-\chi_-(\frac{1}{\pi}\phi(0)-\frac{1}{2})]g(x,0)\lambda_{\rm left}^{-2/3}(\frac{\partial Y_{\rm left}(x,0)}{\partial x})^{-1}A^2(\lambda_{\rm left}^{2/3}Y_{\rm left}(x,0))[\phi'(0)]^{-1}\Big|\ dx$$

$$\le C_*\Lambda^{-1}\lambda_{\rm left}^{-1}B_{\rm left}^2 S_{\rm min}(\frac{\delta x}{B_{\rm left}})^{1/2} . \blacksquare$$

Finally, if $|E_0| > 2\hat{c}S_{\rm min}$, then $g(x,0)\equiv 0$ so obviously we can replace $\chi_-(b)$ by $\chi_-(\frac{1}{\pi}\phi(0)-\frac{1}{2})$ in (102). Combining these observations with (102)…(104), we obtain the following.

$$(108)\quad \sum_{k\in\mathbb{Z}\cap[a,b]} f_{\rm Airey}(x,k)f_{\rm other}(x,k)$$

$$= \frac{1}{\pi}\int_{-\infty}^{0} g(x,E)\lambda_{\rm left}^{-2/3}(\frac{\partial Y_{\rm left}(x,E)}{\partial x})^{-1}A^2(\lambda_{\rm left}^{2/3}Y_{\rm left}(x,E))\ dE$$

$$- g(x,0)\lambda_{\rm left}^{-2/3}(\frac{\partial Y_{\rm left}(x,0)}{\partial x})^{-1}A^2(\lambda_{\rm left}^{2/3}Y_{\rm left}(x,0))[\phi'(0)]^{-1}\chi_-(\frac{1}{\pi}\phi(0)-\frac{1}{2})$$

$$+\ {\rm Error}_8(x)\ , \quad \text{where :}$$

If $|E_0| \le 2\hat{c}S_{\rm min}$ and $\min_{k\in\mathbb{Z}}|\phi(0)-\pi(k+1/2)| > \overline{C}_{\#}\Lambda^{-1}$ then

(109)

$$\int_{I_{\rm BVP}} |{\rm Error}_8(x)|\ dx \le C_*(\Lambda^{-2}+(\frac{\delta x}{B_{\rm left}})^{1/2}\Lambda^{-1})\lambda_{\rm left}^{-1}B_{\rm left}^2 S_{\rm min}(\frac{\delta x}{B_{\rm left}})^{1/2}$$

$$+C_*\Lambda^{-N'}\lambda_{\rm left}^{-1}B_{\rm left}^2 S_{\rm min} .$$

If $|E_0| \leq 2\hat{c}S_{\min}$ and $\min_{k\in\mathbb{Z}} |\phi(0) - \pi(k+1/2)| \leq \overline{C}_{\#}\Lambda^{-1}$, then

$$\int_{I_{\mathrm{BVP}}} |\mathrm{Error}_8(x)|\, dx \leq C_*\Lambda^{-1}\lambda_{\mathrm{left}}^{-1}B_{\mathrm{left}}^2 S_{\min}\Big(\frac{\delta x}{B_{\mathrm{left}}}\Big)^{1/2} + C_*\Lambda^{-N'}\lambda_{\mathrm{left}}^{-1}B_{\mathrm{left}}^2 S_{\min}\ . \tag{110}$$

If $|E_0| \geq 2\hat{c}S_{\min}$, then

$$\int_{I_{\mathrm{BVP}}} |\mathrm{Error}_8(x)|\, dx \leq C_*\Lambda^{-2}\lambda_{\mathrm{left}}^{-1}B_{\mathrm{left}}^2 S_{\min}\Big(\frac{\delta x}{B_{\mathrm{left}}}\Big)^{1/2} + C_*\Lambda^{-N'}\lambda_{\mathrm{left}}^{-1}B_{\mathrm{left}}^2 S_{\min}\ . \tag{111}$$

The left–hand side of (108) is equal to the sum in (49), as one sees from (50). Therefore, (49) and (108)...(111) combine to prove the following estimates.

$$\rho(x,g) = \frac{1}{\pi}\int_{-\infty}^{0} g(x,E)\lambda_{\mathrm{left}}^{-2/3}\Big(\frac{\partial Y_{\mathrm{left}}(x,E)}{\partial x}\Big)^{-1} A^2(\lambda_{\mathrm{left}}^{2/3}Y_{\mathrm{left}}(x,E))\, dE - g(x,0)\lambda_{\mathrm{left}}^{-2/3}\Big(\frac{\partial Y_{\mathrm{left}}(x,0)}{\partial x}\Big)^{-1} A^2(\lambda_{\mathrm{left}}^{2/3}Y_{\mathrm{left}}(x,0))[\phi'(0)]^{-1}\chi_-\Big(\frac{1}{\pi}\phi(0)-\frac{1}{2}\Big) + \mathrm{Error}_9(x), \quad \text{where :} \tag{112}$$

If $|E_0| \leq 2\hat{c}S_{\min}$ and $\min_{k\in\mathbb{Z}} |\phi(0) - \pi(k+1/2)| > \overline{C}_{\#}\Lambda^{-1}$, then

$$\int_{I_{\mathrm{BVP}}} |\mathrm{Error}_9(x)|\, dx \leq C_*\Big(\Lambda^{4\varepsilon-2} + \Big(\frac{\delta x}{B_{\mathrm{left}}}\Big)^{1/2}\Lambda^{-1}\Big)\lambda_{\mathrm{left}}^{-1}B_{\mathrm{left}}^2 S_{\min}\Big(\frac{\delta x}{B_{\mathrm{left}}}\Big)^{1/2} + C_*\Lambda^{-N'}\lambda_{\mathrm{left}}^{-1}B_{\mathrm{left}}^2 S_{\min} + C_*\Lambda^{10-N''}\Gamma\ . \tag{113}$$

If $|E_0| \leq 2\hat{c}S_{\min}$ and $\min_{k\in\mathbb{Z}} |\phi(0) - \pi(k+1/2)| \leq \overline{C}_{\#}\Lambda^{-1}$, then

$$\int_{I_{\mathrm{BVP}}} |\mathrm{Error}_9(x)|\, dx \leq C_*\Lambda^{-1}\lambda_{\mathrm{left}}^{-1}B_{\mathrm{left}}^2 S_{\min}\Big(\frac{\delta x}{B_{\mathrm{left}}}\Big)^{1/2} + C_*\Lambda^{-N'}\lambda_{\mathrm{left}}^{-1}B_{\mathrm{left}}^2 S_{\min} + C_*\Lambda^{10-N''}\Gamma\ . \tag{114}$$

If $|E_0| > 2\hat{c}S_{\min}$, then

$$\int_{I_{\mathrm{BVP}}} |\mathrm{Error}_9(x)|\, dx \leq C_*\Lambda^{4\varepsilon-2}\lambda_{\mathrm{left}}^{-1}B_{\mathrm{left}}^2 S_{\min}\Big(\frac{\delta x}{B_{\mathrm{left}}}\Big)^{1/2} + C_*\Lambda^{-N'}\lambda_{\mathrm{left}}^{-1}B_{\mathrm{left}}^2 S_{\min} + C_*\Lambda^{10-N''}\Gamma\ . \tag{115}$$

Equations (113)...(115) simplify, because

$$\Lambda^{10-N''}\Gamma \geq c_\# \Lambda^{10-N''} S_{\min} S_{\text{left}}^{-1/2} B_{\text{left}} \quad (\text{by } (11)) \qquad = c_\# \Lambda^{10-N''} \lambda_{\text{left}}^{-1} B_{\text{left}}^2 S_{\min}$$
$$> \Lambda^{-N'} \lambda_{\text{left}}^{-1} B_{\text{left}}^2 S_{\min} \ .$$

Hence the term $C_* \Lambda^{-N'} \lambda_{\text{left}}^{-1} B_{\text{left}}^2 S_{\min}$ may be deleted from the right–hand side of (113)...(115).

Also, it will be convenient to note that $\Gamma \leq C_* \int_{E_0 - \hat{c} S_{\min}}^{E_0} \phi'(E)\, dE$ (by (9)), and to substitute this in (113) ...(115). Thus we arrive at the main result of this section.

Lemma. *Suppose $V(x)$, $S(x)$, $B(x)$, $g(x,E)$ satisfy the hypotheses of the WKB Theorems with $E_\infty = 0$, as well as ASSUMPTIONS 1...5 above. Then the microlocalized density for $g(x,E)$ is given by*

$$\rho(x,g) = \frac{1}{\pi} \int_{-\infty}^{0} g(x,E) \lambda_{\text{left}}^{-2/3} \Big(\frac{\partial Y_{\text{left}}(x,E)}{\partial x}\Big)^{-1} A^2(\lambda_{\text{left}}^{2/3} Y_{\text{left}}(x,E))\, dE$$
$$- g(x,0) \lambda_{\text{left}}^{-2/3} \Big(\frac{\partial Y_{\text{left}}(x,0)}{\partial x}\Big)^{-1} A^2(\lambda_{\text{left}}^{2/3} Y_{\text{left}}(x,0)) [\phi'(0)]^{-1} \chi_-\Big(\frac{1}{\pi}\phi(0) - \frac{1}{2}\Big)$$
$$+ \ \text{Error}(x) \ ,$$

where λ_{left}, $Y_{\text{left}}(x,E)$, $\phi(E)$ are as in the WKB Theorems, and Error(x) *satisfies the following estimates.*

(a) *If $|E_0| > 2\hat{c} S_{\min}$, then*

$$\int_{I_{\text{BVP}}} |\text{Error}(x)|\, dx \leq C_* \Lambda^{4\varepsilon - 2} \lambda_{\text{left}}^{-1} B_{\text{left}}^2 S_{\min} \Big(\frac{\delta x}{B_{\text{left}}}\Big)^{1/2}$$
$$+ C_* \Lambda^{10-N''} \int_{E_0 - \hat{c} S_{\min}}^{E_0} \phi'(E)\, dE \ .$$

(b) *If $|E_0| \leq 2\hat{c} S_{\min}$ and $\min_{k \in \mathbb{Z}} |\phi(0) - \pi(k+1/2)| > \overline{C}_\# \Lambda^{-1}$, then*

$$\int_{I_{\text{BVP}}} |\text{Error}(x)|\, dx \leq C_* \Big(\Lambda^{4\varepsilon - 2} + \Big(\frac{\delta x}{B_{\text{left}}}\Big)^{1/2} \Lambda^{-1}\Big) \lambda_{\text{left}}^{-1} B_{\text{left}}^2 S_{\min} \Big(\frac{\delta x}{B_{\text{left}}}\Big)^{1/2}$$
$$+ C_* \Lambda^{10-N''} \int_{E_0 - \hat{c} S_{\min}}^{E_0} \phi'(E)\, dE \ .$$

(c) *If* $|E_0| \leq 2\hat{c}S_{\min}$*, and* $\min_{k\in\mathbb{Z}} |\phi(0) - \pi(k+1/2)| \leq \overline{C}_{\#}\Lambda^{-1}$*, then*

$$\int_{I_{\mathrm{BVP}}} |\mathrm{Error}(x)|\ dx \leq C_* \Lambda^{-1} \lambda_{\mathrm{left}}^{-1} B_{\mathrm{left}}^2 S_{\min} \big(\frac{\delta x}{B_{\mathrm{left}}}\big)^{1/2} + C_* \Lambda^{10-N''} \int_{E_0 - \hat{c}S_{\min}}^{E_0} \phi'(E)\ dE\ .$$

For the definition of $S_{\min}$*, see the section on the WKB Theorems. The constant* $\overline{C}_{\#}$ *depends only on* ε*,* K*,* N*,* c*,* C*,* c_1*,* c_2*,* C_α *in the hypotheses of the WKB Theorems. The constant* C_* *depends only on* ε*,* K*,* N*,* c*,* C*,* c_1*,* c_2*,* C_α *in the hypotheses of the WKB Theorems, and on* $\hat{c}$*,* $\hat{C}_\beta$ *in ASSUMPTIONS 1 and 2.*

Remarks.

1. The term containing $g(x,0)$ in the above lemma is defined to be identically zero whenever $|E_0| \geq 2\hat{c}S_{\min}$.
2. For typical $g(x,E)$ satisfying *ASSUMPTIONS* 1…5, we have

$$\int_{I_{\mathrm{BVP}}} \Big| \int_{-\infty}^{0} g(x,E)(E - V(x))_+^{-1/2}\ dE \Big|\ dx \sim \lambda_{\mathrm{left}}^{-1} B_{\mathrm{left}}^2 S_{\min} \big(\frac{\delta x}{B_{\mathrm{left}}}\big)^{1/2}\ .$$

Thus in inequality (a) above, the Error has L^1–norm roughly $\Lambda^{4\varepsilon-2}$ as large as the L^1–norm of the main term. In (b) the factor is $\Lambda^{4\varepsilon-2} + (\frac{\delta x}{B_{\mathrm{left}}})^{1/2}\Lambda^{-1}$, and in (c) the factor is Λ^{-1}.

THE MICROLOCALIZED DENSITY IN THE AIREY REGION II

Set-Up. We are given a potential $V(x)$ defined on a (possibly unbounded) interval I_{BVP}. On a subinterval $I \subset I_{\text{BVP}}$ we are given positive functions $S(x)$, $B(x)$. We are given positive numbers ε, K, N with $\varepsilon < \frac{1}{100}$ and $K > 100$, $N > K/\varepsilon^{20}$. Set $N' = [\varepsilon N/500]$ and $N'' = \frac{3}{2}\varepsilon N' - 100K - 33$. We are given a function $g(x, E)$ defined on all of $\mathbb{R}^2$. We are given numbers E_0 and δE with $E_0 \leq 0$ and $\delta E > 0$.

Define $H = -\frac{d^2}{dx^2} + V(x)$ on I_{BVP}, with Dirichlet or Neumann boundary conditions. We make the following assumptions.

Assumptions on $V(x)$, $S(x)$, $B(x)$ on I.

(X0) If $x, y \in I$ and $|x - y| < cB(x)$, then $c < \frac{B(y)}{B(x)} < C$ and $c < \frac{S(y)}{S(x)} < C$.

(X1) If $x \in I$ and $\alpha \geq 0$, then $|(\frac{d}{dx})^\alpha V(x)| \leq C_\alpha S(x) B^{-\alpha}(x)$.

(X2) For $E \in \mathcal{J} = [E_0 - \delta E, E_0 + \delta E] \cap (-\infty, 0]$, the equation $V(x) = E$ has two solutions $x_{\text{left}}(E) < x_{\text{rt}}(E)$ in I, and they satisfy $\text{dist}(x_{\text{left}}(E), \partial I) > cB(x_{\text{left}}(E))$, $\text{dist}(x_{\text{rt}}(E), \partial I) > cB(x_{\text{rt}}(E))$.

(X3) For $E \in \mathcal{J}$ and $x \in [x_{\text{left}}(E), x_{\text{left}}(E) + c_1 B(x_{\text{left}}(E))]$ we have $-V'(x) > cS(x)B^{-1}(x)$. Similarly, for $E \in \mathcal{J}$ and $x \in [x_{\text{rt}}(E) - c_1 B(x_{\text{rt}}(E)), x_{\text{rt}}(E)]$ we have $+V'(x) > cS(x)B^{-1}(x)$.

(X4) For $E \in \mathcal{J}$ and $x \in [x_{\text{left}}(E) + c_1 B(x_{\text{left}}(E)), x_{\text{rt}}(E) - c_1 B(x_{\text{rt}}(E))]$ we have $cS(x) < E - V(x) < CS(x)$.

(X5) For $E \in \mathcal{J}$ we have $c(\delta E) < S(x_{\text{left}}(E)) < C(\delta E)$

To state the remaining hypotheses, we set up some notation. Set $\lambda(x) = S^{1/2}(x)B(x)$ as usual, and let

$$B_{\text{left}}(E) = B(x_{\text{left}}(E)) \ , \ B_{\text{rt}}(E) = B(x_{\text{rt}}(E)) \ ,$$

$$S_{\text{left}}(E) = S(x_{\text{left}}(E)) \ , \ S_{\text{rt}}(E) = S(x_{\text{rt}}(E)) \ ,$$

$$\lambda_{\text{left}}(E) = \lambda(x_{\text{left}}(E)) \ , \ \lambda_{\text{rt}}(E) = \lambda(x_{\text{rt}}(E)) \quad \text{for } E \in \mathcal{J} \ .$$

Then define $S_{\min}(E) = \inf_{x_{\mathrm{left}}(E) \le x \le x_{\mathrm{rt}}(E)} S(x)$ and

$$\Lambda(E) = \Big(\int_{x_{\mathrm{left}}(E)}^{x_{\mathrm{rt}}(E)} \frac{dx}{S^{1/2}(x) B^2(x)} \Big)^{-1} , \quad \text{for } E \in \mathcal{J} .$$

Set $\Lambda_{\min} = \inf_{E \in \mathcal{J}} \Lambda(E)$.

Now we can state the rest of our assumptions.

Assumptions on $V(x)$ in all of I_{BVP}.

(X6) For $E \in \mathcal{J}$ and $x \in I_{\mathrm{BVP}} \backslash [x_{\mathrm{left}}(E), x_{\mathrm{rt}}(E)]$ we have $V(x) > E$.

(X7) For $E \in \mathcal{J}$ and $x \in I_{\mathrm{BVP}}$, we have

$$V(x) \ge \frac{100}{|x - x_{\mathrm{left}}(E)|^2} \quad \text{if } x < x_{\mathrm{left}}(E) - \frac{1}{2} \lambda_{\mathrm{left}}^K(E) B_{\mathrm{left}}(E) \ , \ \text{and}$$
$$V(x) \ge \frac{100}{|x - x_{\mathrm{rt}}(E)|^2} \quad \text{if } x > x_{\mathrm{rt}}(E) + \frac{1}{2} \lambda_{\mathrm{rt}}^K(E) B_{\mathrm{rt}}(E) \ .$$

Polynomial Growth Conditions on $S(x)$, $B(x)$.

(X8) We have

$$\max_{x \in I} B(x) < \Lambda_{\min}^K \min_{x \in I} B(x) \ ,$$
$$\max_{x \in I} S(x) < \Lambda_{\min}^K \min_{x \in I} S(x) \ ,$$
$$|I| \le \Lambda_{\min}^K \min_{x \in I} B(x) \ .$$

Assumptions on $g(x, E)$.

(X9) $\mathrm{supp}\, g(x, E) \subset \{ |E - E_0| < \hat{c}(\delta E), \ |x - x_{\mathrm{left}}(E)| \le (\delta x) \}$,
with $\lambda_{\mathrm{left}}^{-\frac{2}{3} + 2\varepsilon}(E_0) B_{\mathrm{left}}(E_0) < (\delta x) < \frac{1}{20} \lambda_{\mathrm{left}}^{-2\varepsilon}(E_0) B_{\mathrm{left}}(E_0)$.

(X10) $|\partial_x^\alpha \partial_E^\beta g(x, E)| \le \hat{C}_{\alpha\beta} (\delta x)^{-\alpha} \left[\frac{S_{\mathrm{left}}(E_0)(\delta x)}{B_{\mathrm{left}}(E_0)} \right]^{-\beta}$ for all x, E.

(X11) The constant $\hat{c}$ in (X9) is bounded above by a certain small positive number determined by ε, K, N, c, c_1, C_α, C in (X0)…(X8) above.

The WKB Hypothesis.

(X12) $\Lambda_{\min}$ is bounded below by a certain large positive number determined by ε, K, N, c, c_1, C_α, C, $\hat{c}$, $\hat{C}_{\alpha\beta}$ in (X0)...(X10) above.

Under these hypotheses, we will use the results of the previous section to compare the microlocalized density $\rho(x, g)$ with its semiclassical approximation.

We write $c_\#$, $C_\#$, $C^\alpha_\#$ etc. for constants that depend only on ε, K, N, c, C, c_1, C_α; and we write C_*, c_*, C^α_* etc. for constants that depend also on $\hat{c}$, $\hat{C}_{\alpha\beta}$.

Define $\mathcal{J}^0 = [E_0 - c_\# \delta E, E_0 + c_\# \delta E] \cap (-\infty, 0]$ for a small constant $c_\#$. Our present assumptions easily imply that for any $E'_0 \in \mathcal{J}^0$, the hypotheses of the WKB Theorems are satisfied, with E'_0 in place of E_0, with $E_\infty = 0$, and with $100K$ in place of K. However, our present assumptions on $g(x, E)$ are different from the assumptions imposed on g in the previous section. Our current $g(x, E)$ is smoother and has larger support than the g considered there. The hypotheses in the previous section were appropriate for applying the WKB Theorems. Our present hypotheses are natural for the study of integrals of squares of Airey functions, such as that in the Lemma of the previous section. The first step in our analysis will be to write $g(x, E)$ as a sum of pieces $g_\nu(x, E)$ with small supports, to which we can apply the Lemma of the previous section. This allows us to approximate $\rho(x, E)$ closely by an integral involving the square of the Airey function. Most of the work in this section is devoted to comparing that integral with the semiclassical density $\rho_{sc}(x, g)$.

Note that $x_{\text{left}}(E)$ and $x_{\text{rt}}(E)$ play different rôles in the hypotheses. This is caused by our desire to study $g(x, E)$ supported as in (X9), which is inherently asymmetrical.

Let us start by cutting $g(x, E)$ into suitable $g_\nu(x, E)$. We use the following result.

Lemma 1. *For all $\hat{c}' < c_\#$ we can produce a sequence of functions $\theta_\nu(E)$ and energies E'_ν with the following properties.*

(a) $\sum_\nu \theta_\nu(E) = 1 \quad \textit{for} \quad E \in \mathcal{J}^0.$

(b) *Each $\theta_\nu(E)$ is supported in $\{|E - E'_\nu| \le \hat{c}' S_{\min}(E'_\nu)\} \equiv \mathcal{J}''_\nu$, and each $E \in \mathbb{R}^1$ belongs to at most $C_\#[\hat{c}']^{-2}$ of the $\mathcal{J}''_\nu$.*

(c) *Each $\theta_\nu(E)$ satisfies $\left|\left(\frac{d}{dE}\right)^\beta \theta_\nu(E)\right| \le C^\beta_\#([\hat{c}']^2 S_{\min}(E'_\nu))^{-\beta}$.*

(d) *We have $|E'_\nu| \le 2\hat{c}' S_{\min}(E'_\nu)$ for at most $C_\#[\hat{c}']^{-2}$ of the E'_ν.*

(e) *Each E'_ν belongs to $\mathcal{J}^0$.*

Sketch of Proof. The point is that if $E_1, E_0 \in \mathcal{J}^0$ with $|E_1 - E_2| < c_\# S_{\min}(E_1)$ then $c_\# < \frac{S_{\min}(E_2)}{S_{\min}(E_1)} < C_\#$. This implies that $\mathcal{J}^0$ can be partitioned into subintervals $\mathcal{J}_\mu$ with the following properties. If $E \in \mathcal{J}^0$ belongs to the double of $\mathcal{J}_\mu$, then length$(\mathcal{J}_\mu) \sim S_{\min}(E)$. Each point $E \in \mathbb{R}^1$ belongs to at most $C_\#$ of the doubles of the $\mathcal{J}_\mu$.

Next, pick an integer $M \sim (\hat{c}')^{-2}$. Then cut each $\mathcal{J}_\mu$ into M equal subintervals, to partition $\mathcal{J}^0$ into smaller subintervals $\mathcal{J}'_\nu$, with the following properties.

If $E \in \mathcal{J}^0$ belongs to the double of $\mathcal{J}'_\nu$, then length $(\mathcal{J}'_\nu) \sim \frac{S_{\min}(E)}{M}$. Consequently, if the doubles of $\mathcal{J}'_{\nu_1}$ and $\mathcal{J}'_{\nu_2}$ meet, then length $\mathcal{J}'_{\nu_1} \sim$ length $\mathcal{J}'_{\nu_2}$. Also, each point $E \in \mathbb{R}^1$ belongs to at most $C_\#$ of the doubles of the $\mathcal{J}'_\nu$.

Now take $\chi_\nu(E)$ supported in the double of $\mathcal{J}'_\nu$, equal to 1 on $\mathcal{J}'_\nu$, and satisfying $\left|\left(\frac{d}{dE}\right)^\beta \chi_\nu(E)\right| \le C^\beta_\# \left[\text{length}(\mathcal{J}'_\nu)\right]^{-\beta}$. For $E \in \mathcal{J}^0$, define $\theta_\nu(E) = \chi^2_\nu(E)\left[\sum_{\nu'} \chi^2_{\nu'}(E)\right]^{-1}$. Certainly $\sum_\nu \theta_\nu(E) = 1$ on $\mathcal{J}^0$; and $\theta_\nu(E) = 0$ if $E \in \mathcal{J}^0$, $|E - E'_\nu| > \frac{1}{2}\hat{c}' S_{\min}(E'_\nu)$. Here, E'_ν is any point in $\mathcal{J}'_\nu \subset \mathcal{J}^0$. Also $\left|\left(\frac{d}{dE}\right)^\beta \theta_\nu(E)\right| \le C^\beta_\#(\text{length } \mathcal{J}'_\nu)^{-\beta} \le C^\beta_\#([\hat{c}']^2 S_{\min}(E'_\nu))^{-\beta}$ for $E \in \mathcal{J}^0$. Since $E = 0$ belongs to at most $C_\#$ of the doubles of the $\mathcal{J}_\mu$, and length $\mathcal{J}_\mu \sim S_{\min}(E'_\nu)$ for $E'_\nu \in \mathcal{J}_\mu$, it follows that $|E'_\nu| \le c_\# S_{\min}(E'_\nu)$ for at most $C_\# M$ of the E'_ν. Extend $\theta_\nu(E)$ from $\mathcal{J}^0$ to $\mathbb{R}^1$, preserving the properties

$$\Big|\big(\frac{d}{dE}\big)^{\beta}\theta_\nu(E)\Big| \le C_{\#}^{\beta}([\hat{c}']^2 S_{\min}(E'_\nu))^{-\beta} \quad \text{and}$$

$$\theta_\nu(E) = 0 \quad \text{for} \quad |E - E'_\nu| > \hat{c}' S_{\min}(E'_\nu)\ .$$

Properties (a)...(e) are then obvious. ■

Now we can write $g(x,E) = \sum_\nu g_\nu(x,E)$ for $E \le 0$. We take $g_\nu(x,E) = \theta_\nu(E) g(x,E)$, where $\theta_\nu(E)$ and E'_ν come from Lemma 1, with $\hat{c}'$ equal to a small enough constant of the form c_*. The hypotheses of the WKB Theorems, and the assumptions of the preceding section are satisfied, with: E'_ν in place of E_0; 0 in place of E_∞; $g_\nu(x,E)$ in place of $g(x,E)$; $100K$ in place of K; and $\hat{c}'$ in place of $\hat{c}$. (Note that our present δE is not the same as the δE in the previous section. Also note that $B_{\text{left}}(E) \sim B_{\text{left}}(E_0)$ and $S_{\text{left}}(E) \sim S_{\text{left}}(E_0)$ for $E \in \mathcal{J}^0$, since $|E - E_0| < c_{\#} S_{\text{left}}(E_0)$ implies $|x_{\text{left}}(E) - x_{\text{left}}(E_0)| < c_{\#} B_{\text{left}}(E_0)$. The last remark is needed to verify *ASSUMPTION 3* in the previous section.) The constants $\hat{C}_\beta$ in the previous section now have the form C_*^β. Hence any constant of the form C_* in that section still has the form C_* here. The lemma of the previous section therefore shows that

$$(1) \quad \rho(x, g_\nu) = \frac{1}{\pi}\int_{-\infty}^{0} g_\nu(x,E)\lambda_\nu^{-2/3}\Big(\frac{\partial Y_\nu(x,E)}{\partial x}\Big)^{-1} A^2(\lambda_\nu^{2/3} Y_\nu(x,E))\, dE$$
$$- g_\nu(x,0)\lambda_\nu^{-2/3}\Big(\frac{\partial Y_\nu}{\partial x}(x,0)\Big)^{-1} A^2\big(\lambda_\nu^{2/3} Y_\nu(x,0)\big)[\phi'(0)]^{-1}\chi_-\big(\frac{1}{\pi}\phi(0) - \frac{1}{2}\big) + \mathcal{E}_\nu^1(x)$$

where $\lambda_\nu = \lambda_{\text{left}}(E'_\nu)$, and $Y_\nu(x,E)$ is the function $Y_{\text{left}}(x,E)$ that comes from applying the WKB Theorems with E'_ν in place of E_0 etc., as above. For the error $\mathcal{E}_\nu^1(x)$ we have the following estimates from the lemma of the previous section.

$$(2) \quad \int_{I_{\text{BVP}}} |\mathcal{E}_\nu^1(x)|\, dx \le C_* \Lambda_{\min}^{4\varepsilon-2}\lambda_{\text{left}}^{-1} B_{\text{left}}^2 S_{\min}(E'_\nu)\big(\frac{\delta x}{B_{\text{left}}}\big)^{1/2}$$
$$+ C_*\Lambda_{\min}^{10-N''}\int_{E'_\nu - \hat{c}' S_{\min}(E'_\nu)}^{E'_\nu} \phi'(E)\, dE$$
$$\text{if} \quad |E'_\nu| > 2\hat{c}' S_{\min}(E'_\nu)\ .$$

$$(3)\quad \int_{I_{\rm BVP}} |\mathcal{E}^1_\nu(x)|\,dx \le C_*\Big(\Lambda_{\rm min}^{4\varepsilon-2} + \big(\frac{\delta x}{B_{\rm left}}\big)^{1/2}\Lambda^{-1}_{\rm min}\Big)\lambda^{-1}_{\rm left}B^2_{\rm left}S_{\rm min}(E'_\nu)\big(\frac{\delta x}{B_{\rm left}}\big)^{1/2}$$

$$+ C_*\Lambda_{\rm min}^{10-N''}\int_{E'_\nu-\hat{c}'S_{\rm min}(E'_\nu)}^{E'_\nu}\phi'(E)\,dE$$

$$\text{if}\quad |E'_\nu| \le 2\hat{c}'S_{\rm min}(E'_\nu) \text{ and } \min_{k\in\mathbb{Z}}|\phi(0)-\pi(k+1/2)| > \overline{C}_\#\Lambda^{-1}_{\rm min}\ .$$

$$(4)\quad \int_{I_{\rm BVP}} |\mathcal{E}^1_\nu(x)|\,dx \le C_*\Lambda^{-1}_{\rm min}\lambda^{-1}_{\rm left}B^2_{\rm left}S_{\rm min}(E'_\nu)\big(\frac{\delta x}{B_{\rm left}}\big)^{1/2}$$

$$+ C_*\Lambda_{\rm min}^{10-N''}\int_{E'_\nu-\hat{c}'S_{\rm min}(E'_\nu)}^{E'_\nu}\phi'(E)\,dE$$

$$\text{if}\quad |E'_\nu| \le 2\hat{c}'S_{\rm min}(E'_\nu) \quad\text{and}\quad \min_{k\in\mathbb{Z}}|\phi(0)-\pi(k+1/2)| \le \overline{C}_\#\Lambda^{-1}_{\rm min}\ .$$

Here we set $\lambda_{\rm left} = \lambda_{\rm left}(E_0)$, $B_{\rm left} = B_{\rm left}(E_0)$, $S_{\rm left} = S_{\rm left}(E_0)$, and we use the fact that $\lambda_\nu \sim \lambda_{\rm left}$, $B_{\rm left}(E'_\nu) \sim B_{\rm left}$, $S_{\rm left}(E'_\nu) \sim S_{\rm left}$, $\Lambda(E'_\nu) \ge \Lambda_{\rm min}$ to deduce (2), (3), (4) from the lemma of the preceding section.

Throughout this section, we make the convention that terms such as the $g_\nu(x,0)$-term in (1) are defined to be identically zero whenever $g_\nu(x,0)\equiv 0$. (Recall that there are cases in which $g_\nu(x,0)\equiv 0$ but $Y_\nu(x,0)$ isn't well-defined.)

Our next task is to sum (1) over ν to get a formula for $\rho(x,g)$. To do so , we need to know that the various $\lambda_\nu^{2/3}Y_\nu(x,E)$ agree closely with one another as ν varies. This follows from the proof of the WKB Theorems, but we prefer to give a more self-contained argument. We use the fact that $\lambda_\nu \sim \lambda_{\rm left}$, and that

$$(5)\quad \big|\lambda^2_\nu(\partial_x Y_\nu)^2 Y_\nu + \{Y_\nu, x\} - (E - V(x))\big| < C_\#\lambda_{\rm left}^{-N'}S_{\rm left}$$

$$\text{for}\quad |E - E'_\nu| < c'_\# S_{\rm min}(E'_\nu)\ , |x - x_{\rm left}(E)| < c_\#\lambda^{-\varepsilon}_{\rm left}B_{\rm left}$$

(6)
$$\big|\partial^\alpha_x Y_\nu(x,E)\big| \le C^\alpha_\# B^{-\alpha}_{\rm left}\quad \text{for } |E - E'_\nu| < \hat{c}'S_{\rm min}(E'_\nu)\ , |x - x_{\rm left}(E)| < c_\# B_{\rm left}\ .$$

Estimates (5) and (6) are part of the conclusion of the WKB Eigenfunction Theorem. We also use the following result.

Lemma 2. *There is a function $Y(x,E)$ satisfying the estimates*

$$(7) \quad \left|\lambda_{\text{left}}^2(\partial_x Y)^2 Y + \{Y,x\} - (E - V(x))\right| \leq C_\# \lambda_{\text{left}}^{-N'} S_{\text{left}}$$

$$\textit{for} \quad |E - E_0| < c_\#(\delta E) \sim c_\# S_{\text{left}} \quad \textit{and} \quad |x - x_{\text{left}}(E)| \leq c_\# \lambda_{\text{left}}^{-\varepsilon} B_{\text{left}}$$

and

$$(8) \quad \left|\partial_x^\alpha \partial_E^\beta Y(x,E)\right| \leq C_\#^{\alpha\beta} B_{\text{left}}^{-\alpha} S_{\text{left}}^{-\beta}$$

$$\textit{for} \quad |E - E_0| < c_\#(\delta E) \quad \textit{and} \quad |x - x_{\text{left}}(E)| < c_\# B_{\text{left}} \ .$$

Also,

$$(9) \quad \left|\frac{\partial}{\partial x} Y(x,E)\right| > c_\# B_{\text{left}}^{-1} \quad \textit{and} \quad \left|\frac{\partial Y}{\partial E}(x,E)\right| > c_\# S_{\text{left}}^{-1}$$

$$\textit{for} \quad |E - E_0| < c_\#(\delta E) \quad \textit{and} \quad |x - x_{\text{left}}(E)| < c_\# B_{\text{left}} \ .$$

This is contained in Lemma 1 from the section on approximate global solutions of ODE's in [FS2]. We compare $Y(x,E)$ with $Y_\nu(x,E)$, using the following.

Lemma 3. *Suppose $y_1(x)$ and $y_2(x)$ are defined on $\{|x| < c\}$ and satisfy there $\left|\left(\frac{d}{dx}\right)^\alpha y_i(x)\right| \leq C_\alpha$, $\frac{d}{dx} y_i(x) > c > 0$, $|y_i(0)| < C\lambda^{-2}$. Suppose also $(\frac{d}{dx} y_1)^2 y_1 + \lambda^{-2}\{y_1, x\} = \left(\frac{d}{dx} y_2\right)^2 y_2 + \lambda^{-2}\{y_2, x\} +$ Error, with $|\text{Error}| \leq C\lambda^{-N'}$ for $|x| < \lambda^{-\varepsilon}$. Then $|(\frac{d}{dx})^\alpha (y_1 - y_2)| < \overline{C}\lambda^{-N''}$ for $|x| < \lambda^{-\varepsilon}$, $0 \leq \alpha \leq 3$, with $\overline{C}$ depending only on c, C, C_α, N, ε.*

Proof. Let $P_i(x)$ be the N^{th} order Taylor expansion of $y_i(x)$ about 0. Taylor's theorem shows that $P_i(x)$ satisfy the hypotheses assumed for $y_i(x)$, (with different c, C, C_α depending on the original c, C, C_α) and that $\left|\left(\frac{d}{dx}\right)^\alpha [P_1 - P_2]\right| \leq \overline{C}\lambda^{-N''}$ for $\alpha \leq 3$ and $|x| < \lambda^{-\varepsilon}$ implies $\left|\left(\frac{d}{dx}\right)^\alpha [y_1 - y_2]\right| \leq \overline{C}\lambda^{-N''}$ for a different $\overline{C}$. Hence, it's enough to prove the lemma assuming the $y_i(x)$ are N^{th} degree polynomials.

By induction on m $(0 \le m \le N'' + 10)$ we will show that $|y_1 - y_2| \le \overline{C}\lambda^{-m}$ for $|x| < \lambda^{-\varepsilon}$, with $\overline{C}$ depending only on c, C, C_α, N, ε. This certainly holds for $m = 0$. Assume it holds for a given m. Then $|\left(\frac{d}{dx}\right)^\alpha y_1 - \left(\frac{d}{dx}\right)^\alpha y_2| \le \overline{C}\lambda^{-m+\varepsilon\alpha}$ for $|x| < \lambda^{-\varepsilon}$ and $0 \le \alpha \le 3$, since the $y_i(x)$ are polynomials. The Schwartzian $\{y_i, x\}$ is a function $\mathcal{F}\left(\frac{d}{dx}y_i, \frac{d^2}{dx^2}y_i, \frac{d^3}{dx^3}y_i\right)$ with $\mathcal{F}(t_1, t_2, t_3)$ smooth on $\{|t_1|, |t_2|, |t_3| \le C$ and $|t_1| \ge c > 0\}$. Hence $|\{y_1, x\} - \{y_2, x\}| \le \overline{C}\lambda^{-m+3\varepsilon}$ for $|x| < \lambda^{-\varepsilon}$, and so by hypothesis,

$$\left|\left(\frac{dy_1}{dx}\right)^2 y_1 - \left(\frac{dy_2}{dx}\right)^2 y_2\right| \le \overline{C}\lambda^{-m+3\varepsilon-2} \quad \text{for} \quad |x| < \lambda^{-\varepsilon} \ . \tag{10}$$

Let x_0 be the zero of y_1 with $|x_0| \le \overline{C}\lambda^{-2}$. Estimate (10) and $\frac{dy_2}{dx} > c$ show that

$$|y_2(x_0)| < \overline{C}\lambda^{-m+3\varepsilon-2} \ . \tag{11}$$

Set $y_3(x) = y_2(x) - y_2(x_0)$, so that $y_1(x)$ and $y_3(x)$ both vanish at x_0, $\frac{d}{dx}y_3 > c > 0$, and

$$\left|\left(\frac{dy_1}{dx}\right)^2 y_1 - \left(\frac{dy_3}{dx}\right)^2 y_3\right| \le \overline{C}\lambda^{-m+3\varepsilon-2} \quad \text{for} \quad |x| < \lambda^{-\varepsilon} \ . \tag{12}$$

Both $y_1(x)$ and $y_3(x)$ are N^{th} order polynomials, bounded a-priori on $\{|x| \le c\}$. If $x_0 < x < \lambda^{-\varepsilon}$, then $\left(\frac{dy_1}{dx}\right)$, $\left(\frac{dy_3}{dx}\right)$, $y_1(x)$ and $y_3(x)$ are all positive, and $y_1(x) \sim (x - x_0)$, $y_3(x) \sim (x - x_0)$. We apply the elementary inequality $|a^{1/2} - b^{1/2}| \le \frac{|a-b|}{a^{1/2}+b^{1/2}}$ with $a = \left(\frac{dy_1}{dx}\right)^2 y_1(x)$ and $b = \left(\frac{dy_3}{dx}\right)^2 y_3(x)$ to derive from (12) that

$$\left|y_1^{1/2}\frac{dy_1}{dx} - y_3^{1/2}\frac{dy_3}{dx}\right| \le \frac{\overline{C}\lambda^{-m+3\varepsilon-2}}{(x - x_0)^{1/2}} \quad \text{for} \quad x_0 < x < \lambda^{-\varepsilon} \ . \tag{13}$$

Since also $y_1(x_0) = y_3(x_0) = 0$, the fundamental theorem of calculus and (13) yield

$$\left|y_1^{3/2} - y_3^{3/2}\right| \le \overline{C}\lambda^{-m+3\varepsilon-2} \quad \text{for} \quad x_0 < x < \lambda^{-\varepsilon} \ . \tag{14}$$

For $\frac{1}{2}\lambda^{-\varepsilon} < x < \lambda^{-\varepsilon}$, we have $y_1(x), y_3(x) \sim (x - x_0) \sim \lambda^{-\varepsilon}$ since $|x_0| < \overline{C}\lambda^{-2}$. Hence (14) implies

$$|y_1 - y_3| \le \overline{C}\lambda^{-m+10\varepsilon-2} \quad \text{for} \quad \frac{1}{2}\lambda^{-\varepsilon} < x < \lambda^{-\varepsilon} \ .$$

Since $y_1(x)$ and $y_3(x)$ are polynomials, it follows that $|y_1 - y_3| \leq \overline{C}\lambda^{-m+10\varepsilon-2}$ for $|x| < \lambda^{-\varepsilon}$. Since $y_3(x) = y_2(x) - y_2(x_0)$ with $|y_2(x_0)| < \overline{C}\lambda^{-m+3\varepsilon-2}$, we conclude that $|y_1 - y_2| \leq \overline{C}\lambda^{-m+10\varepsilon-2}$ for $|x| < \lambda^{-\varepsilon}$. To complete the induction step, all we need is $|y_1 - y_2| \leq \overline{C}\lambda^{-m-1}$ for $|x| < \lambda^{-\varepsilon}$.

Hence we know by induction that $|y_1 - y_2| < \overline{C}\lambda^{-m}$ for $|x| < \lambda^{-\varepsilon}$, $m = [N''+10]$. Since $y_i(x)$ are polynomials, it follows that $\left|\left(\frac{d}{dx}\right)^{\alpha}[y_1 - y_2]\right| \leq \overline{C}\lambda^{-N''}$ for $|x| < \lambda^{-\varepsilon}$, $0 \leq \alpha \leq 3$, which is the conclusion of the lemma. ■

After a rescaling, Lemma 3 applies to $x \mapsto Y(x,E)$ and $x \mapsto (\lambda_\nu/\lambda_{\text{left}})^{2/3}Y_\nu(x,E)$ for fixed E with $|E - E'_\nu| < \hat{c}' S_{\min}(E'_\nu)$. We conclude that

$$\text{(15)} \quad \left|\left(\frac{\partial}{\partial x}\right)^{\alpha}(\lambda_{\text{left}}^{2/3}Y(x,E) - \lambda_\nu^{2/3}Y_\nu(x,E))\right| \leq C_\# \lambda_{\text{left}}^{\frac{2}{3}-N''} B_{\text{left}}^{-\alpha} \quad (0 \leq \alpha \leq 3)$$
$$\text{for} \quad |E - E'_\nu| < \hat{c}' S_{\min}(E'_\nu), |x - x_{\text{left}}(E)| < \lambda_{\text{left}}^{-\varepsilon} B_{\text{left}} \ .$$

Using (15), we can compare the right-hand side of (1) with its analogue in which λ_ν, Y_ν are replaced by λ_{left}, Y. We note that $\left|\frac{d}{d\xi}A^2(\xi)\right| \leq C_\#$, hence

$$\text{(16)} \qquad |A^2(\lambda_{\text{left}}^{2/3}Y) - A^2(\lambda_\nu^{2/3}Y_\nu)| \leq C_\# \lambda_{\text{left}}^{2/3-N''} \text{ in supp } g_\nu \ , \text{ by (15)} \ .$$

Since $\lambda_\nu^{2/3}\frac{\partial Y_\nu}{\partial x} \sim \lambda_{\text{left}}^{2/3} B_{\text{left}}^{-1}$ and $|\lambda_{\text{left}}^{2/3}\frac{\partial Y}{\partial x} - \lambda_\nu^{2/3}\frac{\partial Y_\nu}{\partial x}| \leq C_\# \lambda_{\text{left}}^{\frac{2}{3}-N''} B_{\text{left}}^{-1}$ in supp g_ν by (15), it follows that

$$\text{(17)} \qquad \left|\lambda_{\text{left}}^{-2/3}\left(\frac{\partial Y}{\partial x}\right)^{-1} - \lambda_\nu^{-2/3}\left(\frac{\partial Y_\nu}{\partial x}\right)^{-1}\right| \leq C_\# \lambda_{\text{left}}^{-2/3-N''} B_{\text{left}} \quad \text{in supp } g_\nu \ .$$

Hence in supp g_ν we have

$$\begin{aligned}
\text{(18)} \quad & \left|\lambda_{\text{left}}^{-2/3}\left(\frac{\partial Y}{\partial x}\right)^{-1} A^2(\lambda_{\text{left}}^{2/3}Y) - \lambda_\nu^{-2/3}\left(\frac{\partial Y_\nu}{\partial x}\right)^{-1} A^2(\lambda_\nu^{2/3}Y_\nu)\right| \\
& \leq \left|\lambda_{\text{left}}^{-2/3}\left(\frac{\partial Y}{\partial x}\right)^{-1}\right| |A^2(\lambda_{\text{left}}^{2/3}Y) - A^2(\lambda_\nu^{2/3}Y_\nu)| \\
& + \left|\lambda_{\text{left}}^{-2/3}\left(\frac{\partial Y}{\partial x}\right)^{-1} - \lambda_\nu^{-2/3}\left(\frac{\partial Y_\nu}{\partial x}\right)^{-1}\right| A^2(\lambda_\nu^{2/3}Y_\nu) \\
& \leq C_\# \lambda_{\text{left}}^{-2/3} B_{\text{left}} \cdot \lambda_{\text{left}}^{\frac{2}{3}-N''} + C_\# \lambda_{\text{left}}^{-2/3-N''} B_{\text{left}} \\
& \text{(by (16), (17) and boundedness of the Airey function)} \ \leq C_\# \lambda_{\text{left}}^{-N''} B_{\text{left}} \ .
\end{aligned}$$

Integrating (18) over the support of g_ν, we get

$$(19)\quad \int_{I_{\rm BVP}} \Big| \int_{-\infty}^{0} g_\nu(x,E)\lambda_\nu^{-2/3}\big(\frac{\partial Y_\nu}{\partial x}\big)^{-1} A^2(\lambda_\nu^{2/3} Y_\nu)\, dE - \int_{-\infty}^{0} g_\nu(x,E)\lambda_{\rm left}^{-2/3}\big(\frac{\partial Y}{\partial x}\big)^{-1} A^2(\lambda_{\rm left} Y)\, dE \Big|\, dx \le C_* S_{\rm min}(E'_\nu)\lambda_{\rm left}^{-N''} B_{\rm left}^2 \ .$$

Also, suppose $|E'_\nu| \le 2\hat{c}' S_{\rm min}(E'_\nu)$. Then using (18) and the estimate

$$(19\ \text{bis})\qquad \phi'(0) = \frac{1}{2}\int_{I_{\rm BVP}} (-V(x))_+^{-1/2}\, dx \ge c_\# S_{\rm left}^{-1/2} B_{\rm left} \ ,$$

we get

$$(20)\quad \int_{I_{\rm BVP}} \Big| g_\nu(x,0)\lambda_\nu^{-2/3}\big(\frac{\partial Y_\nu(x,0)}{\partial x}\big)^{-1} A^2(\lambda_\nu^{2/3} Y_\nu(x,0))[\phi'(0)]^{-1} - g_\nu(x,0)\lambda_{\rm left}^{-2/3}\big(\frac{\partial Y(x,0)}{\partial x}\big)^{-1} A^2(\lambda_{\rm left}^{2/3} Y(x,0))[\phi'(0)]^{-1} \Big|\, dx$$
$$\le C_\# \lambda_{\rm left}^{-N''} B_{\rm left}^2 [\phi'(0)]^{-1} \le C_* \lambda_{\rm left}^{-N''} B_{\rm left}^2 (S_{\rm left}^{-1/2} B_{\rm left})^{-1} = C_* \lambda_{\rm left}^{1-N''}$$
$$\text{for}\quad |E'_\nu| \le 2\hat{c}' S_{\rm min}(E'_\nu) \ .$$

For $|E'_\nu| > 2\hat{c}' S_{\rm min}(E'_\nu)$ we have $g_\nu(x,0) \equiv 0$, so the left side of (20) is equal to zero.

Combining (19) and (20) with (1)...(4), we obtain the following results.

$$(21)\quad \rho(x, g_\nu) = \frac{1}{\pi}\int_{-\infty}^{0} g_\nu(x,E)\lambda_{\rm left}^{-2/3}\big(\frac{\partial Y(x,E)}{\partial x}\big)^{-1} A^2(\lambda_{\rm left}^{2/3} Y(x,E))\, dE$$
$$- g_\nu(x,0)\lambda_{\rm left}^{-2/3}\big(\frac{\partial Y(x,0)}{\partial x}\big)^{-1} A^2(\lambda_{\rm left}^{2/3} Y(x,0))[\phi'(0)]^{-1}\chi_-\big(\frac{1}{\pi}\phi(0) - \frac{1}{2}\big) + \mathcal{E}_\nu^2(x) \ .$$

(22) If $|E'_\nu| > 2\hat{c}' S_{\rm min}(E'_\nu)$, then

$$\int_{I_{\rm BVP}} |\mathcal{E}_\nu^2(x)|\, dx \le C_* \Lambda_{\rm min}^{4\varepsilon-2} \lambda_{\rm left}^{-1} B_{\rm left}^2 S_{\rm min}(E'_\nu)\cdot\big(\frac{\delta x}{B_{\rm left}}\big)^{1/2} + C_* \Lambda_{\rm min}^{10-N''} \int_{E'_\nu - \hat{c}' S_{\rm min}(E'_\nu)}^{E'_\nu} \phi'(E)\, dE + C_* S_{\rm min}(E'_\nu)\lambda_{\rm left}^{-N''} B_{\rm left}^2 \ .$$

(23) If $|E'_\nu| \le 2\hat{c}' S_{\rm min}(E'_\nu)$ and $\min_{k\in\mathbb{Z}} |\phi(0) - \pi(k+1/2)| > \overline{C}_\# \Lambda_{\rm min}^{-1}$, then

$$\int_{I_{\rm BVP}} |\mathcal{E}_\nu^2(x)|\, dx \le C_* \big(\Lambda_{\rm min}^{4\varepsilon-2} + \big(\frac{\delta x}{B_{\rm left}}\big)^{1/2}\Lambda_{\rm min}^{-1}\big)\lambda_{\rm left}^{-1} B_{\rm left}^2 S_{\rm min}(E'_\nu)\cdot\big(\frac{\delta x}{B_{\rm left}}\big)^{1/2}$$
$$+ C_* \Lambda_{\rm min}^{10-N''} \int_{E'_\nu - \hat{c}' S_{\rm min}(E'_\nu)}^{E'_\nu} \phi'(E)\, dE + C_* S_{\rm min}(E'_\nu)\lambda_{\rm left}^{-N''} B_{\rm left}^2 + C_* \lambda_{\rm left}^{1-N''} \ .$$

(24) If $|E'_\nu| \le 2\hat{c}' S_{\min}(E'_\nu)$ and $\min_{k\in\mathbb{Z}} |\phi(0) - \pi(k+1/2)| \le \overline{C}_\# \Lambda_{\min}^{-1}$, then

$$\int_{I_{\mathrm{BVP}}} |\mathcal{E}^2_\nu(x)|dx \le C_* \Lambda_{\min}^{-1} \lambda_{\mathrm{left}}^{-1} B^2_{\mathrm{left}} S_{\min}(E'_\nu) \cdot \Big(\frac{\delta x}{B_{\mathrm{left}}}\Big)^{1/2}$$

$$+ C_* \Lambda_{\min}^{10-N''} \int_{E'_\nu - \hat{c}' S_{\min}(E'_\nu)}^{E'_\nu} \phi'(E) dE + C_* S_{\min}(E'_\nu) \lambda_{\mathrm{left}}^{-N''} B^2_{\mathrm{left}} + C_* \lambda_{\mathrm{left}}^{1-N''} \ .$$

We now sum (21)...(24) over all ν, to get a formula for $\rho(x,g)$. To sum the error terms on the right of (22)...(24), we use the following observations. First of all, $\sum_\nu S_{\min}(E'_\nu) \le \sum_\nu C_* |\mathcal{J}''_\nu| \le C_* |\bigcup_\nu \mathcal{J}''_\nu| \le C_* S_{\mathrm{left}}$ and $\sum_\nu \int_{E'_\nu - \hat{c}' S_{\min}(E'_\nu)}^{E'_\nu} \phi'(E)\, dE$ $\le \sum_\nu \int_{\mathcal{J}''_\nu \cap \mathcal{J}} \phi'(E)\, dE \le C_* \int_{\mathcal{J}} \phi'(E)\, dE \le C_* \phi(\max \mathcal{J})$ by Lemma 1. Secondly, there are at most C_* distinct ν for which (23) or (24) applies. This also follows from Lemma 1. If $|E_0| > 2\hat{c}(\delta E)$, then (23), (24) will not be needed at all, provided we take $\hat{c}'$ small enough depending on $\hat{c}$. From these remarks, we get the following results by summing (21)...(24) over ν.

$$(25) \quad \rho(x,g) = \frac{1}{\pi} \int_{-\infty}^{0} g(x,E) \lambda_{\mathrm{left}}^{-2/3} \Big(\frac{\partial Y(x,E)}{\partial x}\Big)^{-1} A^2(\lambda_{\mathrm{left}}^{2/3} Y(x,E))\, dE$$

$$- g(x,0) \lambda_{\mathrm{left}}^{-2/3} \Big(\frac{\partial Y(x,0)}{\partial x}\Big)^{-1} A^2(\lambda_{\mathrm{left}}^{2/3} Y(x,0)) [\phi'(0)]^{-1} \chi_-\Big(\frac{1}{\pi}\phi(0) - \frac{1}{2}\Big) + \mathcal{E}_3(x) \ .$$

(26) If $|E_0| > 2\hat{c}(\delta E)$, then

$$\int_{I_{\mathrm{BVP}}} |\mathcal{E}_3(x)|dx \le C_* \Lambda_{\min}^{4\varepsilon-2} \lambda_{\mathrm{left}}^{-1} B^2_{\mathrm{left}} S_{\mathrm{left}} \cdot \Big(\frac{\delta x}{B_{\mathrm{left}}}\Big)^{1/2} + C_* \Lambda_{\min}^{10-N''} \phi(\max \mathcal{J})$$

$$+ C_* S_{\mathrm{left}} \lambda_{\mathrm{left}}^{-N''} B^2_{\mathrm{left}} \le C_* \Lambda_{\min}^{4\varepsilon-2} \lambda_{\mathrm{left}} \Big(\frac{\delta x}{B_{\mathrm{left}}}\Big)^{1/2} + C_* \Lambda_{\min}^{10-N''} \phi(\max \mathcal{J})$$

$$+ C_* \Lambda_{\min}^{2-N''} \equiv \Omega \ .$$

(27) If $|E_0| \le 2\hat{c}(\delta E)$ and $\min_{k\in\mathbb{Z}} |\phi(0) - \pi(k+1/2)| > \overline{C}_\# \Lambda_{\min}^{-1}$, then

$$\int_{I_{\mathrm{BVP}}} |\mathcal{E}_3(x)|dx \le \Omega + C_* \Big(\frac{\delta x}{B_{\mathrm{left}}}\Big)^{1/2} \Lambda_{\min}^{-1} \lambda_{\mathrm{left}}^{-1} B^2_{\mathrm{left}} S_{\mathrm{left}} \Big(\frac{\delta x}{B_{\mathrm{left}}}\Big)^{1/2} + C_* \lambda_{\mathrm{left}}^{1-N''}$$

$$\le C_* \Omega + C_* \Lambda_{\min}^{-1} \lambda_{\mathrm{left}} \Big(\frac{\delta x}{B_{\mathrm{left}}}\Big) \ .$$

(28) If $|E_0| \le 2\hat{c}(\delta E)$ and $\min_{k\in\mathbb{Z}} |\phi(0) - \pi(k+1/2)| \le \overline{C}_{\#}\Lambda_{\min}^{-1}$, then

$$\int_{I_{\mathrm{BVP}}} |\mathcal{E}_3(x)|dx \le C_*\Lambda_{\min}^{-1}\lambda_{\mathrm{left}}^{-1}B_{\mathrm{left}}^2 S_{\mathrm{left}}\big(\frac{\delta x}{B_{\mathrm{left}}}\big)^{1/2} + C_*\Lambda_{\min}^{10-N''}\phi(\max \mathcal{J})$$
$$+ C_*\lambda_{\mathrm{left}}^{-N''}S_{\mathrm{left}}B_{\mathrm{left}}^2 + C_*\lambda_{\mathrm{left}}^{1-N''} \le C_*\Lambda_{\min}^{-1}\lambda_{\mathrm{left}}\big(\frac{\delta x}{B_{\mathrm{left}}}\big)^{1/2} + C_*\Lambda_{\min}^{10-N''}\phi(\max \mathcal{J})$$
$$+ C_*\Lambda_{\min}^{2-N''} .$$

Equations (25)...(28) reduce the study of $\rho(x,g)$ to that of an integral involving the Airey function. It remains to relate (25) to the semiclassical density $\rho_{sc}(x,g)$. Our goal is to control $\rho(x,g) - \rho_{sc}(x,g)$ in the Sobolev norm H^{-1}.

The main step is to understand the integrals

$$(29)\quad \mathcal{F}(x) = \int_{-\infty}^{x}\int_{-\infty}^{0} g(x',E)\lambda_{\mathrm{left}}^{-2/3}\big(\frac{\partial Y(x',E)}{\partial x'}\big)^{-1}[A^2(\lambda_{\mathrm{left}}^{2/3}Y(x',E))$$
$$- \frac{1}{2}\lambda_{\mathrm{left}}^{-1/3}(Y(x',E))_+^{-1/2}]\,dE\,dx'$$

and

$$(30)\quad \mathcal{G}(x) = \int_{-\infty}^{x} g(x',0)\lambda_{\mathrm{left}}^{-2/3}\big(\frac{\partial Y(x',0)}{\partial x'}\big)^{-1}[A^2(\lambda_{\mathrm{left}}^{2/3}Y(x',0))$$
$$- \frac{1}{2}\lambda_{\mathrm{left}}^{-1/3}(Y(x',0))_+^{-1/2}]\,dx' .$$

We begin with $\mathcal{F}$, and suppose that $|E_0| \le 2\hat{c}(\delta E)$. Set $y = Y(x',E)$ for fixed x', and define $h(y,x')$ by $h(y,x') = g(x',E)\big(\frac{\partial Y(x',E)}{\partial x'}\big)^{-1}\big(\frac{\partial Y(x',E)}{\partial E}\big)^{-1}$ for $y = Y(x',E)$. Thus $g(x',E)\big(\frac{\partial Y(x',E)}{\partial x'}\big)^{-1}dE = h(y,x')dy$ for fixed x'. Also note that $E \to y = Y(x',E)$ is increasing for $|E - E_0| < \hat{c}(\delta E)$, $|x - x_{\mathrm{left}}(E)| < c_{\#}B_{\mathrm{left}}$. Changing variables from (x',E) to (x',y) in (29), we get

$$\mathcal{F}(x) = \int_{-\infty}^{x}\int_{-\infty}^{Y(x',0)} \lambda_{\mathrm{left}}^{-2/3}h(y,x')\big[A^2(\lambda_{\mathrm{left}}^{2/3}y) - \frac{1}{2}(\lambda_{\mathrm{left}}^{2/3}y)_+^{-1/2}\big]\,dy\,dx' .$$

Changing the order of integration, we have

$$(31)\quad \mathcal{F}(x) = \int_{-\infty}^{Y(x,0)}\Big[\int_{y<Y(x',0)<Y(x,0)} \lambda_{\mathrm{left}}^{-2/3}h(y,x')dx'\Big]$$
$$\cdot\big[A^2(\lambda_{\mathrm{left}}^{2/3}y) - \frac{1}{2}(\lambda_{\mathrm{left}}^{2/3}y)_+^{-1/2}\big]\,dy .$$

To find the domain of integration in (31), we used the fact that $x' \mapsto Y(x',0)$ is strictly increasing for $|x' - x_{\text{left}}(0)| < c_{\#} B_{\text{left}}$. (Recall that $\lambda^2_{\text{left}}(\partial_x Y)^2 Y$ is a small perturbation of $E - V(x)$ with $-V'(x) > 0$. So at $x = x_{\text{left}}(E)$ we have $\frac{\partial Y}{\partial E}, \frac{\partial Y}{\partial x} > 0$.)

Let us estimate the inner integral in (31). We know that

$$|\partial^\alpha_{x'} \partial^\beta_E g(x',E)| \le \hat{C}_{\alpha\beta}(\delta x)^{-\alpha}\Big(\frac{S_{\text{left}}}{B_{\text{left}}}\delta x\Big)^{-\beta}$$
$$|\partial^\alpha_{x'} \partial^\beta_E \Big(\frac{\partial Y}{\partial x'}(x',E)\Big)^{-1}| \le C^{\alpha\beta}_{\#} B_{\text{left}} \cdot B^{-\alpha}_{\text{left}} S^{-\beta}_{\text{left}} \le C^{\alpha\beta}_{\#} B_{\text{left}}(\delta x)^{-\alpha}\Big(\frac{S_{\text{left}}}{B_{\text{left}}}\delta x\Big)^{-\beta}$$
$$|\partial^\alpha_{x'} \partial^\beta_E \Big(\frac{\partial Y}{\partial E}(x',E)\Big)^{-1}| \le C^{\alpha\beta}_{\#} S_{\text{left}} \cdot B^{-\alpha}_{\text{left}} S^{-\beta}_{\text{left}} \le C^{\alpha\beta}_{\#} S_{\text{left}}(\delta x)^{-\alpha}\Big(\frac{S_{\text{left}}}{B_{\text{left}}}\delta x\Big)^{-\beta} ,$$

so $f(x',E) \equiv g(x',E)\big(\frac{\partial Y}{\partial x'}(x',E)\big)^{-1}\big(\frac{\partial Y}{\partial E}(x',E)\big)^{-1}$ satisfies

$$(32) \qquad |\partial^\alpha_{x'} \partial^\beta_E f(x',E)| \le C^{\alpha\beta}_{*} S_{\text{left}} B_{\text{left}}(\delta x)^{-\alpha}\Big(\frac{S_{\text{left}}}{B_{\text{left}}}\delta x\Big)^{-\beta} .$$

Now $y = Y(x',E)$ is a smooth function of $\frac{x' - x_{\text{left}}(0)}{B_{\text{left}}}$ and $\frac{E}{S_{\text{left}}}$. That smooth function has its C^∞ seminorms bounded a-priori, and the derivative with respect to the second argument ($\frac{E}{S_{\text{left}}}$) is bounded a-priori away from zero. Hence $\frac{E}{S_{\text{left}}}$ is a smooth function of $\frac{x' - x_{\text{left}}(0)}{B_{\text{left}}}$ and y, i.e.

$$(33) \quad |\partial^\alpha_{x'} \partial^\beta_y E(y,x')| \le C^{\alpha\beta}_{\#} S_{\text{left}} B^{-\alpha}_{\text{left}} \text{ with } E(y,x') \text{ the solution of } Y(x',E) = y .$$

By definition, $h(y,x') = f(x',E(y,x'))$. Hence the derivative $\partial^\alpha_{x'}\partial^\beta_y h(y,x')$ is a sum of terms

$$(34) \qquad \partial^{\alpha_0}_{x'} \partial^\gamma_E f(x',E)\ \Big|_{E=E(y,x')} \cdot \prod_{\nu=1}^{\gamma} [\partial^{\alpha_\nu}_{x'} \partial^{\beta_\nu}_y E(y,x')]$$

with $\alpha_0 + \alpha_1 + \ldots + \alpha_\gamma = \alpha$, $\beta_1 + \ldots + \beta_\gamma = \beta$, $\alpha_\nu + \beta_\nu \ge 1$. (In particular, $0 \le \gamma \le \beta + (\alpha_1 + \ldots + \alpha_\gamma)$.) The term (34) is dominated by $C^{\alpha\beta}_{*} S_{\text{left}} B_{\text{left}}(\delta x)^{-\alpha_0}\big(\frac{S_{\text{left}}}{B_{\text{left}}}\delta x\big)^{-\gamma} \cdot S^\gamma_{\text{left}} B^{-(\alpha_1 + \ldots + \alpha_\gamma)}_{\text{left}}$ (by (32) and (33)), which is dominated by $C^{\alpha\beta}_{*} S_{\text{left}} B_{\text{left}}(\delta x)^{-\alpha}\big(\frac{\delta x}{B_{\text{left}}}\big)^{-\beta}$ since $0 \le \gamma \le \beta + (\alpha_1 + \ldots + \alpha_\gamma)$. Thus,

$$(35) \qquad |\partial^\alpha_{x'} \partial^\beta_y h(y,x')| \le C^{\alpha\beta}_{*} S_{\text{left}} B_{\text{left}}(\delta x)^{-\alpha}\Big(\frac{\delta x}{B_{\text{left}}}\Big)^{-\beta} .$$

For $y < Y(x,0)$, the inner integral in (31) may be written as

$$(36) \quad H(y,x) = \lambda_{\text{left}}^{-2/3} \int_y^{Y(x,0)} h(y,X(t)) \frac{dX(t)}{dt}\, dt$$

$$\text{with} \quad X(t) \quad \text{defined as the solution of } Y(x,0) = t\ .$$

For $X(t)$ we have the estimates

$$(37) \qquad \Big|\big(\frac{d}{dt}\big)^{\gamma} X(t)\Big| \le C_{\#}^{\gamma} B_{\text{left}} \quad (\gamma \ge 1)\ ,$$

since $Y(x,0)$ is a smooth function of $\frac{x - x_{\text{left}}(0)}{B_{\text{left}}}$, and that smooth function has first derivative bounded below. The derivative $\partial_y^{\beta}\partial_t^{\sigma} h(y,X(t))$ is a sum of terms of the form

(38)

$$\partial_{x'}^{\alpha}\partial_y^{\beta} h(y,x')\ \big|_{x'=X(t)} \cdot \prod_{\nu=1}^{\alpha}\Big[\big(\frac{d}{dt}\big)^{\sigma_\nu} X(t)\Big] \quad \text{with} \quad \sigma_\nu \ge 1 \text{ and } \sigma_1 + \ldots + \sigma_\alpha = \sigma\ .$$

The term (38) is dominated by $C_*^{\beta\sigma} S_{\text{left}} B_{\text{left}} (\delta x)^{-\alpha} \big(\frac{\delta x}{B_{\text{left}}}\big)^{-\beta} (B_{\text{left}})^{\alpha}$. Thus, $|\partial_y^{\beta}\partial_t^{\sigma} h(y,X(t))| \le C_*^{\beta\sigma} S_{\text{left}} B_{\text{left}} \big(\frac{\delta x}{B_{\text{left}}}\big)^{-\sigma-\beta}$. Together with (37), this gives

$$(39) \quad |\partial_y^{\beta}\partial_t^{\sigma}\{h(y,X(t))\frac{dX(t)}{dt}\}| \le C_*^{\beta\sigma} S_{\text{left}} B_{\text{left}}^2 \big(\frac{\delta x}{B_{\text{left}}}\big)^{-\sigma-\beta}$$

$$= C_*^{\beta\sigma} \lambda_{\text{left}}^2 \big(\frac{\delta x}{B_{\text{left}}}\big)^{-\sigma-\beta}\ .$$

Equation (36) shows that

$$\lambda_{\text{left}}^{2/3}\partial_y^{\gamma} H(y,x) = \int_y^{Y(x,0)} \partial_y^{\gamma}\{h(y,X(t))\frac{dX}{dt}\}\, dt$$

$$+ \sum_{\gamma_1+\gamma_2=\gamma-1} \text{coeff}(\gamma_1,\gamma_2)\partial_y^{\gamma_1}\partial_t^{\gamma_2}\{h(y,X(t))\frac{dX}{dt}\}\ \big|_{t=y}\ .$$

From (39) we get therefore

$$(40) \qquad |\partial_y^{\gamma} H(y,x)| \le C_*^{\gamma} \lambda_{\text{left}}^{4/3} \big(\frac{\delta x}{B_{\text{left}}}\big)^{-\gamma}\Big[|y - Y(x,0)| + \big(\frac{\delta x}{B_{\text{left}}}\big)\Big]\ .$$

We check the range of y in which $H(y,x)$ is supported. Since $g(x',E)$ is supported in $\{|x' - x_{\text{left}}(E)| \le \delta x\}$, and since $h(y,x') = g(x',E(y,x')) \cdot \{\text{STUFF}\}$, the support of $h(y,x')$ is contained in $\{|x'-x_{\text{left}}(E)|\le \delta x\}$ with $E=E(y,x')$, i.e. $Y(x',E)=y$. For such (y,x') we have $|y|\le |Y(x',E)-Y(x_{\text{left}}(E),E)|+|Y(x_{\text{left}}(E),E)|\le \frac{C_\#}{B_{\text{left}}}|x'-x_{\text{left}}(E)|+C_\#\lambda_{\text{left}}^{-2}\le C_\#\frac{\delta x}{B_{\text{left}}}$, so $h(y,x')$ is supported in $\{|y|\le C_\#\frac{\delta x}{B_{\text{left}}}\}$. So (40) implies

$$(41)\quad |\partial_y^\gamma H(y,x)|\le C_*^\gamma\Big[\lambda_{\text{left}}^{4/3}\big(\frac{\delta x}{B_{\text{left}}}\big)\Big]\cdot\big(\frac{\delta x}{B_{\text{left}}}\big)^{-\gamma}\qquad \text{if}\quad |Y(x,0)|<C_\#\frac{\delta x}{B_{\text{left}}}$$

$$(42)\qquad |\partial_y^\gamma H(y,x)|\le C_*^\gamma\lambda_{\text{left}}^{4/3}\big(\frac{\delta x}{B_{\text{left}}}\big)^{-\gamma}\quad \text{if}\quad |x-x_{\text{left}}(0)|<c_\# B_{\text{left}}\ ;$$

and we know that

$$(43)\qquad \text{supp } H(y,x)\subset\{|y|\le C_\#\big(\frac{\delta x}{B_{\text{left}}}\big)\}\quad \text{and}\quad H(Y(x,0),x)=0 \text{ by } (36)\ .$$

We are interested in x in the interval $\{|x-x_{\text{left}}(0)|<c_\# B_{\text{left}}\}$. Equations (31) and (36) give

$$(44)\quad \mathcal{F}(x)=\int_{-\infty}^{Y(x,0)} H(y,x)\cdot\big[A^2(\lambda_{\text{left}}^{2/3}y)-\frac12(\lambda_{\text{left}}^{2/3}y)_+^{-1/2}\big]\,dy$$
$$=\lambda_{\text{left}}^{-2/3}\tilde{\mathcal{F}}(\lambda_{\text{left}}^{2/3}Y(x,0))\quad \text{with}$$

$$(45)\qquad \tilde{\mathcal{F}}(\tilde x)=\int_{-\infty}^{\tilde x}\Theta(\tilde y,\tilde x)\big[A^2(\tilde y)-\frac12(\tilde y)_+^{-1/2}\big]\,d\tilde y\quad \text{and}$$

$$(46)\qquad \Theta(\tilde y,\tilde x)=H(\lambda_{\text{left}}^{-2/3}\tilde y,x)\quad \text{with } x \text{ obtained by solving } \lambda_{\text{left}}^{2/3}Y(x,0)=\tilde x\ .$$

Note that

$$(46\text{ bis})\qquad \Theta(\tilde y,\tilde x)\quad \text{is supported in}\quad \{|\tilde y|<C_\#\lambda_{\text{left}}^{2/3}\frac{\delta x}{B_{\text{left}}}\}\quad \text{and satisfies}$$

$$(47)\quad |\partial_{\tilde y}^\gamma\Theta(\tilde y,\tilde x)|\le C_*^\gamma\big[\lambda_{\text{left}}^{4/3}\big(\frac{\delta x}{B_{\text{left}}}\big)\big]\big(\lambda_{\text{left}}^{2/3}\frac{\delta x}{B_{\text{left}}}\big)^{-\gamma}\qquad \text{if}\quad |\tilde x|\le C_\#\big(\lambda_{\text{left}}^{2/3}\frac{\delta x}{B_{\text{left}}}\big)\ .$$

$$(48) \qquad |\partial_{\tilde{y}}^{\gamma}\Theta(\tilde{y},\tilde{x})| \le C_*^{\gamma}\lambda_{\text{left}}^{4/3} \cdot \big(\lambda_{\text{left}}^{2/3}\frac{\delta x}{B_{\text{left}}}\big)^{-\gamma} \quad \text{for} \quad |\tilde{x}| \le c_\# \lambda_{\text{left}}^{2/3} \; .$$

$$(49) \qquad \Theta(\tilde{y},\tilde{x}) = 0 \quad \text{when} \quad \tilde{y} = \tilde{x} \; .$$

We shall study integrals of the form

$$(50) \qquad F(x) = \int_{-\infty}^{x} \theta(x,y)[A^2(y) - \frac{1}{2}y_+^{-1/2}]\, dy \; , \quad \text{where we assume}$$

$$(51) \qquad |\partial_y^{\gamma}\theta(x,y)| \le C_\gamma R^{-\gamma} \; , \; \text{supp}\; \theta(x,y) \subset \{|y| \le R\} \; , \theta(x,x) \equiv 0 \; , R > 10 \; .$$

We look separately at the cases $x > R$, $10 < x \le R$, $|x| \le 10$, and $x < -10$. Property (A3) from the section "Review of Earlier Results" gives

$$(52) \qquad \Big|\int_{-\infty}^{\infty} \theta(x,y)[A^2(y) - \frac{1}{2}y_+^{-1/2}]\, dy\Big| \le C' R^{-5/2} \; ,$$

$$\text{with} \quad C' \quad \text{depending only on} \quad C_\gamma \quad \text{in (51)} \; .$$

Since supp $\theta(x,y) \subset \{|y| \le R\}$, estimate (52) shows that

$$(53) \qquad |F(x)| \le C' R^{-5/2} \quad \text{for} \quad x > R \; , \quad \text{and that}$$

$$(54) \qquad F(x) = -\int_{x}^{\infty} \theta(x,y)[A^2(y) - \frac{1}{2}y_+^{-1/2}]\, dy + O(R^{-5/2}) \quad \text{for} \quad 10 < x \le R \; .$$

Recall that $A(y) = \text{Re}\Big[\frac{e^{\pm i\frac{\pi}{4}}\, e^{\frac{2}{3}iy^{3/2}}}{y^{1/4}}\big(1 \pm \frac{5i}{48}y^{-3/2} + O(y^{-3})\big)\Big]$ for $y > 10$. Hence

$$A^2(y) - \frac{1}{2}y_+^{-1/2} = \text{Re}\Big[\frac{e^{\pm i\frac{\pi}{2}}\, e^{\frac{4}{3}iy^{3/2}}}{2y^{1/2}}\Big] + \text{Re}\Big[\frac{(\text{const})e^{\frac{4}{3}iy^{3/2}}}{y^2}\Big] + O(y^{-7/2}) \; .$$

Here the $O(y^{-7/2})$ error term contributes to the right-hand side of (54) at most $\int_x^{\infty}(R^{-1}|x-y|)y^{-7/2}dy \le C'R^{-1}x^{-3/2}$, since $|\theta(x,y)| \le CR^{-1}|x-y|$. Therefore, (54) implies

(55)

$$F(x) = \sum_{k=\pm 1} c_k \int_x^{\infty} \theta(x,y)\frac{e^{\frac{4}{3}iky^{3/2}}}{y^{1/2}}\, dy + \sum_{k=\pm 1} \tilde{c}_k \int_x^{\infty} \theta(x,y)\frac{e^{\frac{4}{3}iky^{3/2}}}{y^{2}}\, dy + O(R^{-1}x^{-3/2})$$

$$\text{for} \quad 10 < x \le R \; .$$

Let $q = \frac{1}{2}$ or 2.

We use the elementary identity

$$\frac{\partial}{\partial y}\left[e^{\frac{4}{3}iky^{3/2}} y^{-p}\partial_y^m \theta(x,y)\right] = 2ike^{\frac{4}{3}iky^{3/2}} y^{\frac{1}{2}-p}\partial_y^m \theta(x,y) + e^{\frac{4}{3}iky^{3/2}} y^{-p}\partial_y^{m+1}\theta(x,y) - pe^{\frac{4}{3}iky^{3/2}} y^{-p-1}\partial_y^m \theta(x,y)$$

and induction on M to show that

$$\frac{e^{\frac{4}{3}iky^{3/2}}}{y^q}\theta(x,y) + \partial_y\Big[\sum_{\substack{m\ge 0\\ p\ge 1}} a^k_{mp} e^{\frac{4}{3}iky^{3/2}} y^{-p}\partial_y^m\theta(x,y)\Big] = \sum_{\substack{m\ge 0\\ p\ge M}} b^k_{mp} e^{\frac{4}{3}iky^{3/2}} y^{-p}\partial_y^m \theta(x,y) \quad (k=\pm 1) \tag{56}$$

for a finite collection of coefficients a^k_{mp} and b^k_{mp} depending on M and q.

We estimate the right-hand side, by recalling that $|\partial_y^m\theta(x,y)| \le \begin{cases} CR^{-1} & \text{for } m\ge 1 \\ CR^{-1}|x-y| & \text{for } m = 0\end{cases}$. Integrating (56), we therefore find that

$$\Big|\int_x^\infty \frac{e^{\frac{4}{3}iky^{3/2}}}{y^q}\theta(x,y)dy - \sum_{\substack{m\ge 0\\ p\ge 1}} a^k_{mp} e^{\frac{4}{3}ikx^{3/2}} x^{-p}\partial_y^m\theta(x,y)\Big|_{y=x}\Big| \le \int_x^\infty \frac{C'_M}{y^M}\left(R^{-1} + R^{-1}|x-y|\right)dy \le C'R^{-1}x^{-3/2} \quad \text{for} \quad M > 100\ .$$

Since $\partial_y^m\theta(x,y)\,|_{y=x} = \begin{cases} O(R^{-1}) & \text{for } m\ge 1\\ 0 & \text{for } m=0\end{cases}$, this yields

$$\Big|\int_x^\infty \frac{e^{\frac{4}{3}iky^{3/2}}}{y^q}\theta(x,y)dy\Big| \le C'R^{-1}x^{-1} \quad \text{for} \quad 10 < x \le R\ .$$

From (55) we get

$$|F(x)| \le C'R^{-1}x^{-1} \quad \text{for} \quad 10 < x \le R\ . \tag{57}$$

Next we study $F(x)$ for $|x| < 10$, where we write

$$F(x) = \int_{-\infty}^{\min(x,0)} \theta(x,y)A^2(y)dy + \int_0^{\max(x,0)} \theta(x,y)\left[A^2(y) - \frac{1}{2}y^{-1/2}\right]dy\ . \tag{58}$$

Since $|\theta(x,y)| \leq C'R^{-1}|x-y| \leq C'R^{-1}(10+|y|)$ for $|x| \leq 10$, while $A^2(y) \leq C_M(1+|y|)^{-M}$ for $y<0$, $|A^2(y) - \frac{1}{2}y^{-1/2}| \leq Cy^{-1/2}$ for $0<y\leq 10$, (58) gives $|F(x)| \leq \int_{-\infty}^{\min(x,0)} C'_M R^{-1}(10+|y|)\cdot(1+|y|)^{-M}dy + \int_0^{\max(x,0)} C'y^{-1/2}R^{-1}(10+|y|)dy \leq C'R^{-1}$ for $|x|\leq 10$. Thus,

$$|F(x)| \leq C'R^{-1} \quad \text{for} \quad |x| \leq 10\ . \tag{59}$$

Similarly, for $-\infty < y < x \leq -10$ we have $|\theta(x,y)| \leq C'R^{-1}|x-y|$ and $A^2(y) \leq C_m(1+|y|)^{-M}$, so

$$\begin{aligned}|F(x)| = \left|\int_{-\infty}^{x} \theta(x,y)A^2(y)dy\right| &\leq C'_M \int_{-\infty}^{x} R^{-1}|x-y|(1+|y|)^{-M}dy \\ &\leq C'_m R^{-1}|x|^{-100} \\ &\qquad \text{for large} \quad M \quad \text{and} \quad -\infty < x \leq 10\ .\end{aligned}$$

Thus,

$$|F(x)| \leq C'R^{-1}|x|^{-100} \quad \text{for} \quad -\infty < x \leq -10\ . \tag{60}$$

Combining (53), (57), (59), (60) and throwing away information, we get

$$|F(x)| \leq C'R^{-1}(1+|x|)^{-1} \quad \text{for} \quad |x| \leq R \tag{61}$$

$$|F(x)| \leq C'R^{-5/2} \quad \text{for} \quad |x| \geq R \tag{61 bis}$$

We use (61) to control $\tilde{\mathcal{F}}(\tilde{x})$ in (45).

Take

$$\begin{aligned}\theta_{\text{in}}(\tilde{x},\tilde{y}) &= \frac{\Theta(\tilde{y},\tilde{x})}{\lambda_{\text{left}}^{4/3}\left(\frac{\delta x}{B_{\text{left}}}\right)}\ \chi_{|\tilde{x}|<C_\# \lambda_{\text{left}}^{2/3}\frac{\delta x}{B_{\text{left}}}} \\ \theta_{\text{out}}(\tilde{x},\tilde{y}) &= \frac{\Theta(\tilde{y},\tilde{x})}{\lambda_{\text{left}}^{4/3}}\ \chi_{|\tilde{x}|<c_\# \lambda_{\text{left}}^{2/3}} \\ R &= C_\# \lambda_{\text{left}}^{2/3}\frac{\delta x}{B_{\text{left}}} > 10\ .\end{aligned}$$

Estimates (46 bis)... (49) show that $\theta_{\text{in}}, \theta_{\text{out}}$ both satisfy hypothesis (51). Applying (50) and (61), (61 bis) for $\theta = \theta_{\text{in}}, \theta_{\text{out}}$, and comparing with (45), we learn that

$$|\tilde{\mathcal{F}}(\tilde{x})| \le C'_* \Big[\lambda_{\text{left}}^{4/3} \frac{\delta x}{B_{\text{left}}}\Big] \Big[C_\# \lambda_{\text{left}}^{2/3} \frac{\delta x}{B_{\text{left}}}\Big]^{-1} (1+|\tilde{x}|)^{-1}$$
$$= C_* \lambda_{\text{left}}^{2/3} (1+|\tilde{x}|)^{-1} \quad \text{for} \quad |\tilde{x}| \le C_\# \lambda_{\text{left}}^{2/3} \frac{\delta x}{B_{\text{left}}} \ ,$$

and

$$|\tilde{\mathcal{F}}(\tilde{x})| \le C'_* [\lambda_{\text{left}}^{4/3}] \Big[\lambda_{\text{left}}^{2/3} \frac{\delta x}{B_{\text{left}}}\Big]^{-5/2} \quad \text{for} \quad C_\# \lambda_{\text{left}}^{2/3} \frac{\delta x}{B_{\text{left}}} < |\tilde{x}| < c_\# \lambda_{\text{left}}^{2/3} \ .$$

Substituting these estimates in (44) gives

(62) $$|\mathcal{F}(x)| \le C_* (1 + \lambda_{\text{left}}^{2/3} |Y(x,0)|)^{-1} \quad \text{if} \quad |x - x_{\text{left}}(0)| < C_\# (\delta x)$$

(63) $$|\mathcal{F}(x)| \le C_* \lambda_{\text{left}}^{-1} \big(\frac{\delta x}{B_{\text{left}}}\big)^{-5/2} \quad \text{if} \quad C_\# (\delta x) < |x - x_{\text{left}}(0)| < c_\# B_{\text{left}} \ .$$

Since $1 + \lambda_{\text{left}}^{2/3} |Y(x,0)| \sim 1 + \lambda_{\text{left}}^{2/3} |x - x_{\text{left}}(0)| B_{\text{left}}^{-1}$, (62) and (63) imply

(64) $$\int_{|x - x_{\text{left}}(0)| < c_\# B_{\text{left}}} |\mathcal{F}(x)|^2 \, dx \le C_* \lambda_{\text{left}}^{-2/3} B_{\text{left}} + C_* \lambda_{\text{left}}^{-2} \big(\frac{\delta x}{B_{\text{left}}}\big)^{-5} B_{\text{left}} \ .$$

We proved (64) assuming $|E_0| < 2\hat{c}(\delta E)$. Since $\{|x - x_{\text{left}}(E_0)| < c'_\# B_{\text{left}}\} \subset \{|x - x_{\text{left}}(0)| < c_\# B_{\text{left}}\}$ when $|E_0| < 2\hat{c}(\delta E)$, estimate (64) gives

(65) $$\int_{|x - x_{\text{left}}(E_0)| < c_\# B_{\text{left}}} |\mathcal{F}(x)|^2 dx \le C_* \Big(\lambda_{\text{left}}^{-2/3} + \lambda_{\text{left}}^{-2} \big(\frac{\delta x}{B_{\text{left}}}\big)^{-5}\Big) B_{\text{left}}$$
$$\text{if} \quad |E_0| < 2\hat{c}(\delta E) \ .$$

On the other hand, suppose $|E_0| \ge 2\hat{c}(\delta E)$. Then as in (31) we set $y = Y(x', E)$ and $h(y, x') = g(x', E) \big(\frac{\partial Y}{\partial x'}(x', E)\big)^{-1} \big(\frac{\partial Y}{\partial E}(x', E)\big)^{-1}$. This time, since $g(x', E)$ is supported in $\{|E - E_0| < \hat{c}(\delta E)\}$, we have

$$\mathcal{F}(x) = \int_{-\infty}^{x} \int_{-\infty}^{\infty} h(y, x') \lambda_{\text{left}}^{-2/3} [A^2(\lambda_{\text{left}}^{2/3} y) - \frac{1}{2} (\lambda_{\text{left}}^{2/3} y)_+^{-1/2}] \, dy dx'$$

(66) $$= \int_{-\infty}^{\infty} \big[\lambda_{\text{left}}^{-4/3} \int_{-\infty}^{x} h(y, x') dx'\big] \cdot \big[A^2(\lambda_{\text{left}}^{2/3} y) - \frac{1}{2} (\lambda_{\text{left}}^{2/3} y)_+^{-1/2}\big] \lambda_{\text{left}}^{2/3} \, dy \ .$$

The proof of estimate (35) is still valid (with trivial changes). Also, supp $h(y,x')$ $\subset \{|x'-x_{\text{left}}(E_0)| < c_\# B_{\text{left}}, |y| < C_\# \frac{\delta x}{B_{\text{left}}}\}$, since $g(x',E)$ appears in h as a factor and $y = Y(x',E)$. Hence for $|x - x_{\text{left}}(E_0)| < c_\# B_{\text{left}}$, we have

$$\begin{aligned}\Big|\partial_y^\gamma \big[\lambda_{\text{left}}^{-4/3}\int_{-\infty}^{x} h(y,x')dx'\big]\Big| &\le C_* B_{\text{left}} \cdot \lambda_{\text{left}}^{-4/3} S_{\text{left}} B_{\text{left}} \big(\frac{\delta x}{B_{\text{left}}}\big)^{-\gamma} \\ &= C_* \lambda_{\text{left}}^{2/3} \big(\frac{\delta x}{B_{\text{left}}}\big)^{-\gamma} ,\end{aligned}$$

i.e.

$$(67)\quad |\partial_{\tilde y}^\gamma H(\tilde y, x)| \le C_* \lambda_{\text{left}}^{2/3} \big(\lambda_{\text{left}}^{2/3} \frac{\delta x}{B_{\text{left}}}\big)^{-\gamma} \quad \text{for} \quad |x - x_{\text{left}}(E_0)| < c_\# B_{\text{left}} ,$$
$$\text{with} \quad H(\tilde y, x) = \lambda_{\text{left}}^{-4/3} \int_{-\infty}^{x} h(\lambda_{\text{left}}^{-2/3}\tilde y, x')dx' ,$$

so that

$$(68)\qquad \mathcal{F}(x) = \int_{-\infty}^{\infty} H(\tilde y, x)[A^2(\tilde y) - \frac{1}{2}\tilde y_+^{-1/2}]d\tilde y \quad \text{by (66) ;} \quad \text{and}$$

$$(69)\qquad \text{supp } H(\tilde y, x) \subset \{|\tilde y| < C_\# \lambda_{\text{left}}^{+2/3} \frac{\delta x}{B_{\text{left}}}\} .$$

As above, we may apply (A3) from the section "Review of Earlier Results" to deduce from (67), (68), (69) that $|\mathcal{F}(x)| \le$ $C_* \lambda_{\text{left}}^{2/3} \cdot \big(\lambda_{\text{left}}^{2/3} \frac{\delta x}{B_{\text{left}}}\big)^{-5/2}$ for $|x - x_{\text{left}}(E_0)| < c_\# B_{\text{left}}$, i.e. $|\mathcal{F}(x)| \le C_* \lambda_{\text{left}}^{-1} \big(\frac{\delta x}{B_{\text{left}}}\big)^{-5/2}$ for $|x - x_{\text{left}}(E_0)| < c_\# B_{\text{left}}$, and thus

$$(70)\quad \int_{|x - x_{\text{left}}(E_0)| < c_\# B_{\text{left}}} |\mathcal{F}(x)|^2 dx \le C_* \lambda_{\text{left}}^{-2} \big(\frac{\delta x}{B_{\text{left}}}\big)^{-5} B_{\text{left}}$$
$$\text{if} \quad |E_0| \ge 2\hat c(\delta E) .$$

Next we suppose $|E_0| < 2\hat c(\delta E)$ and estimate $\mathcal{G}(x)$ in (30). By definition,

$$\begin{aligned}\mathcal{G}(x) &= \int_{-\infty}^{x} g(x',0)\lambda_{\text{left}}^{-4/3}\big(\frac{\partial Y(x',0)}{\partial x'}\big)^{-2}\big[A^2(\lambda_{\text{left}}^{2/3} Y(x',0)) - \frac{1}{2}(\lambda_{\text{left}}^{2/3} Y(x',0))_+^{-1/2}\big] \\ &\qquad \cdot \lambda_{\text{left}}^{2/3}\big(\frac{\partial Y}{\partial x'}(x',0)\big)\, dx' \\ &= \int_{-\infty}^{\lambda_{\text{left}}^{2/3} Y(x,0)} \tilde g(\xi)\big[A^2(\xi) - \frac{1}{2}\xi_+^{-1/2}\big]\, d\xi \text{ for} \quad |x - x_{\text{left}}(0)| < c_\# B_{\text{left}}\end{aligned}$$

$$(71)\qquad \text{with} \quad \tilde g(\xi) = \lambda_{\text{left}}^{-4/3} g(x',0)\big(\frac{\partial Y(x',0)}{\partial x'}\big)^{-2} \quad \text{for} \quad \xi = \lambda_{\text{left}}^{2/3} Y(x',0) .$$

We estimate the derivatives of $\tilde{g}(\xi)$. Since $|\partial_{x'}^{\alpha} g(x',0)| \le C_*^{\alpha}(\delta x)^{-\alpha}$ and $|\partial_{x'}^{\alpha}\big(\frac{\partial Y(x',0)}{\partial x'}\big)^{-2}| \le C_\# B_{\text{left}}^2 \cdot B_{\text{left}}^{-\alpha}$, we have

$$(72)\qquad \Big|\partial_{x'}^{\alpha}\{g(x',0)\big(\frac{\partial Y}{\partial x'}(x',0)\big)^{-2}\}\Big| \le C_*^{\alpha} B_{\text{left}}^2 (\delta x)^{-\alpha}\ .$$

On the other hand, $\big|\partial_{x'}^{\alpha}[\lambda_{\text{left}}^{2/3} Y(x',0)]\big| \le C_\#^{\alpha}\lambda_{\text{left}}^{2/3} B_{\text{left}}^{-\alpha}$ and $\partial_{x'}[\lambda_{\text{left}}^{2/3} Y(x',0)] > c_\# \lambda_{\text{left}}^{2/3} B_{\text{left}}^{-1}$, so

$$(73)\quad \Big|\big(\frac{d}{d\xi}\big)^{\gamma} x'\Big| \le C_\#^{\gamma} B_{\text{left}} \lambda_{\text{left}}^{-\frac{2}{3}\gamma}\ (\gamma \ge 1)\ ,$$
$$\text{where}\quad x' = x'(\xi)\quad \text{is the solution of } \lambda_{\text{left}}^{2/3} Y(x',0) = \xi\ .$$

The derivative $\lambda_{\text{left}}^{4/3}\big(\frac{d}{d\xi}\big)^m \tilde{g}(\xi)$ is a sum of terms

$$(74)\quad \Big[\partial_{x'}^{\alpha}\big\{g(x',0)\big(\frac{\partial Y}{\partial x'}(x',0)\big)^{-2}\big\}\Big]\ \Big|_{x'=x'(\xi)} \cdot \prod_{\nu=1}^{\alpha}\Big[\big(\frac{d}{d\xi}\big)^{m_\nu} x'\Big]\ ,$$
$$\text{with}\quad m_\nu \ge 1\quad \text{and}\quad m_1 + \ldots + m_\alpha = m\ .$$

By (72) and (73), the term (74) is dominated by $C_*^m B_{\text{left}}^2 (\delta x)^{-\alpha} \cdot B_{\text{left}}^{\alpha} \lambda_{\text{left}}^{-\frac{2}{3}m}$, which in turn is at most $C_*^m B_{\text{left}}^2 \big(\lambda_{\text{left}}^{2/3}\frac{\delta x}{B_{\text{left}}}\big)^{-m}$, since $0 \le \alpha \le m$. Hence,

$$(75)\qquad \Big|\big(\frac{d}{d\xi}\big)^m \tilde{g}(\xi)\Big| \le C_*^m \lambda_{\text{left}}^{-4/3} B_{\text{left}}^2 \big(\lambda_{\text{left}}^{2/3}\frac{\delta x}{B_{\text{left}}}\big)^{-m}\ ,\quad m \ge 0\ .$$

Also, since $g(x',0)$ is supported in $|x' - x_{\text{left}}(0)| < \delta x$, it follows that $\tilde{g}(\xi)$ is supported in $\{|\xi| \le C_\# \lambda_{\text{left}}^{2/3}\frac{\delta x}{B_{\text{left}}}\}$.

We investigate in general integrals of the form

$$(76)\quad G(x) = \int_{-\infty}^{x} \theta(\xi)[A^2(\xi) - \frac{1}{2}\xi_+^{-1/2}]\, d\xi\ ,\quad \text{with}\quad \big|\big(\frac{d}{d\xi}\big)^m \theta\big| \le C_m R^{-m}\ ,$$
$$\operatorname{supp}\theta \subset \{|\xi| < R\}\ , R > 10\ .$$

Taking $\theta(\xi) = \frac{\tilde{g}(\xi)}{\lambda_{\text{left}}^{-4/3} B_{\text{left}}^2}$ and $R = C_\# \lambda_{\text{left}}^{2/3}\frac{\delta x}{B_{\text{left}}}$, we satisfy the assumptions of (76); and (71), (76) together give

$$(77)\qquad \mathcal{G}(x) = \lambda_{\text{left}}^{-4/3} B_{\text{left}}^2 G(\lambda_{\text{left}}^{2/3} Y(x,0))\quad \text{for}\quad |x - x_{\text{left}}(0)| < c_\# B_{\text{left}}\ .$$

We study $G(x)$ separately in the cases $x > R$, $10 < x \le R$, $|x| \le 10$, $x < -10$. For $x > R$ we have $G(x) = \int_{-\infty}^{\infty} \theta(\xi)[A^2(\xi) - \frac{1}{2}\xi_+^{-1/2}]\,d\xi = O(R^{-5/2})$, by (A3) from the section "Review of Earlier Results". For $10 < x < R$ we get from (A3) that

$$G(x) = O(R^{-5/2}) - \int_x^{\infty} \theta(\xi)[A^2(\xi) - \frac{1}{2}\xi_+^{-1/2}]\,d\xi \ . \tag{78}$$

Since $[A^2(\xi) - \frac{1}{2}\xi_+^{-1/2}] = \mathrm{Re}\,\big[\frac{e^{\pm i\frac{\pi}{2}}\,e^{\frac{4}{3}i\xi^{3/2}}}{2\xi^{1/2}}\big] + \ \mathrm{Re}\ \big[\frac{(\mathrm{const})e^{\frac{4}{3}i\xi^{3/2}}}{\xi^2}\big] + O(\xi^{-7/2})$ as we saw before, it follows from (78) that

$$G(x) = O(x^{-5/2}) + \sum_{k=\pm 1} c_k \int_x^{\infty} \theta(\xi)\frac{e^{\frac{4}{3}ik\xi^{3/2}}}{\xi^{1/2}}\,d\xi + \sum_{k=\pm 1} \tilde{c}_k \int_x^{\infty} \theta(\xi)\frac{e^{\frac{4}{3}ik\xi^{3/2}d\xi}}{\xi^2}\,d\xi \ . \tag{78bis}$$

The proof of (56) gives here

$$\frac{e^{\frac{4}{3}ik\xi^{3/2}}}{\xi^q}\theta(\xi) + \frac{d}{d\xi}\Big[\sum_{\substack{m\ge 0\\ p\ge 1}} a_{kmp}e^{\frac{4}{3}ik\xi^{3/2}}\Big[\big(\frac{d}{d\xi}\big)^m\theta(\xi)\Big]\xi^{-p}\Big] = \sum_{\substack{p\ge M\\ m\ge 0}} b_{kmp}e^{\frac{4}{3}ik\xi^{3/2}}\xi^{-p}\big(\frac{d}{d\xi}\big)^m\theta(\xi) \quad (k=\pm 1)\ , \tag{79}$$

$$\text{with} \quad q = \frac{1}{2} \text{ or } 2\ .$$

The right-hand side is $O(\xi^{-M})$, so integrating (79) over $[x,\infty)$ yields

$$\int_x^{\infty} \frac{e^{\frac{4}{3}ik\xi^{3/2}}}{\xi^q}\theta(\xi)d\xi - \sum_{\substack{m\ge 0\\ p\ge 1}} a_{kmp}e^{\frac{4}{3}ikx^{3/2}}x^{-p}\theta^{(m)}(x) = O(x^{1-M}) \quad (10 < x < R)\ .$$

In particular,

$$\Big|\int_x^{\infty} \frac{e^{\frac{4}{3}ik\xi^{3/2}}}{\xi^q}\theta(\xi)d\xi\Big| \le C\,x^{-1} \quad \text{for} \quad 10 < x < R\ ,$$

so (78 bis) implies

$$|G(x)| \le Cx^{-1} \quad \text{for} \quad 10 < x < R\ . \tag{80}$$

For $|x| \leq 10$ we have

$$|G(x)| \leq \Big| \int_{\infty}^{\min(0,x)} \theta(\xi) A^2(\xi) d\xi \Big| + \Big| \int_0^{\max(0,x)} \theta(\xi)[A^2(\xi) - \frac{1}{2}\xi^{-1/2}]\, d\xi \Big|$$
$$\leq C \int_{-\infty}^{0} A^2(\xi) d\xi + C \int_0^{10} |A^2(\xi) - \frac{1}{2}\xi^{-1/2}| d\xi \leq C' \ .$$

For $x < -10$ and $-\infty < \xi < x$ we have $|A^2(\xi)| \leq \frac{C_M}{|\xi|^M}$, so $|G(x)| \leq \int_{-\infty}^{x} |\theta(\xi)| \frac{C_M}{|\xi|^M} d\xi \leq \frac{C_M}{|x|^{M-1}}$. Combining our results for the various cases, we get $|G(x)| \leq \frac{C}{1+|x|} + CR^{-5/2}$. Using this in (77) gives

$$\text{(81)} \quad |\mathcal{G}(x)| \leq C_* \frac{\lambda_{\text{left}}^{-4/3} B_{\text{left}}^2}{1 + \lambda_{\text{left}}^{2/3} |Y(x,0)|} + C_* \lambda_{\text{left}}^{-4/3} B_{\text{left}}^2 \big(\lambda_{\text{left}}^{2/3} \frac{\delta x}{B_{\text{left}}}\big)^{-5/2}$$
$$\text{for} \quad |x - x_{\text{left}}(0)| < c_{\#} B_{\text{left}} \ .$$

Also $\phi'(0) = \frac{1}{2} \int_{V<0} (-V)^{-1/2} \geq c_{\#} S_{\text{left}}^{-1/2} B_{\text{left}}$, so

$$|\mathcal{G}(x)[\phi'(0)]^{-1}| \leq C_* \lambda_{\text{left}}^{-4/3} B_{\text{left}}^2 \cdot C_{\#} S_{\text{left}}^{1/2} B_{\text{left}}^{-1} \cdot \Big[\frac{1}{1 + \lambda_{\text{left}}^{2/3} |Y(x,0)|} + \big(\lambda_{\text{left}}^{2/3} \frac{\delta x}{B_{\text{left}}}\big)^{-5/2}\Big]$$
$$= C_* \lambda_{\text{left}}^{-1/3} \cdot \Big[\frac{1}{1 + \lambda_{\text{left}}^{2/3} |Y(x,0)|} + \big(\lambda_{\text{left}}^{2/3} \frac{\delta x}{B_{\text{left}}}\big)^{-5/2}\Big]$$
$$\text{for} \quad |x - x_{\text{left}}(0)| < c_{\#} B_{\text{left}} \ .$$

Hence

$$\text{(82)} \quad \int_{|x - x_{\text{left}}(0)| < c_{\#} B_{\text{left}}} |\mathcal{G}(x)[\phi'(0)]^{-1}|^2 \, dx$$
$$\leq C_* \lambda_{\text{left}}^{-2/3} B_{\text{left}} \big[\lambda_{\text{left}}^{-2/3} + \big(\lambda_{\text{left}}^{2/3} \frac{\delta x}{B_{\text{left}}}\big)^{-5}\big] \ ,$$

again using $1 + \lambda_{\text{left}}^{2/3} |Y(x,0)| \sim 1 + \lambda_{\text{left}}^{2/3} |x - x_{\text{left}}(0)| B_{\text{left}}^{-1}$. Since $\lambda_{\text{left}}^{2/3} \frac{\delta x}{B_{\text{left}}} > 10$, we can throw away information from (82) to get

$$\text{(83)} \quad \int_{|x - x_{\text{left}}(E_0)| < c_{\#} B_{\text{left}}} |\mathcal{G}(x)[\phi'(0)]^{-1}|^2 dx \leq C_* \lambda_{\text{left}}^{-2/3} B_{\text{left}}$$
$$\text{for} \quad |E_0| < 2\hat{c}(\delta E) \ .$$

Again we have used the inclusion $\{|x - x_{\text{left}}(E_0)| < c_{\#} B_{\text{left}}\} \subset \{|x - x_{\text{left}}(0)| < c'_{\#} B_{\text{left}}\}$ for $|E_0| < 2\hat{c}(\delta E)$.

The information we just discarded is useless, since (83) will be combined with (65). Note that

$$\text{(83bis)} \qquad \text{If} \quad |E_0| \geq 2\hat{c}(\delta E)\ , \quad \text{then} \quad g(x',0) \equiv 0 \quad \text{and so} \quad \mathcal{G}(x) = 0\ .$$

We are ready to combine our results on $\mathcal{F}(x)$, $\mathcal{G}(x)$ with estimates (25)...(28) on $\rho(x,g)$. From (25), (29), (30) we get

$$\text{(84)} \quad \rho(x,g) = \frac{1}{2\pi}\int_{-\infty}^{0} g(x,E)\big(\lambda_{\text{left}}^2\big(\frac{\partial Y}{\partial x}(x,E)\big)^2 Y(x,E)\big)_+^{-1/2}\,dE - \frac{1}{2}g(x,0)\big(\lambda_{\text{left}}^2\big(\frac{\partial Y(x,0)}{\partial x}\big)^2 Y(x,0)\big)_+^{-1/2}[\phi'(0)]^{-1}\chi_-\big(\frac{1}{\pi}\phi(0)-\frac{1}{2}\big) + \frac{d}{dx}H(x)$$

with

$$\text{(85)} \qquad H(x) = \frac{1}{\pi}\mathcal{F}(x) - \chi_-\big(\frac{1}{\pi}\phi(0)-\frac{1}{2}\big)\mathcal{G}(x)[\phi'(0)]^{-1} + \int_{\inf\ I_{\text{BVP}}}^{x}\mathcal{E}_3(x')dx'\ .$$

Note that $\int_{|x-x_{\text{left}}(E_0)|<c_\# B_{\text{left}}}\big|\int_{\inf\ I_{\text{BVP}}}^{x}\mathcal{E}_3(x')dx'\big|^2 dx \ \leq\ B_{\text{left}}\big(\int_{I_{\text{BVP}}}|\mathcal{E}_3(x')|dx'\big)^2$. Hence (65), (70), (83), (83 bis) and (85) show the following.

(86) If $|E_0| < 2\hat{c}(\delta E)$, then

$$\int_{|x-x_{\text{left}}(E_0)|<c_\# B_{\text{left}}}|H(x)|^2 dx \leq C_*\big(\lambda_{\text{left}}^{-2/3} + \lambda_{\text{left}}^{-2}\big(\frac{\delta x}{B_{\text{left}}}\big)^{-5}\big)B_{\text{left}} + C_*\big(\int_{I_{\text{BVP}}}|\mathcal{E}_3(x')|dx'\big)^2 B_{\text{left}}\ .$$

(87) If $|E_0| \geq 2\hat{c}(\delta E)$, then

$$\int_{|x-x_{\text{left}}(E_0)|<c_\# B_{\text{left}}}|H(x)|^2 dx \leq C_*\lambda_{\text{left}}^{-2}\big(\frac{\delta x}{B_{\text{left}}}\big)^{-5}B_{\text{left}} + C_*\big(\int_{I_{\text{BVP}}}|\mathcal{E}_3(x')|dx'\big)^2 B_{\text{left}}\ .$$

Combining (86), (87) with (26), (27), (28), we get the following estimates for $H(x)$.

(88) If $|E_0| > 2\hat{c}(\delta E)$ then

$$\frac{1}{B_{\text{left}}}\int_{|x-x_{\text{left}}(E_0)|<c_\# B_{\text{left}}}|H(x)|^2 dx \leq C_*\lambda_{\text{left}}^{-2}\big(\frac{\delta x}{B_{\text{left}}}\big)^{-5} + C_*\Lambda_{\min}^{8\varepsilon-4}\lambda_{\text{left}}^2\big(\frac{\delta x}{B_{\text{left}}}\big) + C_*\Lambda_{\min}^{20-2N''}(\phi(\max\mathcal{J}))^2 + C_*\Lambda_{\min}^{4-2N''}\ .$$

(89) If $|E_0| \le 2\hat{c}(\delta E)$ and $\min_{k\in\mathbb{Z}} |\phi(0) - \pi(k+\frac{1}{2})| > \overline{C}_\# \Lambda_{\min}^{-1}$, then

$$\frac{1}{B_{\text{left}}} \int_{|x - x_{\text{left}}(E_0)| < c_\# B_{\text{left}}} |H(x)|^2 dx \le C_* \lambda_{\text{left}}^{-2/3} + C_* \lambda_{\text{left}}^{-2} \left(\frac{\delta x}{B_{\text{left}}}\right)^{-5}$$

$$+ C_* \Lambda_{\min}^{8\varepsilon - 4} \lambda_{\text{left}}^2 \left(\frac{\delta x}{B_{\text{left}}}\right) + C_* \Lambda_{\min}^{-2} \lambda_{\text{left}}^2 \left(\frac{\delta x}{B_{\text{left}}}\right)^2$$

$$+ C_* \Lambda_{\min}^{20 - 2N''} (\phi(\max \mathcal{J}))^2 + C_* \Lambda_{\min}^{4 - 2N''} .$$

(90) If $|E_0| \le 2\hat{c}(\delta E)$ and $\min_{k\in\mathbb{Z}} |\phi(0) - \pi(k+1/2)| \le \overline{C}_\# \Lambda_{\min}^{-1}$, then

$$\frac{1}{B_{\text{left}}} \int_{|x - x_{\text{left}}(E_0)| < c_\# B_{\text{left}}} |H(x)|^2 dx \le C_* \lambda_{\text{left}}^{-2/3} + C_* \lambda_{\text{left}}^{-2} \left(\frac{\delta x}{B_{\text{left}}}\right)^{-5}$$

$$+ C_* \Lambda_{\min}^{-2} \lambda_{\text{left}}^2 \left(\frac{\delta x}{B_{\text{left}}}\right) + C_* \Lambda_{\min}^{20 - 2N''} (\phi(\max \mathcal{J}))^2 + C_* \Lambda_{\min}^{4 - 2N''} .$$

If in (84) we could replace $\lambda_{\text{left}}^2 (\frac{\partial Y}{\partial x})^2 Y(x, E)$ by $E - V(x)$, then we would have $\rho(x, g) = \rho_{sc}(x, g) + \frac{dH}{dx}$, with estimates (88)…(90) for H. On the other hand, equation (7) shows that $\lambda_{\text{left}}^2 (\frac{\partial Y}{\partial x})^2 Y(x, E)$ is close to $E - V(x)$, so we proceed as follows.

Lemma 4. *Let $X(t, \tau)$ be smooth on $\{|t|, |\tau| \le 1\}$, with $\frac{\partial X}{\partial t}(t, 0) > c > 0$ and $|X(0, 0)| << 1$. Let $\varphi(t)$ be supported in $\{|t| \le \delta\}$ and satisfy $|\varphi^{(m)}(t)| \le C_m \delta^{-m}$. Set $F(s, \tau) = \int_{-\infty}^{s} \varphi(t)(X(t, \tau))_+^{-1/2} dt$. Then $\int_{|s|<1} |F(s, 0) - F(s, \tau)|^2 ds \le C\tau^2(\delta^{-1} + \ell n \frac{1}{|\tau|})$ for $|\tau| << 1$.*

Proof. Set $\xi = X(t, \tau)$, and define $t = T(\xi, \tau)$ to be the solution of $X(t, \tau) = \xi$. Changing variable from t to ξ in the definition of F gives $F(s, \tau) = \int_{\{0 < \xi < X(s, \tau)\}} \varphi(T(\xi, \tau)) \xi^{-1/2} (\frac{\partial T(\xi, \tau)}{\partial \xi}) d\xi$. Hence

$$
\begin{aligned}
&|F(s,\tau)-F(s,0)| \\
&\quad \le \int_{\min\{X(s,\tau),X(s,0)\}<\xi<\max\{X(s,\tau),X(s,0)\}} |\varphi(T(\xi,\tau))|\ |\xi|^{-1/2}\ |\frac{\partial T(\xi,\tau)}{\partial \xi}|\, d\xi \\
&\qquad\qquad + \int_{0<\xi<X(s,0)} |\varphi(T(\xi,\tau))-\varphi(T(\xi,0))| \xi^{-1/2} |\frac{\partial T(\xi,\tau)}{\partial \xi}|\, d\xi \\
&\qquad + \int_{0<\xi<X(s,0)} |\varphi(T(\xi,0))| \xi^{-1/2} |\frac{\partial T(\xi,\tau)}{\partial \xi} - \frac{\partial T(\xi,0)}{\partial \xi}|\, d\xi \equiv A_1+A_2+A_3\ .
\end{aligned}
$$

In A_1, the region of integration is contained in $\{|\xi - X(s,0)| < C|\tau|\}$, and the integrand is dominated by $C|\xi|^{-1/2}$. Hence $A_1 \le \frac{C|\tau|}{(|X(s,0)|+|\tau|)^{1/2}}$.

To handle A_2, note that $\xi \mapsto \varphi(T(\xi,\tau))$ and $\xi \mapsto \varphi(T(\xi,0))$ are supported in intervals of length $\le C\delta$, and that $|\varphi'| < C\delta^{-1}$, while $|T(\xi,\tau) - T(\xi,0)| \le C\tau$. Hence the integrand in A_2 is at most $C\delta^{-1}|\tau||\xi|^{-1/2}$, and the support of integrand is contained in a union of two intervals of length $\le C\delta$. Therefore, $A_2 \le$ $C\delta^{-1}|\tau| \max_{|J|<C\delta} \int_J |\xi|^{-1/2} d\xi$, i.e. $A_2 \le C\delta^{-1/2}|\tau|$.

In A_3, the integrand is dominated by $C|\tau|\xi^{-1/2}$, and the region of integration is contained in $(0,C)$. Thus, $A_3 \le C|\tau|$. Combining our estimates for A_1, A_2, A_3, we get $|F(s,\tau) - F(s,0)| \le C|\tau|(|X(s,0)| + |\tau|)^{-1/2} + C\delta^{-1/2}|\tau|$ for $|s| \le 1$. Since $\frac{\partial}{\partial s} X(s,0) > c > 0$ for $|s| \le 1$, it follows that

$$
\begin{aligned}
\int_{|s|\le 1} |F(s,\tau)-F(s,0)|^2 ds &\le C\tau^2 \int_{|s|\le 1} (|X(s,0)|+|\tau|)^{-1} ds + C\delta^{-1}\tau^2 \\
&\le C\tau^2(\delta^{-1} + \ell\mathrm{n}\frac{1}{|\tau|})\ .
\end{aligned}
$$

■

To apply the lemma, fix E with $|E - E_0| < 2\hat{c}(\delta E)$, and set $X(t,\tau)$ $= B^2_{\text{left}}\big[\big(\frac{\partial Y(x,E)}{\partial x}\big)^2 Y(x,E) + \tau\{Y(x,E),x\}\big]$ with $x = x_{\text{left}}(E) + B_{\text{left}}t$. Also, we set $\varphi(t) = g(x_{\text{left}}(E) + B_{\text{left}}t, E)$ and $\delta = \big(\frac{\delta x}{B_{\text{left}}}\big)$. The hypotheses of Lemma 4 are then satisfied, and we use the conclusion of Lemma 4 with $\tau = \lambda^{-2}_{\text{left}}$. Note that

$\delta^{-1} + \ell\text{n}\frac{1}{|\tau|} \sim \delta^{-1} = \left(\frac{\delta x}{B_{\text{left}}}\right)^{-1}$ since we take $\frac{\delta x}{B_{\text{left}}} < \lambda_{\text{left}}^{-\varepsilon}$. Hence, Lemma 4 gives

$$(91) \qquad C_*\left(\frac{\delta x}{B_{\text{left}}}\right)^{-1}\lambda_{\text{left}}^{-4} \geq \int_{|s|\leq 1} |F(s,\lambda_{\text{left}}^{-2}) - F(s,0)|^2 ds \ , \quad \text{with}$$

$$F(s,\tau) = \int_{-\infty}^{s} g(x_{\text{left}}(E) + B_{\text{left}}t, E)\cdot\left[B_{\text{left}}^2((\partial_x Y)^2 Y + \tau\{Y,x\})\right]_+^{-1/2} dt$$
$$= \int_{-\infty}^{x_{\text{left}}(E)+B_{\text{left}}s} g(x,E)B_{\text{left}}^{-1}\left[(\partial_x Y)^2 Y + \tau\{Y,x\}\right]_+^{-1/2} B_{\text{left}}^{-1} dx \ .$$
(92)

Hence, making the change of variable $x' = x_{\text{left}}(E) + B_{\text{left}}s$ in (91), (92), we get

$$(93) \quad \frac{1}{B_{\text{left}}}\int_{|x'-x_{\text{left}}(E)|<B_{\text{left}}} \Big| \int_{-\infty}^{x'} g(x,E)(\lambda_{\text{left}}^2(\partial_x Y)^2 Y)_+^{-1/2} dx$$
$$- \int_{-\infty}^{x'} g(x,E)(\lambda_{\text{left}}^2(\partial_x Y)^2 Y + \{Y,x\})_+^{-1/2} dx\Big|^2 dx'$$
$$\leq C_* B_{\text{left}}^{+4}\cdot\lambda_{\text{left}}^{-2}\cdot\left(\frac{\delta x}{B_{\text{left}}}\right)^{-1}\lambda_{\text{left}}^{-4} \quad \text{for} \quad |E-E_0| < 2\hat{c}(\delta E) \ .$$

In (93) we may shrink the region of the integration from $\{|x'-x_{\text{left}}(E)| < B_{\text{left}}\}$ to $\{|x'-x_{\text{left}}(E_0)| < c_{\#}B_{\text{left}}\}$. Since $g(x,E)$ is supported in $\{|E-E_0| \leq \hat{c}(\delta E)\}$, it follows from (93) and the triangle inequality that

$$(94) \quad \frac{1}{2\pi}\int_{-\infty}^{x'}\int_{-\infty}^{0} g(x,E)(\lambda_{\text{left}}^2(\partial_x Y)Y)_+^{-1/2} dE dx$$
$$= \frac{1}{2\pi}\int_{-\infty}^{x'}\int_{-\infty}^{0} g(x,E)\left(\lambda_{\text{left}}^2(\partial_x Y)^2 Y + \{Y,x\}\right)_+^{-1/2} dE dx + \quad \text{Error}\,(x')$$

with

$$(95) \quad \frac{1}{B_{\text{left}}}\int_{|x-x_{\text{left}}(E_0)|<c_{\#}B_{\text{left}}} |\,\text{Error}(x)|^2 dx \leq C_* B_{\text{left}}^4 \lambda_{\text{left}}^{-6}\left(\frac{\delta x}{B_{\text{left}}}\right)^{-1}(\delta E)^2$$
$$\leq C_* S_{\text{left}}^2 B_{\text{left}}^4 \lambda_{\text{left}}^{-6}\left(\frac{\delta x}{B_{\text{left}}}\right)^{-1} = C_*\lambda_{\text{left}}^{-2}\left(\frac{\delta x}{B_{\text{left}}}\right)^{-1} \ .$$

Also if $|E_0| \leq 2\hat{c}(\delta E)$, then (93) gives

$$(96) \quad \frac{1}{2}\int_{-\infty}^{x'} g(x,0)\left(\lambda_{\text{left}}^2\left(\frac{\partial}{\partial x}Y(x,0)\right)^2 Y(x,0)\right)_+^{-1/2} dx[\phi'(0)]^{-1}$$
$$= \frac{1}{2}\int_{-\infty}^{x'} g(x,0)\left(\lambda_{\text{left}}^2\left(\frac{\partial}{\partial x}Y(x,0)\right)^2 Y(x,0) + \{Y(x,0),x\}\right)_+^{-1/2}[\phi'(0)]^{-1} dx +$$
$$\text{Error}'(x') \ ,$$

with

$$(97)\quad \frac{1}{B_{\text{left}}}\int_{|x-x_{\text{left}}(E_0)|<c_\# B_{\text{left}}} |\,\text{Error}'(x)|^2 dx \le C_* B_{\text{left}}^4 \lambda_{\text{left}}^{-6}\Big(\frac{\delta x}{B_{\text{left}}}\Big)^{-1}[\phi'(0)]^{-2}$$

$$\le C_* B_{\text{left}}^4 \lambda_{\text{left}}^{-6}\Big(\frac{\delta x}{B_{\text{left}}}\Big)^{-1}\cdot S_{\text{left}} B_{\text{left}}^{-2} \quad \text{(by (19 bis))}$$

$$= C_* S_{\text{left}} B_{\text{left}}^2 \lambda_{\text{left}}^{-6}\Big(\frac{\delta x}{B_{\text{left}}}\Big)^{-1} = C_* \lambda_{\text{left}}^{-4}\Big(\frac{\delta x}{B_{\text{left}}}\Big)^{-1}.$$

If $|E_0| > 2\hat{c}(\delta E)$, then (97) holds trivially, since $g(x,0)\equiv 0$.

The right-hand sides of (95) and (97) are dominated by the right-hand sides of (88), (89), (90), since the latter all contain $\lambda_{\text{left}}^{-2}\big(\frac{\delta x}{B_{\text{left}}}\big)^{-5}$. Hence in (84), we may replace $\lambda_{\text{left}}^2(\partial_x Y)^2 Y$ by $\lambda_{\text{left}}^2(\partial_x Y)^2 Y + \{Y,x\}$ on the right, without affecting the error estimates (88)…(90). That is,

$$(98)\quad \rho(x,g) = \frac{1}{2\pi}\int_{-\infty}^{0} g(x,E)\Big(\lambda_{\text{left}}^2\Big(\frac{\partial Y(x,E)}{\partial x}\Big)^2 Y(x,E) + \{Y(x,E),x\}\Big)_+^{-1/2} dE$$

$$-\frac{1}{2} g(x,0)\Big(\lambda_{\text{left}}^2\Big(\frac{\partial Y(x,0)}{\partial x}\Big)^2 Y(x,0) + \{Y(x,0),x\}\Big)_+^{-1/2}[\phi'(0)]^{-1}\chi_-\Big(\frac{1}{\pi}\phi(0)-\frac{1}{2}\Big)$$

$$+\frac{d}{dx}\mathcal{H}(x)\,,\quad \text{with}\quad \mathcal{H}(x)\quad \text{satisfying the following}\,.$$

(99)

If $|E_0| > 2\hat{c}(\delta E)$, then

$$\frac{1}{B_{\text{left}}}\int_{|x-x_{\text{left}}(E_0)|<c_\# B_{\text{left}}} |\mathcal{H}(x)|^2 dx \le C_*\lambda_{\text{left}}^{-2}\Big(\frac{\delta x}{B_{\text{left}}}\Big)^{-5} + C_*\Lambda_{\min}^{8\varepsilon-4}\lambda_{\text{left}}^2\Big(\frac{\delta x}{B_{\text{left}}}\Big) + C_*\Lambda_{\min}^{20-2N''}(\phi(\max\mathcal{J}))^2 + C_*\Lambda_{\min}^{4-2N''}\,.$$

(100) If $|E_0| \le 2\hat{c}(\delta E)$ and $\min_{k\in\mathbb{Z}}|\phi(0)-\pi(k+1/2)| > \overline{C}_\#\Lambda_{\min}^{-1}$, then

$$\frac{1}{B_{\text{left}}}\int_{|x-x_{\text{left}}(E_0)|<c_\# B_{\text{left}}} |\mathcal{H}(x)|^2 dx \le C_*\lambda_{\text{left}}^{-2/3} + C_*\lambda_{\text{left}}^{-2}\Big(\frac{\delta x}{B_{\text{left}}}\Big)^{-5}$$

$$+ C_*\Lambda_{\min}^{8\varepsilon-4}\lambda_{\text{left}}^2\Big(\frac{\delta x}{B_{\text{left}}}\Big)$$

$$+ C_*\Lambda_{\min}^{-2}\lambda_{\text{left}}^2\Big(\frac{\delta x}{B_{\text{left}}}\Big)^2 + C_*\Lambda_{\min}^{20-2N''}(\phi(\max\mathcal{J}))^2 + C_*\Lambda_{\min}^{4-2N''}\,.$$

(101) If $|E_0| \le 2\hat{c}(\delta E)$ and $\min_{k\in\mathbb{Z}} |\phi(0) - \pi(k+1/2)| \le \overline{C}_{\#}\Lambda_{\min}^{-1}$, then

$$\frac{1}{B_{\text{left}}} \int_{|x - x_{\text{left}}(E_0)| < c_{\#} B_{\text{left}}} |\mathcal{H}(x)|^2 dx \le C_* \lambda_{\text{left}}^{-2/3} + C_* \lambda_{\text{left}}^{-2} \left(\frac{\delta x}{B_{\text{left}}}\right)^{-5} + C_* \Lambda_{\min}^{-2} \lambda_{\text{left}}^{2} \left(\frac{\delta x}{B_{\text{left}}}\right) + C_* \Lambda_{\min}^{20-2N''} (\phi(\max \mathcal{J}))^2 + C_* \Lambda_{\min}^{4-2N''} .$$

The terms other than $\frac{d\mathcal{H}}{dx}$ on the right in (98) are very close to the semiclassical density $\rho_{sc}(x,g)$, by virtue of (7). In fact (7) shows that

$$\lambda_{\text{left}}^2 \left(\frac{\partial Y}{\partial x}\right)^2 Y + \{Y, x\} = E - V(x) + f(x,E) \quad \text{with} \quad |f(x,E)| \le C_{\#} \lambda_{\text{left}}^{-N'} S_{\text{left}} \quad \text{in supp } g(x,E) .$$

Since also $|E - V(x)| \sim \frac{S_{\text{left}}}{B_{\text{left}}} |x - x_{\text{left}}(E)|$ in supp $g(x,E)$, and since (8), (9) yield $|E - V(x) + f(x,E)| \sim \frac{S_{\text{left}}}{B_{\text{left}}} |x - x_{\text{perturbed}}(E)|$ in supp g for some $x_{\text{perturbed}}(E)$ we have for fixed E (satisfying $|E - E_0| < 2\hat{c}\delta E$) the following.

(102)
$$\begin{aligned} &\int_{x \in I_{\text{BVP}}} |g(x,E)| \left| \big(E - V(x) + f(x,E)\big)_+^{-1/2} - \big(E - V(x)\big)_+^{-1/2} \right| dx \\ &\le \int_{(\delta x) > |x - x_{\text{left}}(E)| > \lambda_{\text{left}}^{-\frac{2}{3}N'} B_{\text{left}}} C_* \lambda_{\text{left}}^{-\frac{1}{3}N'} \left(\frac{S_{\text{left}}}{B_{\text{left}}} |x - x_{\text{left}}(E)|\right)^{-1/2} dx \\ &\quad + \int_{|x - x_{\text{left}}(E)| < \lambda_{\text{left}}^{-\frac{2}{3}N'} B_{\text{left}}} \Big\{ C_* \left(\frac{S_{\text{left}}}{B_{\text{left}}} |x - x_{\text{left}}(E)|\right)^{-1/2} \\ &\qquad + C_* \left(\frac{S_{\text{left}}}{B_{\text{left}}} |x - x_{\text{perturbed}}(E)|\right)^{-1/2} \Big\} dx \\ &\le C_* \lambda_{\text{left}}^{-\frac{1}{3}N'} S_{\text{left}}^{-1/2} B_{\text{left}} . \end{aligned}$$

Since $g(x,E)$ is supported inside $\{|E - E_0| < \hat{c}\delta E\}$, we may integrate (102) to obtain

(103)
$$\begin{aligned} &\int_{x \in I_{\text{BVP}}} \left| \int_{-\infty}^{0} g(x,E) \big(E - V(x) + f(x,E)\big)_+^{-1/2} dE \right. \\ &\qquad \left. - \int_{-\infty}^{0} g(x,E) (E - V(x))_+^{-1/2} dE \right| dx \\ &\le C_* \lambda_{\text{left}}^{-\frac{1}{3}N'} S_{\text{left}}^{-1/2} B_{\text{left}} (\delta E) \le C_* \lambda_{\text{left}}^{-\frac{1}{3}N'} S_{\text{left}}^{+1/2} B_{\text{left}} = C_* \lambda_{\text{left}}^{1 - \frac{1}{3}N'} . \end{aligned}$$

Also, (19 bis) and (102) imply

(104)
$$\int_{x\in I_{\mathrm{BVP}}} |g(x,0)\big(-V(x)+f(x,0)\big)_+^{-1/2}[\phi'(0)]^{-1} - g(x,0)\big(-V(x)\big)_+^{-1/2}[\phi'(0)]^{-1}|dx$$
$$\leq C_*\lambda_{\mathrm{left}}^{-\frac{1}{3}N'} \quad \text{for} \quad |E_0| < 2\hat{c}(\delta E) \ .$$

If $|E_0| \geq 2\hat{c}(\delta E)$, then (104) holds trivially, since $g(x,0)\equiv 0$ in that case. Putting (103), (104) into (98) and recalling the definition of $\rho_{sc}(x,g)$, we get the main result of this section.

Airey Density Lemma. *Suppose the potential $V(x)$, the weight functions $S(x)$, $B(x)$, and the cutoff function $g(x,E)$ satisfy assumptions (X0)…(X12). Then the microlocalized density is given by $\rho(x,g)=\rho_{sc}(x,g)+\frac{d}{dx}H(x)$, where $\rho_{sc}(x,g)$ is the semiclassical approximation, and $H(x)$ satisfies the following estimates on $I_{\mathrm{left}}=\{|x-x_{\mathrm{left}}(E_0)|<c_{\#}B_{\mathrm{left}}\}$.*

(A) If $|E_0|>2\hat{c}(\delta E)$, then

$$\frac{1}{B_{\mathrm{left}}}\int_{I_{\mathrm{left}}}|H(x)|^2dx \leq C_*\lambda_{\mathrm{left}}^{-2}\Big(\frac{\delta x}{B_{\mathrm{left}}}\Big)^{-5} + C_*\Lambda_{\mathrm{min}}^{8\varepsilon-4}\lambda_{\mathrm{left}}^2\Big(\frac{\delta x}{B_{\mathrm{left}}}\Big)$$
$$+ C_*\Lambda_{\mathrm{min}}^{-N''}(\phi(max\ \mathcal{J}))^2 + C_*\Lambda_{\mathrm{min}}^{-N''} \ .$$

(B) If $|E_0|\leq 2\hat{c}(\delta E)$ and $\min_{k\in\mathbb{Z}}|\phi(0)-\pi(k+1/2)|>\overline{C}_{\#}\Lambda_{\mathrm{min}}^{-1}$, then

$$\frac{1}{B_{\mathrm{left}}}\int_{I_{\mathrm{left}}}|H(x)|^2dx \leq C_*\lambda_{\mathrm{left}}^{-2/3} + C_*\lambda_{\mathrm{left}}^{-2}\Big(\frac{\delta x}{B_{\mathrm{left}}}\Big)^{-5} + C_*\Lambda_{\mathrm{min}}^{8\varepsilon-4}\lambda_{\mathrm{left}}^2\Big(\frac{\delta x}{B_{\mathrm{left}}}\Big)$$
$$+ C_*\Lambda_{\mathrm{min}}^{-2}\lambda_{\mathrm{left}}^2\Big(\frac{\delta x}{B_{\mathrm{left}}}\Big)^2 + C_*\Lambda_{\mathrm{min}}^{-N''}(\phi(max\ \mathcal{J}))^2 + C_*\Lambda_{\mathrm{min}}^{-N''} \ .$$

(C) If $|E_0|\leq 2\hat{c}(\delta E)$ and $\min_{k\in\mathbb{Z}}|\phi(0)-\pi(k+1/2)|\leq\overline{C}_{\#}\Lambda_{\mathrm{min}}^{-1}$, then

$$\frac{1}{B_{\mathrm{left}}}\int_{I_{\mathrm{left}}}|H(x)|^2dx \leq C_*\lambda_{\mathrm{left}}^{-2/3} + C_*\lambda_{\mathrm{left}}^{-2}\Big(\frac{\delta x}{B_{\mathrm{left}}}\Big)^{-5}$$
$$+ C_*\Lambda_{\mathrm{min}}^{-2}\lambda_{\mathrm{left}}^2\Big(\frac{\delta x}{B_{\mathrm{left}}}\Big) + C_*\Lambda_{\mathrm{min}}^{-N''}(\phi(max\ \mathcal{J}))^2 + C_*\Lambda_{\mathrm{min}}^{-N''} \ .$$

Here, the constants $c_\#$ *and* $\overline{C}_\#$ *are determined by* ε, K, N, c, C, c_1, C_α *in (X0)... (X12), while* C_* *is determined by* ε, K, N, c, C, c_1, C_α, $\hat{c}$, $\hat{C}_{\alpha\beta}$ *in (X0)... (X12).*

Remark. Since $E_{\text{upper}} \equiv \max \mathcal{J} \le 0$, we have $\phi(\max\ \mathcal{J}) = \int_I (E_{\text{upper}} - V(x))_+^{1/2} dx \le \int_{I_{\text{BVP}}} (-V(x))_+^{1/2} dx$.

THE MICROLOCALIZED DENSITY IN THE OSCILLATORY REGION

In this section we study $\rho(x,g)$ for $g(x,E)$ supported in the region where eigenfunctions are given approximately by

$$u(x) \sim \mathrm{Re}\Big[e^{\pm i\frac{\pi}{4}} \exp\big(i\int_{x_{\mathrm{left}}(E)}^{x} (E-V(t))^{1/2}\ dt\big) \cdot \frac{(1+u_{\mathrm{left}}(x))}{(E-V(x))^{1/4}}\Big] \ ,$$

or by

$$u(x) \sim \mathrm{Re}\Big[e^{\mp i\frac{\pi}{4}} \exp\big(-i\int_{x}^{x_{\mathrm{rt}}(E)} (E-V(t))^{1/2}\ dt\big) \cdot \frac{(1+u_{\mathrm{rt}}(x))}{(E-V(x))^{1/4}}\Big] \ .$$

Our set-up and assumptions on $V(x)$ and $g(x,E)$ are as follows.

Set-Up: We are given the following data.

A potential $V(x)$ defined on an interval I_{BVP};

Weight functions $S(x)$, $B(x) > 0$ defined on a subinterval $I \subset I_{\mathrm{BVP}}$;

An interval $[E_{\ell o}, E_{hi}] \subset (-\infty, 0]$;

A point $x_0 \in I$;

A function $g(x,E)$ defined on $I_{\mathrm{BVP}} \times \mathbb{R}^1$;

Positive constants ε, K, N, c, C, c_1, c_2, c_3, C_α, $\hat{c}$, $\hat{c}_1$, $\hat{C}$, $\hat{C}_{\alpha\beta}$.

Assumptions:

(A1) For all $E_0 \in [E_{\ell o}, E_{hi}]$, the hypotheses of the WKB Theorems are satisfied, with $E_\infty = 0$, and with ε, K, N, c, C, c_1, c_2, C_α as in our present *set-up*.

Denote by $\Lambda(E_0)$, $S_{\min}(E_0)$ the quantities called Λ, $S_{\min}$ in the section on the WKB Theorems. Set $\Lambda_{\min} = \inf\{\Lambda(E_0)\colon E_0 \in [E_{\ell o}, E_{hi}]\}$.

(A2) There is a point $\tilde{x}_0 \in I$ with $|\tilde{x}_0 - x_0| < cB(x_0)$ and $E_{\ell o} - V(\tilde{x}_0) > c_3 S(\tilde{x}_0)$.

(A3) Let $x \in I_{\mathrm{BVP}}$ and $E \in \mathbb{R}^1$ be given. Then $g(x,E) = 0$ unless the following conditions are satisfied: $|x - x_0| < \hat{c}B(x_0)$; either $E_{\ell o} + \hat{c}_1 S_{\min}(E_{\ell o}) \le E \le E_{hi}$ or else $E > 0$;

$$\hat{c}\tau S(x_0) < E - V(x) < \hat{C}\tau S(x_0) \ .$$

Moreover, $\hat{c}$, $\hat{c}_1 > 0$ are bounded above by a small, positive number determined by ε, K, N, c, C, c_1, c_2, c_3, C_α.

(A4) $\Lambda_{\min}^{\varepsilon-\frac{2}{3}} < \tau \leq 1$.

(A5) $|\partial_x^\alpha \partial_E^\beta g(x,E)| \leq \hat{C}_{\alpha\beta}(\tau B(x_0))^{-\alpha}(\tau S(x_0))^{-\beta}$.

(A6) $\Lambda_{\min}$ is bounded below by a large positive number determined by ε, K, N, c, C, c_1, c_2, c_3, C_α, $\hat{c}$, $\hat{c}_1$, $\hat{C}$, $\hat{C}_{\alpha\beta}$.

Let $c_\#$, $C_\#$, $C'_\#$ etc. denote constants determined by ε, K, N, c, C, c_1, c_2, c_3, C_α. Let c_*, C_*, C'_* etc. denote constants determined by ε, K, N, c, C, c_1, c_2, c_3, C_α, $\hat{c}$, $\hat{c}_1$, $\hat{C}$, $\hat{C}_{\alpha\beta}$.

(That is, $C_\#$ is determined by the constants appearing in our hypotheses on $V(x)$, $S(x)$, $B(x)$ and $[E_{\ell o}, E_{hi}]$. Constants C_* may depend also on the constants appearing in our assumptions on the function $g(x,E)$.)

We may assume that the set $G = \{(x,E) \mid E_{\ell o} + \hat{c}_1 S_{\min}(E_{\ell o}) \leq E \leq E_{hi}, |x - x_0| < \hat{c}B(x_0), \hat{c}\tau S(x_0) < E - V(x) < \hat{C}\tau S(x_0)\}$ is non-empty. For, if G were empty, then by (A3) we would have $g(x,E) = 0$ for $E \leq 0$, so that the results of this section on $\rho(x,g) - \rho_{sc}(x,g)$ would hold trivially.

If $G \neq \emptyset$, then one of the following must hold:

(i) $\tau \geq c_* > 0$.

(ii) $G \subset \{(x,E) \mid |x - x_0| < \hat{c}B(x_0),\ x - x_{\text{left}}(E) \sim \tau B(x_0)\}$

(iii) $G \subset \{(x,E) \mid |x - x_0| < \hat{c}B(x_0),\ x_{\text{rt}}(E) - x \sim \tau B(x_0)\}$.

Here $A \sim B$ means $c_* < A/B < C_*$.

We give a discussion that applies to cases (i) and (ii). Case (iii) may be reduced to case (ii) by just studying $V(-x)$ in place of $V(x)$.

Next we recall what we have learned about the eigenvalues and eigenfunctions of $-\frac{d^2}{dx^2} + V(x)$. Applying for each $E_0 \in [E_{\ell o}, E_{hi}]$ the WKB Eigenfunction Theorem and the lemma on phases in the section on the WKB Theorems, as well as the Corollary to the Reformulated Eigenvalue Theorem, we obtain the following results.

The eigenvalues in $[E_{\ell o}, E_{hi}]$ may be written as $E_{k_{\ell o}}, \dots, E_{k_{hi}}$ with

$$|\phi(E_k) - \pi(k+1/2)| \leq C_\# \Lambda_{\min}^{-1} \quad \text{for} \quad k_{\ell o} \leq k \leq k_{hi} \tag{1}$$

$$\{k \mid k_{\ell o} \leq k \leq k_{hi}\} = \mathbb{Z} \cap [a,b], \quad \text{where} \tag{2}$$

$$a = a_0 + \omega_{\ell o}, \ b = b_0 + \omega_{hi} \ ; \quad |\omega_{\ell o}|, |\omega_{hi}| \leq C_\# \Lambda_{\min}^{-1} \tag{3}$$

$$a_0 = \frac{1}{\pi}\phi(E_{\ell o}) - \frac{1}{2}, \quad b_0 = \frac{1}{\pi}\phi(E_{hi}) - \frac{1}{2} \ . \tag{4}$$

The (normalized) eigenfunctions $u_k(x)$ corresponding to the E_k may be written as

$$u_k(x) = b_k \ \mathrm{Re}\Big[\frac{e^{i\eta(x,E_k)}}{(E_k - V(x))^{1/4}} \ (1 + w_k(x))\Big] + \varepsilon_k(x) \quad \text{for } (x, E_k) \in \text{supp } g \ , \tag{5}$$

with

$$\eta(x,E) = \pm\frac{\pi}{4} + \int_{x_{\text{left}}(E)}^{x} (E - V(t))^{1/2} \ dt \tag{6}$$

$$|w_k(x)| \leq C_* \Lambda_{\min}^{-1} \tau^{-3/2}, \ |\mathrm{Re} \ w_k(x)| \leq C_* \Lambda_{\min}^{-2} \tau^{-3} \quad \text{for } (x, E_k) \in \text{supp } g \tag{7}$$

$$|\partial_x^\alpha w_k(x)| \leq C_*^\alpha (\tau B(x_0))^{-\alpha} \quad \text{for } (x, E_k) \in \text{supp } g \tag{8}$$

$$|b_k^2 \phi'(E_k) - 1| \leq C_\# \Lambda_{\min}^{4\varepsilon - 2} \tag{9}$$

$$\int_{I_{\text{BVP}}} |\varepsilon_k(x)|^2 \ dx \leq \Lambda_{\min}^{-N''} \ . \tag{10}$$

The phase $\phi(E) = \int_I (E - V(x))_+^{1/2} dx$ satisfies the estimates

$$(11) \quad \left|\left(\frac{d}{dE}\right)^m \phi(E)\right| \leq C_\#^m (S_{\min}(E)\phi'(E))(S_{\min}(E))^{-m} \ (m \geq 1)$$
$$\text{for} \quad E \in [E_{\min}, E_{\max}] \equiv \bigcup_{E_0 \in [E_{\ell o}, E_{hi}]} \{E \mid |E - E_0| < c_\# S_{\min}(E_0)\} \ .$$

To see (1)...(4), we invoke the Corollary to the Reformulated Eigenvalue Theorem. (The hypotheses of that Corollary hold, by *ASSUMPTION* (A1) and $G \neq \emptyset$. In particular, $G \neq \emptyset$ implies $\hat{c}_1 S_{\min}(E_{\ell o}) + E_{\ell o} \leq E_{hi}$, which yields $\phi(E_{hi}) - \phi(E_{\ell o}) > 100$.)

To deduce (5)...(10) from the WKB Eigenfunction Theorem, we use the fact that $|x - x_{\text{left}}(E)| > c_* \tau B(x_{\text{left}}(E))$, $|x - x_{\text{rt}}(E)| > c_* \tau B(x_{\text{rt}}(E))$ for $(x, E) \in \text{supp } g$, by virtue of our assumption (A3) on the support of g.

In (11) we take $S_{\min}(E) = \inf_{x_{\text{left}}(E) < x < x_{\text{rt}}(E)} S(x)$. This is slightly more general than our previous definition of $S_{\min}(E)$, since we do not assume $E \leq 0$.

Note that on $[a_{\min}, b_{\max}] =$ image of $[E_{\min}, E_{\max}]$ under $E \mapsto \frac{1}{\pi}\phi(E) - \frac{1}{2} = t$, we may solve for E as a function of t, and the solution $E(t)$ satisfies

$$(12) \quad \left|\left(\frac{d}{dt}\right)^m E(t)\right| \leq C_\#^m S_{\min}(E) \cdot (S_{\min}(E)\phi'(E))^{-m} \ (m \geq 1)$$
$$\text{for } E = E(t), t \in [a_{\min}, b_{\max}].$$

This follows from (11). Also from (11) we get

$$(13) \quad \phi'(E) \text{ has constant order of magnitude on } \{E \mid |E - E_0| < c_\# S_{\min}(E_0)\} \ .$$

It will be useful to work with energy intervals of the form $J(E) = \{E' \mid |E' - E| < c_\# (\phi'(E))^{-1}\}$.

Since $\phi'(E) \geq c_\# S^{-1/2}(\tilde{x}) B(\tilde{x})$ for $x_{\text{left}}(E) < \tilde{x} < x_{\text{rt}}(E)$, $E \in [E_{\min}, E_{\max}]$, we have

$$(14) \quad \phi'(E) \geq c_\# S^{-1/2}(x_0) B(x_0) \quad \text{for} \quad E \in [E_{\min}, E_{\max}] \ ,$$

$$(15) \qquad \phi'(E)S_{\min}(E) \geq c_{\#}\Lambda_{\min} \quad \text{for} \quad E \in [E_{\min}, E_{\max}] \ .$$

(To prove (14), we take $\tilde{x}$ to be the point $\tilde{x}_0$ given in (A2). To prove (15), take $\tilde{x}$ so that $S(\tilde{x}) \sim S_{\min}(E)$, and then recall that $\lambda(\tilde{x}) \geq \Lambda(E) \geq c_{\#}\Lambda_{\min}$.)

¿From (13) and (15), we get $\phi'(E') \sim \phi'(E)$ for $E' \in J(E)$, and so the number of $E_k \in J(E)$ is at most $C_{\#}$, by (1). If $|\tilde{E} - E| \leq c'_{\#}(\phi'(E))^{-1}$, then $E \in J(\tilde{E})$ because $\phi'(\tilde{E}) \sim \phi'(E)$.

Another consequence of (15) is that $\phi(E_{\min}) \leq \phi(E_{\ell o} - c_{\#}S_{\min}(E_{\ell o})) \leq \phi(E_{\ell o}) - c_{\#}\Lambda_{\min}$. Similarly, $\phi(E_{\max}) \geq \phi(E_{hi}) + c_{\#}\Lambda_{\min}$. These inequalities and (3), (4) show that a_0, b_0, a, b all belong to $[a_{\min}, b_{\max}]$.

Note that $\frac{dE(t)}{dt}$ has constant order of magnitude for t between a_0 and a, and thus $|E(a) - E_{\ell o}| = |E(a) - E(a_0)| \leq C_{\#}\frac{dE(t)}{dt}\ |_{t=a_0} \cdot |a - a_0| \leq C_{\#}(\phi'(E_{\ell o}))^{-1}\Lambda_{\min}^{-1} \leq C_{\#}S_{\min}(E_{\ell o})\Lambda_{\min}^{-2}$ by (15). So $E(a) \leq E_{\ell o} + \hat{c}_1 S_{\min}(E_{\ell o})$. Therefore $g(x, E) = 0$ whenever $E \leq E(a)$, by virtue of (A3). In particular, $g(x, E(t)) = 0$ to infinite order at $t = a$.

We use the above information to estimate

$$\rho(x, g) = \sum_{k=k_{\ell o}}^{k_{hi}} g(x, E_k)u_k^2(x) \ .$$

First of all,

$$\int_{(x,E_k)\in \text{ supp } g} \left| u_k(x) - b_k \text{Re}\Big[\frac{e^{i\eta(x,E_k)}}{(E_k - V(x))^{1/4}}(1 + w_k(x))\Big]\right|^2 dx \leq \|\mathcal{E}_k(x)\|^2_{L^2(I_{\text{BVP}})}$$
$$\leq \Lambda_{\min}^{-N''} \ .$$

So

$$\begin{aligned}
&\int\limits_{(x,E_k)\in \text{ supp } g} \left|u_k^2(x) - b_k^2\Big(\mathrm{Re}\Big[\frac{e^{i\eta(x,E_k)}}{(E_k - V(x))^{1/4}}\ (1 + w_k(x))\Big]\Big)^2\right|\, dx \\
&\le \Big(\int\limits_{(x,E_k)\in \text{ supp } g} |u_k(x) - b_k\ \mathrm{Re}[\mathrm{etc}]|^2\ dx\Big)^{1/2} \\
&\qquad \cdot \Big(\int\limits_{(x,E_k)\in \text{ supp } g_k} |u_k(x) + b_k\ \mathrm{Re}[\mathrm{etc}]|^2\ dx\Big)^{1/2} \\
&\le \Lambda_{\min}^{-\frac{1}{2}N''}\Big\{\Big(\int\limits_{(x,E_k)\ \text{ supp } g_k} |u_k(x) - b_k \mathrm{Re}[\mathrm{etc}]|^2\ dx\Big)^{1/2} + 2\|u_k\|_{L^2(I_{\mathrm{BVP}})}\Big\} \\
&\le C_{\#}\Lambda_{\min}^{-\frac{1}{2}N''}\ .
\end{aligned}$$

Summing over k, we get

$$\begin{aligned}
&\int_{I_{\mathrm{BVP}}} \Big|\sum_k g(x,E_k)u_k^2(x) - \sum_k g(x,E_k)b_k^2\Big(\mathrm{Re}\Big[\frac{e^{i\eta(x,E_k)}}{(E_k - V(x))^{1/4}}(1 + w_k(x))\Big]\Big)^2\Big|\ dx \\
&\qquad\qquad \le C_*\Lambda_{\min}^{-\frac{1}{2}N''} \cdot (\text{Number of } k) \le C_*\Lambda_{\min}^{-\frac{1}{2}N''}(\phi(0)+1) \quad \text{by } (1)\ .
\end{aligned}$$

That is,

$$\int_{I_{\mathrm{BVP}}} |\rho(x,g) - \rho_0(x,g) - \sum_{\pm} \rho_{\pm}(x,g)|\ dx \le C_*\Lambda_{\min}^{-\frac{1}{2}N''}(\phi(0)+1) \tag{16}$$

with

$$\rho_0(x,g) = \sum_{k=k_{\ell o}}^{k_{hi}} g(x,E_k)b_k^2\frac{|1 + w_k(x)|^2}{2(E_k - V(x))^{1/2}} \tag{17}$$

and

$$\rho_+(x,g) = \sum_{k=k_{\ell o}}^{k_{hi}} g(x,E_k)\frac{b_k^2 e^{2i\eta(x,E_k)}}{4(E_k - V(x))^{1/2}}(1 + w_k(x))^2 \tag{18}$$

$$\rho_-(x,g) = \sum_{k=k_{\ell o}}^{k_{hi}} g(x,E_k)b_k^2\, e^{-2i\eta(x,E_k)}(1 + \overline{w_k(x)})^2/\{4(E_k - V(x))^{1/2}\}\ . \tag{19}$$

We will see that $\rho_\pm(x,g)$ are small in H^{-1}–norm, while $\rho_0(x,g)$ is closely approximated by $\rho_{sc}(x,g)$ in L^1. We begin with ρ_0. In supp $g(x,E_k)$ we have $|1+w_k(x)|^2 = 1+O(\Lambda_{\min}^{-2}\tau^{-3})$ by (7); and $b_k^2 = [\phi'(E_k)]^{-1}\cdot(1+O(\Lambda_{\min}^{4\varepsilon-2}))$ by (9). Hence $\frac{b_k^2|1+w_k(x)|^2}{2(E_k-V(x))^{1/2}} = \frac{[\phi'(E_k)]^{-1}}{2(E_k-V(x))^{1/2}}\cdot(1+O(\Lambda_{\min}^{4\varepsilon-2}\tau^{-3}))$, so that

$$|\rho_0(x,g)-\rho_1(x,g)| \le C_* \sum_k |g(x,E_k)| \frac{[\phi'(E_k)]^{-1}}{(E_k-V(x))^{1/2}}\ \Lambda_{\min}^{4\varepsilon-2}\tau^{-3} \tag{20}$$

with

$$\rho_1(x,g) = \sum_{k=k_{\ell o}}^{k_{hi}} g(x,E_k)\frac{[\phi'(E_k)]^{-1}}{2(E_k-V(x))^{1/2}}\ . \tag{21}$$

To control the right–hand side of (20), we use the properties of the energy intervals $J(E)$. For each E_k we have

$$1 \le C_\# \int_{E_{\min}}^{E_{\max}} \chi_{E_k\in J(E)} \frac{dE}{|J(E)|} \quad \text{since} \quad |E-E_k| < c'_\#[\phi'(E_k)]^{-1} \tag{21a}$$

implies $E_k\in J(E)$ and $|J(E)|\sim(\phi'(E_k))^{-1}$. Hence

$$\begin{aligned} &\sum_k |g(x,E_k)|\frac{[\phi'(E_k)]^{-1}\Lambda_{\min}^{4\varepsilon-2}\tau^{-3}}{(E_k-V(x))^{1/2}} \\ &\le C_\# \int_{E_{\min}}^{E_{\max}} \left[\max_{\tilde E\in J(E)} \frac{|g(x,\tilde E)|(\phi'(\tilde E))^{-1}}{(\tilde E-V(x))^{1/2}}\ \Lambda_{\min}^{4\varepsilon-2}\tau^{-3}\right] \frac{\sum_k \chi_{E_k\in J(E)}}{|J(E)|}\ dE \\ &\le C_\# \int_{E_{\min}}^{E_{\max}} \left[\max_{\tilde E\in J(E)} \frac{|g(x,\tilde E)|}{(\tilde E-V(x))^{1/2}}\Lambda_{\min}^{4\varepsilon-2}\tau^{-3}\right]\ dE\ , \end{aligned} \tag{22}$$

since $[\phi'(\tilde E)]^{-1}\sim|J(E)|$ for $\tilde E\in J(E)$, and since $J(E)$ contains at most $C_\#$ of the E_k.

Estimate (15) gives

$$|J(E)| \sim (\phi'(E))^{-1} \le C_\#\Lambda_{\min}^{-1}S_{\min}(E) << \tau S(x_0)\ . \tag{22a}$$

On the other hand, $\hat{c}\tau S(x_0) < \tilde{E} - V(x) < \hat{C}\tau S(x_0)$ whenever $g(x,\tilde{E}) \neq 0$. (Here we used (A2) and (A4).) Hence the right–hand side of (22) is dominated by

$$C_* \Lambda_{\min}^{4\varepsilon-2} \tau^{-3} \int\limits_{\frac{\hat{c}}{2}\tau S(x_0) < E - V(x) < 2\hat{C}\tau S(x_0)} \frac{dE}{(E-V(x))^{1/2}} \leq C_* \Lambda_{\min}^{4\varepsilon-2} \tau^{-3} \cdot (\tau S(x_0))^{1/2} \ .$$

Therefore (20) and (22) yield $|\rho_0(x,g) - \rho_1(x,g)| \leq C_* \Lambda_{\min}^{4\varepsilon-2} \tau^{-5/2} S^{1/2}(x_0)$, so that

$$\int\limits_{|x-x_0|<c_\# B(x_0)} |\rho_0(x,g) - \rho_1(x,g)| \, dx \leq C_* \Lambda_{\min}^{4\varepsilon-2} \tau^{-5/2} \lambda(x_0) \ . \tag{23}$$

Next we look at the function $E \mapsto \frac{g(x,E)[\phi'(E)]^{-1}}{2(E-V(x))^{1/2}} \equiv F_x(E)$. We know that

$$|\partial_E^\beta g(x,E)| \leq C_*^\beta (\tau S(x_0))^{-\beta} \qquad \text{for all} \quad (x,E) \ .$$

Also, $|\partial_E^\beta \phi'(E)| \leq C_\#^\beta \phi'(E) (S_{\min}(E))^{-\beta}$ for $E \in [E_{\min}, E_{\max}]$, so $|\partial_E^\beta [\phi'(E)]^{-1}| \leq C_\#^\beta [\phi'(E)]^{-1} (S_{\min}(E))^{-\beta}$ for $E \in [E_{\min}, E_{\max}]$; and $|\partial_E^\beta (E - V(x))^{-1/2}| \leq C_*^\beta (\tau S(x_0))^{-1/2} (\tau S(x_0))^{-\beta}$ for $E - V(x) \sim \tau S(x_0)$. Together, these estimates yield

$$\left|\partial_E^\beta F_x(E)\right| \leq \frac{C_*^\beta [\phi'(E)]^{-1}}{(\tau S(x_0))^{1/2}} \big(\min\{\tau S(x_0), S_{\min}(E)\}\big)^{-\beta} \quad \text{in supp } g \ . \tag{24}$$

Also, since $|\{\frac{1}{\pi}\phi(E_k) - \frac{1}{2}\} - k| \leq C_\# \Lambda_{\min}^{-1}$ by (1), we have $|E_k - E(k)| \leq C_\# \Lambda_{\min}^{-1} \cdot \max\{[\phi'(E)]^{-1} \mid E \text{ between } E_k \text{ and } E(k)\}$.
By (13), this means

$$|E_k - E(k)| \leq C_\# \Lambda_{\min}^{-1} [\phi'(E_k)]^{-1} \ . \tag{25}$$

Estimates (13), (22a), (24) and (25) imply

$$\sum_{k=k_{\ell o}}^{k_{hi}} |F_x(E_k) - F_x(E(k))| \tag{26}$$
$$\leq C_* \Lambda_{\min}^{-1} \sum_{k \in \mathcal{K}_x} \frac{[\phi'(E_k)]^{-2}}{(\tau S(x_0))^{1/2}} \Big\{ \frac{1}{\tau S(x_0)} + \frac{1}{S_{\min}(E_k)} \Big\} \ ,$$

where $\mathcal{K}_x = \{k \text{ between } k_{\ell o}, k_{hi} \mid E_k - V(x) \sim \tau S(x_0)\}$. Recall that $\phi'(E_k) \geq c_\# S^{-1/2}(x_0)B(x_0)$ and $S_{\min}(E_k)\phi'(E_k) \geq c_\# \Lambda_{\min}$. Hence $[\phi'(E_k)]^{-1}\{\frac{1}{\tau S(x_0)} + \frac{1}{S_{\min}(E_k)}\} \leq C_\#\{\frac{1}{\tau\lambda(x_0)} + \frac{1}{\Lambda_{\min}}\} \leq \frac{C_\#}{\tau\Lambda_{\min}}$, so that (26) implies

$$\sum_{k=k_{\ell o}}^{k_{hi}} |F_x(E_k) - F_x(E(k))| \leq C_* \tau^{-1}\Lambda_{\min}^{-2} \sum_{k\in\mathcal{K}_x} \frac{[\phi'(E_k)]^{-1}}{(\tau S(x_0))^{1/2}} \ . \tag{27}$$

To estimate the right–hand side of (27), we use the energy intervals $J(E)$. In fact, (21a) and (22a) yield

$$\text{(27 bis)} \quad \sum_{k\in\mathcal{K}_x} [\phi'(E_k)]^{-1} \leq C_\# \sum_{k\in\mathcal{K}_x} \int_{\substack{E-V(x)\sim\tau S(x_0)\\ E\in[E_{\min},E_{\max}]}} [\phi'(E_k)]^{-1}\chi_{E_k\in J(E)} \frac{dE}{|J(E)|}$$

$$\leq C'_\# \int_{\substack{E-V(x)\sim\tau S(x_0)\\ E\in[E_{\min},E_{\max}]}} \{\text{Number of } E_k \in J(E)\} \cdot \left\{\frac{\max_{\tilde{E}\in J(E)}[\phi'(\tilde{E})]^{-1}}{|J(E)|}\right\} dE$$

$$\leq C''_\# \int_{E-V(x)\sim\tau S(x_0)} dE \text{ (since the factors in curly brackets are } \leq C_\#)$$

$$= C''_* \tau S(x_0) \ .$$

Putting this into (27), we get

$$\sum_{k=k_{\ell o}}^{k_{hi}} |F_x(E_k) - F_x(E(k))| \leq C_* \tau^{-1/2}\Lambda_{\min}^{-2}(S(x_0))^{1/2} \ ,$$

so that

$$\int_{|x-x_0|<c_\# B(x_0)} \sum_{k=k_{\ell o}}^{k_{hi}} |F_x(E_k) - F_x(E(k))| \, dx \leq C_* \tau^{-1/2}\Lambda_{\min}^{-2}\lambda(x_0) \ . \tag{28}$$

The definitions of $\rho_1(x,g)$ and $F_x(E)$ show that $\rho_1(x,g) = \sum_{k=k_{\ell o}}^{k_{hi}} F_x(E_k)$. So, defining

$$\rho_2(x,g) = \sum_{k=k_{\ell o}}^{k_{hi}} F_x(E(k)) = \sum_{k\in\mathbb{Z}\cap[a,b]} F_x(E(k)) \ , \quad \text{(see (2))} \tag{29}$$

we learn from (28) that

$$\int_{|x-x_0|<c_\# B(x_0)} |\rho_1(x,g) - \rho_2(x,g)| \, dx \leq C_* \tau^{-1/2}\Lambda_{\min}^{-2}\lambda(x_0) \ . \tag{30}$$

We prepare to compute $\rho_2(x,g)$ using the lemma on Riemann sums. This requires estimates for $(\frac{d}{dt})^m F_x(E(t))$, which is a sum of terms of the form

$$(31)\quad [\partial_E^\beta F_x\, |_{E=E(t)}] \cdot \prod_{\nu=1}^{\beta} \left(\frac{d}{dt}\right)^{m_\nu} E(t)\ ,\quad \text{with } m_\nu \geq 1 \text{ and } m_1+\ldots+m_\beta = m\ .$$

In particular, $0 \leq \beta \leq m$.

By (24) and (12), the term (31) is dominated by

$$(32)\quad \frac{C_*[\phi'(E)]^{-1}}{(\tau S(x_0))^{1/2}} \big(\min\{\tau S(x_0), S_{\min}(E)\}\big)^{-\beta} \cdot (S_{\min}(E))^\beta \cdot (S_{\min}(E)\phi'(E))^{-m}$$
$$\text{with } E = E(t),\ \text{for } (x,E) \in \text{supp } g \text{ and } t \in [a_{\min}, b_{\max}]\ .$$

The expression (32) may be rewritten in the form

$$\begin{aligned}
&\frac{C_*[\phi'(E)]^{-1}}{(\tau S(x_0))^{1/2}} \Big(\min\Big\{\frac{\tau S(x_0)}{S_{\min}(E)}, 1\Big\}\Big)^{-\beta} \cdot (S_{\min}(E)\phi'(E))^{-m} \\
&\leq \frac{C_*[\phi'(E)]^{-1}}{(\tau S(x_0))^{1/2}} \Big(\min\Big\{\frac{\tau S(x_0)}{S_{\min}(E)}, 1\Big\}\Big)^{-m} \cdot (S_{\min}(E)\phi'(E))^{-m} \\
&= \frac{C_*[\phi'(E)]^{-1}}{(\tau S(x_0))^{1/2}} \Big(\min\big\{\tau S(x_0)\phi'(E), S_{\min}(E)\phi'(E)\big\}\Big)^{-m} \\
&\leq \frac{C_*[\phi'(E)]^{-1}}{(\tau S(x_0))^{1/2}} (\tau\Lambda_{\min})^{-m}\ ,\ \text{again because } S(x_0)\phi'(E) \geq c_\# \lambda(x_0) \\
&\qquad\qquad \text{and } S_{\min}(E)\phi'(E) \geq c_\# \Lambda_{\min}\ .
\end{aligned}$$

Hence,

$$(33)\qquad \Big|\left(\frac{d}{dt}\right)^m F_x(E(t))\Big| \leq C_*^m \frac{[\phi'(E(t))]^{-1}}{(\tau S(x_0))^{1/2}} (\tau\Lambda_{\min})^{-m} \quad \text{for} \quad t \in [a_{\min}, b_{\max}]\ .$$

(We have dropped the requirement $(x, E(t)) \in$ supp g, since the left side of (33) vanishes unless that requirement is met.)

¿From (13), (15) we conclude that $\phi'(E(t))$ has constant order of magnitude when t varies over an interval of length $c_\#\Lambda_{\min}$ inside $[a_{\min}, b_{\max}]$. Note also that $\tau\Lambda_{\min} > \Lambda_{\min}^{1/3} > 1$ by (A4) and (A5). These remarks and (33) show that the hypotheses of the lemma on Riemann sums are satisfied. Applying that lemma, and noting that $g(x, E(a)) \equiv 0$, we obtain

$$(34) \qquad \rho_2(x,g) = \sum_{k \in \mathbb{Z} \cap [a,b]} F_x(E(k)) = \int_a^b F_x(E(t))\, dt - F_x(E(b))\chi_-(b) + \mathcal{E}(x)\ ,$$

with

$$(35) \quad |\mathcal{E}(x)| \le C_* \frac{[\phi'(E(b))]^{-1}}{(\tau S(x_0))^{1/2}} (\tau\Lambda_{\min})^{-1} + \frac{C_*[\phi'(E(a))]^{-1}}{(\tau S(x_0))^{1/2}} (\tau\Lambda_{\min})^{-1} + C_*^{\overline{N}} \int_a^b \frac{[\phi'(E(t))]^{-1}}{(\tau S(x_0))^{1/2}} (\tau\Lambda_{\min})^{-\overline{N}}\, dt$$

($\overline{N}$ as large as we please.)

Since $[\phi'(E(a))]^{-1}$, $[\phi'(E(b))]^{-1} \le C_\# S^{1/2}(x_0) B^{-1}(x_0)$, the first two terms on the right of (35) are at most $\frac{C_*}{\tau^{3/2}} \Lambda_{\min}^{-1} B^{-1}(x_0)$. The last term on right of (35) has the order of magnitude $\frac{(\tau\Lambda_{\min})^{-\overline{N}}(E(b)-E(a))}{(\tau S(x_0))^{1/2}} \le \frac{C_*(\Lambda_{\min})^{-\overline{N}/3}}{\tau^{1/2}} (S(x_0))^{1/2}$. Hence, $|\mathcal{E}(x)| \le \frac{C_*^{\overline{N}} \Lambda_{\min}^{-\overline{N}/3}}{\tau^{1/2}} (S(x_0))^{1/2} + \frac{C_* \Lambda_{\min}^{-1}}{\tau^{3/2}} (B(x_0))^{-1}$, so that

$$\int_{|x-x_0| < c_\# B(x_0)} |\mathcal{E}(x)|\, dx \le \frac{C_*^{\overline{N}}}{\tau^{3/2}} \Lambda_{\min}^{-\overline{N}/3} \lambda(x_0) + \frac{C_*}{\tau^{3/2}} \Lambda_{\min}^{-1} \le \frac{C_*}{\tau^{3/2}} \Lambda_{\min}^{-2} \lambda(x_0)\ .$$

Combining this with (34) and setting

$$(36) \qquad \rho_3(x,g) = \int_a^b F_x(E(t))\, dt - F_x(E(b))\chi_-(b)\ ,$$

we obtain

$$(37) \qquad \int_{|x-x_0| < c_\# B(x_0)} |\rho_2(x,g) - \rho_3(x,g)|\, dx \le C_* \tau^{-3/2} \Lambda_{\min}^{-2} \lambda(x_0)\ .$$

Next we compare $\rho_3(x,g)$ with

$$(38) \qquad \rho_4(x,g) = \int_a^{b_0} F_x(E(t))\, dt - F_x(E(b_0))\chi_-(b)\ .$$

Since $|F_x(E)| \le C_* \frac{[\phi'(E)]^{-1}}{(\tau S(x_0))^{1/2}} \chi_{E-V(x)\sim\tau S(x_0)}$ and $|b-b_0| \le \frac{C_\#}{\Lambda_{\min}}$, and since $\phi'(E(t))$ has constant order of magnitude for t between b_0 and b, we have

(39)

$$\Big|\int_{b_0}^{b} F_x(E(t))\ dt\Big| \le \frac{C_*}{\Lambda_{\min}} \frac{[\phi'(E(b_0))]^{-1}}{(\tau S(x_0))^{1/2}}\ \chi_{\begin{cases} E-V(x)\sim\tau S(x_0) \text{ for some} \\ E \text{ between } E(b_0) \text{ and } E(b)\end{cases}} .$$

As t varies between b_0 and b, $E(t)$ varies by at most $C_*[\phi'(E(b_0))]^{-1}|b-b_0| \le C_*[\phi'(E(b_0))]^{-1}\Lambda_{\min}^{-1} \le C_*\lambda^{-1}(x_0)S(x_0)\Lambda_{\min}^{-1}$ (by (14)) $<< \tau S(x_0)$, so (39) implies

$$\Big|\int_{b_0}^{b} F_x(E(t))\ dt\Big| \le \frac{C_*}{\Lambda_{\min}} \frac{[\phi'(E_{hi})]^{-1}}{(\tau S(x_0))^{1/2}} \chi_{E_{hi}-V(x)\sim\tau S(x_0)} .$$

(Recall that $E(b_0) = E_{hi}$.) Hence,

$$\begin{aligned}
&\int\limits_{|x-x_0|<c_\# B(x_0)} \Big|\int\limits_{b_0}^{b} F_x(E(t))\ dt\Big|\ dx \\
&\le \frac{C_*}{\Lambda_{\min}} \frac{[\phi'(E_{hi})]^{-1}}{(\tau S(x_0))^{1/2}} \int\limits_{|x-x_0|<c_\# B(x_0)} \chi_{E_{hi}-V(x)\sim\tau S(x_0)}\ dx \\
&\le \frac{C_*}{\Lambda_{\min}} \frac{[\phi'(E_{hi})]^{-1}}{(\tau S(x_0))^{1/2}} \cdot (\tau B(x_0)) \le \frac{C_*}{\Lambda_{\min}} \frac{[S^{1/2}(x_0)B^{-1}(x_0)]}{(\tau S(x_0))^{1/2}} (\tau B(x_0)) \\
&\le \frac{C_*\tau^{1/2}}{\Lambda_{\min}} \le C_*\Lambda_{\min}^{-2}\lambda(x_0) .
\end{aligned} \tag{40}$$

Similarly,

$$\begin{aligned}
&|F_x(E(b)) - F_x(E(b_0))| \le |b-b_0| \max_{t \text{ between } b,b_0} \Big|\Big(\frac{d}{dt}\Big) F_x(E(t))\Big| \\
&\le C_\#\Lambda_{\min}^{-1} \cdot \max_{t \text{ between } b,b_0} C_* \frac{[\phi'(E(t))]^{-1}}{(\tau S(x_0))^{1/2}} (\tau\Lambda_{\min})^{-1} \chi_{E(t)-V(x)\sim\tau S(x_0)} \\
&\le C_*\tau^{-1}\Lambda_{\min}^{-2} \frac{[\phi'(E(b_0))]^{-1}}{(\tau S(x_0))^{1/2}} \chi_{E(b_0)-V(x)\sim\tau S(x_0)} ,
\end{aligned}$$

since $[\phi'(E(t))]^{-1}$ has constant order of magnitude and $E(t)$ varies by $<< \tau S(x_0)$

as t varies from b_0 to b. Integrating in x, we get

$$\int_{|x-x_0|<c_\# B(x_0)} |F_x(E(b)) - F_x(E(b_0))|\ dx$$

$$\leq C_* \frac{\Lambda_{\min}^{-2}[\phi'(E_{hi})]^{-1}}{\tau(\tau S(x_0))^{1/2}} \int_{|x-x_0|<c_\# B(x_0)} \chi_{E_{hi}-V(x)\sim\tau S(x_0)}\ dx$$

$$\leq \frac{C_*\Lambda_{\min}^{-2}[\phi'(E_{hi})]^{-1}}{\tau(\tau S(x_0))^{1/2}}(\tau B(x_0)) \leq C_* \frac{\Lambda_{\min}^{-2}[S^{1/2}(x_0)B^{-1}(x_0)]}{\tau(\tau S(x_0))^{1/2}}\ (\tau B(x_0))$$

$$= C_*\Lambda_{\min}^{-2}\tau^{-1/2} \leq C_*\Lambda_{\min}^{-2}\lambda(x_0)\tau^{-1/2}\ . \tag{41}$$

Estimates (40), (41) and the definitions of $\rho_3(x,g)$ and $\rho_4(x,g)$ show that

$$\int_{|x-x_0|<c_\# B(x_0)} |\rho_3(x,g) - \rho_4(x,g)|\ dx \leq C_*\Lambda_{\min}^{-2}\lambda(x_0)\tau^{-1/2}\ . \tag{42}$$

Next, we compare $\rho_4(x,g)$ with

$$\rho_5(x,g) = \int_a^{b_0} F_x(E(t))\ dt - F_x(0)\chi_-(\frac{1}{\pi}\phi(0) - \frac{1}{2})\ . \tag{43}$$

Recall that $E(b_0) = E_{hi}$, and either $E_{hi} = 0$ or else $g(x, E_{hi}) = g(x,0) = 0$ for all x. Therefore to compare ρ_4, ρ_5 we may assume $E_{hi} = 0$, since otherwise $\rho_4(x,g) = \rho_5(x,g)$ for all x. Under the assumption $E_{hi} = 0$ we have $\frac{1}{\pi}\phi(0) - \frac{1}{2} = b_0$, and

$$\rho_4(x,g) - \rho_5(x,g) = F_x(E(b_0)) \cdot [\chi_-(b) - \chi_-(b_0)]\ . \tag{44}$$

If $\min_{k\in\mathbb{Z}} |\phi(0) - \pi(k+\frac{1}{2})| \geq \overline{C}_\#\Lambda_{\min}^{-1}$, then $|\chi_-(b) - \chi_-(b_0)| \leq C_\#\Lambda_{\min}^{-1}$, since $b_0 = \frac{1}{\pi}\phi(0) - \frac{1}{2}$ isn't too near the jumps of χ_-, while $|b - b_0| \leq \frac{C_\#}{\Lambda_{\min}}$. In any case, $|\chi_-(b) - \chi_-(b_0)| \leq C_\#$.

Also,

$$\int_{|x-x_0|<c_\# B(x_0)} |F_x(E(b_0))|\ dx$$

$$\leq \int_{|x-x_0|<c_\# B(x_0)} C_* \frac{[\phi'(E(b_0))]^{-1}}{(\tau S(x_0))^{1/2}} \chi_{E(b_0)-V(x)\sim\tau S(x_0)}\ dx$$

$$\leq C_* \frac{[\phi'(E_{hi})]^{-1}}{(\tau S(x_0))^{1/2}} \cdot \tau B(x_0) \leq C_* \frac{[S^{1/2}(x_0)B^{-1}(x_0)]}{(\tau S(x_0))^{1/2}} \tau B(x_0) = C_*\tau^{1/2}\ .$$

Putting these remarks into (44), we get

$$\int\limits_{|x-x_0|<c_\# B(x_0)} |\rho_4(x,g)-\rho_5(x,g)|\,dx \le \frac{C_*\tau^{1/2}}{\Lambda_{\min}} \le C_*\Lambda_{\min}^{-2}\lambda(x_0) \tag{45}$$

$$\text{if}\quad \min_{k\in\mathbb{Z}}|\phi(0)-\pi(k+\frac{1}{2})| \ge \overline{C}_\#\Lambda_{\min}^{-1}\ ,$$

$$\int\limits_{|x-x_0|<c_\# B(x_0)} |\rho_4(x,g)-\rho_5(x,g)|\,dx \le C_*\tau^{1/2} \tag{46}$$

$$\text{if}\ \min_{k\in\mathbb{Z}}\ |\phi(0)-\pi(k+\frac{1}{2})| \le \overline{C}_\#\Lambda_{\min}^{-1}\ .$$

Recalling the definition of $F_x(E)$ and changing variable from t to $E=E(t)$ in the integral in (43), we have

$$\rho_5(x,g) = \frac{1}{2\pi}\int\limits_{E(a)}^{E_{hi}} \frac{g(x,E)\,dE}{(E-V(x))^{1/2}} - \tag{47}$$

$$\frac{1}{2}\frac{g(x,0)}{(-V(x))^{1/2}}[\phi'(0)]^{-1}\chi_-(\frac{1}{\pi}\phi(0)-\frac{1}{2})\ .$$

Since $g(x,E)=0$ for $E\in(-\infty,0]\setminus[E(a),E_{hi}]$, this agrees with our definition of $\rho_{sc}(x,g)$. (Note that $[\phi'(0)]$ contains a factor of $\frac{1}{2}$.) Hence $\rho_5(x,g)=\rho_{sc}(x,g)$. Estimates (23), (30), (37), (42), (45), (46) therefore show that

$$\int\limits_{|x-x_0|<c_\# B(x_0)} |\rho_0(x,g)-\rho_{sc}(x,g)|\,dx \le C_*\Lambda_{\min}^{4\varepsilon-2}\tau^{-5/2}\lambda(x_0) \tag{48}$$

$$\text{if}\ \min_{k\in\mathbb{Z}}|\phi(0)-\pi(k+\frac{1}{2})| \ge \frac{\overline{C}_\#}{\Lambda_{\min}}$$

(49)

$$\int\limits_{|x-x_0|<c_\# B(x_0)} |\rho_0(x,g)-\rho_{sc}(x,g)|\,dx \le C_*\Lambda_{\min}^{4\varepsilon-2}\tau^{-5/2}\lambda(x_0) + C_*\tau^{1/2}$$

$$\text{if}\ \min_{k\in\mathbb{Z}}|\phi(0)-\pi(k+\frac{1}{2})| \le \frac{\overline{C}_\#}{\Lambda_{\min}}\ .$$

So $\rho_0(x,g)$ lies near $\rho_{sc}(x,g)$ in L^1–norm. Next we show that the oscillating terms $\rho_+(x,g)$ and $\rho_-(x,g)$ are small in H^{-1}–norm. We study $\rho_+(x,g)$; the discussion of $\rho_-(x,g)$ is analogous. With $g_k(x) = \frac{g(x,E_k)b_k^2(1+w_k(x))^2}{4(E_k-V(x))^{1/2}}$, we have

(50) $$\rho_+(x,g) = \sum_{k=k_{\ell o}}^{k_{hi}} g_k(x) e^{2i\eta(x,E_k)} \quad \text{by (18) ;}$$

(51) $$\text{supp } g_k \subset \{x \mid |x-x_0| < \hat{c}B(x_0), E_k - V(x) \sim \tau S(x_0)\} \quad \text{by } (A3) \text{ ;}$$

(52)
$$|(\frac{d}{dx})^\alpha g_k(x)| \le C_*^\alpha \frac{[\phi'(E_k)]^{-1}}{(\tau S(x_0))^{1/2}} (\tau B(x_0))^{-\alpha} \quad \text{by (51), (9), (8), } (A5) \text{ ,}$$
$$\text{and } |(\frac{d}{dx})^\alpha V(x)| \le C_\#^\alpha S(x_0) B^{-\alpha}(x_0) \text{ for } |x-x_0| < \hat{c}B(x_0) \text{ .}$$

Recall that

(53) $$\frac{\partial}{\partial x}\eta(x,E_k) = (E_k - V(x))^{1/2} \text{ in supp } g_k(x) \text{ .}$$

We will show that any sum (50) with g_k and η satisfying (51)...(53) is small in H^{-1}–norm.

To estimate the H^{-1}–norm, we compute an indefinite integral for $\rho_+(x,g)$, using the following observation.

Suppose $h_k(x)$ is supported in supp $g_k \subset \{|x-x_0| < \hat{c}B(x_0), E_k - V(x) \sim \tau S(x_0)\}$ and satisfies $|(\frac{d}{dx})^\alpha h_k(x)| \le C_*^\alpha A(\tau B(x_0))^{-\alpha}$. Then

$$\frac{d}{dx}\Big(\frac{h_k(x)e^{2i\eta(x,E_k)}}{2i(E_k - V(x))^{1/2}}\Big) = h_k(x)e^{2i\eta(x,E_k)} + \tilde{h}_k(x)e^{2i\eta(x,E_k)} \text{ ,}$$

with $\tilde{h}_k(x) = \frac{d}{dx}\big(\frac{h_k(x)}{2i(E_k-V(x))^{1/2}}\big)$ satisfying

$$|(\frac{d}{dx})^\alpha \tilde{h}_k(x)| \le \frac{C_*^\alpha A(\tau B(x_0))^{-1-\alpha}}{(\tau S(x_0))^{1/2}} = C_*^\alpha \tilde{A}(\tau B(x_0))^{-\alpha},$$
$$\tilde{A} = \frac{A}{\tau^{3/2}\lambda(x_0)} \text{ .}$$

Note that $|\tilde{A}| \leq \Lambda_{\min}^{-\varepsilon}|A|$, since we assume $\tau > \Lambda_{\min}^{\varepsilon-2/3} \geq \Lambda_{\min}^{\varepsilon}(\lambda(x_0))^{-2/3}$.

We make repeated use of this observation, starting with $h_k = g_k$, to construct by successive approximation a function $\sigma_k(x)$ with the following properties.

$$\text{(54)} \qquad \text{supp}\ \sigma_k(x) \subset \ \text{supp}\ g_k \subset \{|x-x_0| < \hat{c}B(x_0), E_k - V(x) \sim \tau S(x_0)\}$$

$$\text{(55)} \qquad |(\frac{d}{dx})^\alpha \sigma_k(x)| \leq C_*^\alpha \frac{[\phi'(E_k)]^{-1}}{\tau S(x_0)}(\tau B(x_0))^{-\alpha}$$

$$\text{(56)} \qquad \frac{d}{dx}\{\sigma_k(x)e^{2i\eta(x,E_k)}\} = g_k(x)e^{2i\eta(x,E_k)} + \ \text{err}_k(x)\ , \ \text{with}$$

$$\text{(57)} \qquad |\text{err}_k| \leq C_* \Lambda_{\min}^{-N} \frac{[\phi'(E_k)]^{-1}}{(\tau S(x_0))^{1/2}}\ \chi_{E_k - V(x) \sim \tau S(x_0)}\ .$$

Thus,

$$\text{(58)} \qquad \rho_+(x,g) = \frac{d}{dx}\{\sum_{k=k_{\ell o}}^{k_{hi}} \sigma_k(x)e^{2i\eta(x,E_k)}\} - \sum_{k=k_{\ell o}}^{k_{hi}} \ \text{err}_k(x)\ .$$

We have

$$|\sum_k \ \text{err}_k(x)| \leq C_* \sum_k \frac{[\phi'(E_k)]^{-1}\Lambda_{\min}^{-N}}{(\tau S(x_0))^{1/2}}\chi_{E_k - V(x)\sim\tau S(x_0)} \leq \frac{C_*\Lambda_{\min}^{-N}}{(\tau S(x_0))^{1/2}} \cdot C_*\tau S(x_0)$$

by (27 bis), and therefore $\int\limits_{|x-x_0|<c_\# B(x_0)} |\sum\limits_k \ \text{err}_k(x)|\ dx \leq C_*' \Lambda_{\min}^{-N} \tau^{1/2}\lambda(x_0)$. For the indefinite integral $\text{Err}(x) = \int\limits_{x_0}^{x} \sum\limits_k \ \text{err}_k(x')\ dx'$ this implies

$$\text{(59)} \qquad |\text{Err}(x)| \leq C_*\Lambda_{\min}^{-N}\lambda(x_0) \quad \text{for} \quad |x-x_0| < c_\# B(x_0)\ ,$$

and (58) becomes

$$\text{(60)} \qquad \rho_+(x,g) = \frac{d}{dx}\{\sum_k \sigma_k(x)e^{2i\eta(x,E_k)}\} - \frac{d}{dx}\ \text{Err}(x)\ .$$

Thus, H^{-1}–estimates for $\rho_+(x,g)$ are reduced to L^2–estimates for $\sum_k \sigma_k(x)e^{2i\eta(x,E_k)}$. To make the L^2–estimates, we study a single term

$$\int_{|x-x_0|<c_\# B(x_0)} \sigma_k(x)\overline{\sigma_m(x)}e^{2i[\eta(x,E_k)-\eta(x,E_m)]}\, dx \equiv U_{km}\ . \tag{61}$$

For E_k, E_m far apart, we will estimate U_{km} using the following standard

Stationary Phase Lemma. *Suppose $a \in C_0^\infty(|x-\hat{x}_0| < b)$, $\phi \in C^\infty$ on a neighborhood of supp a, with ϕ real and $|\phi^{(k)}| \le C_k\alpha b^{-k}$ $(k \ge 1)$ on supp a, and $|a^{(k)}| \le C_k\beta b^{-k}$ $(k \ge 0)$. Assume also $|\phi'| \ge c\alpha b^{-1}$ on supp a. Then $|\int_{-\infty}^{\infty} ae^{i\phi}dx| \le C'_m\beta b\alpha^{-m}$, with C'_m depending only on m and on the constants C_m, c.*

Proof. Extend $\frac{a}{i\phi'}$ to a C^∞ function by setting it equal to zero outside supp a. Then $\frac{d}{dx}\{\frac{\alpha a}{i\phi'}\}$ satisfies estimates analogous to a, and $\int_{-\infty}^{\infty} ae^{i\phi}dx = \int_{-\infty}^{\infty}\frac{a}{i\phi'}\frac{d}{dx}\{e^{i\phi}\}dx = -\frac{1}{\alpha}\int_{-\infty}^{\infty}\left[\frac{d}{dx}\{\frac{\alpha a}{i\phi'}\}\right]e^{i\phi}\, dx$. (The integration by parts is justified, because ϕ is smooth on a neighborhood of supp a.) Repeated applications of this identity yield the conclusion of the lemma. ■

To apply the stationary phase lemma to U_{km}, we take $\hat{x}_0 \in$ supp σ_k, $\phi(x) = 2\eta(x,E_k) - 2\eta(x,E_m)$, $a(x) = \sigma_k(x)\overline{\sigma_m(x)}$, $b = C_*\tau B(x_0)$. Evidently, $a(x)$ satisfies the hypotheses of the lemma, with $\beta = [\phi'(E_k)]^{-1}[\phi'(E_m)]^{-1}(\tau S(x_0))^{-2}$, so it remains to investigate ϕ. In a neighborhood of supp $a(x)$ we have $E_k - V(x)$, $E_m - V(x) \sim \tau S(x_0)$. Therefore, in a neighborhood of supp a, we have:

(61 bis)

$$\phi'(x) = 2(E_k - V(x))^{1/2} - 2(E_m - V(x))^{1/2} = \int_{E_m}^{E_k} (E - V(x))^{-1/2}\, dE\ , \qquad \text{hence}$$

$$|\phi'(x)| \ge c_*(\tau S(x_0))^{-1/2}|E_k - E_m| \quad \text{in supp } a\ . \tag{62}$$

Also, $\partial_x^\alpha (E-V(x))^{-1/2}$ is a sum of terms $(E-V(x))^{-\frac{1}{2}-s} \prod_{\nu=1}^{s} \partial_x^{\alpha_\nu} V(x)$, with $\alpha_\nu \geq 1$ and $\alpha_1+\ldots+\alpha_s = \alpha$. In particular, $0 \leq s \leq \alpha$. For $x \in$ supp a and E between E_k and E_m, this term is dominated by $C_*^\alpha (\tau S(x_0))^{-\frac{1}{2}-s} \prod_{\nu=1}^{s} \{S(x_0) B^{-\alpha_\nu}(x_0)\} = \frac{C_*^\alpha}{(\tau S(x_0))^{1/2}} \tau^{-s} B^{-\alpha}(x_0) \leq C_*^\alpha (\tau S(x_0))^{-\frac{1}{2}} (\tau B(x_0))^{-\alpha}$. Putting this into (61 bis), we get

$$
(63) \qquad |(\frac{d}{dx})^\alpha \phi'(x)| \leq C_*^\alpha (\tau S(x_0))^{-1/2} |E_k - E_m| \cdot (\tau B(x_0))^{-\alpha}
$$

in a neighborhood of supp a .

Estimates (62) and (63) show that ϕ satisfies the hypotheses of the stationary phase lemma, with $\alpha = (\tau S(x_0))^{-1/2} |E_k - E_m| \cdot (\tau B(x_0)) = \tau^{1/2} \lambda(x_0) \frac{|E_k - E_m|}{S(x_0)}$. The stationary phase lemma shows therefore that

$$
|U_{km}| \leq C_*^{\overline{N}} \, \frac{[\phi'(E_k)]^{-1} [\phi'(E_m)]^{-1}}{(\tau S(x_0))^2} \cdot (\tau B(x_0)) \cdot [\frac{\tau^{1/2} \lambda(x_0) |E_k - E_m|}{S(x_0)}]^{-\overline{N}} ,
$$

with $\overline{N}$ as large as we please. Also, replacing the integrand in the definition by its absolute value, we get the trivial estimate $|U_{km}| \leq C_* \frac{[\phi'(E_k)]^{-1} [\phi'(E_m)]^{-1}}{(\tau S(x_0))^2} \tau B(x_0)$. Hence,

$$
(64) \quad |U_{km}| \leq \frac{C_*^{\overline{N}} [\phi'(E_k)]^{-1} [\phi'(E_m)]^{-1} B(x_0)}{\tau S^2(x_0)} [1 + \tau^{1/2} \lambda(x_0) \frac{|E_k - E_m|}{S(x_0)}]^{-\overline{N}}
$$

with $\overline{N}$ as large as we please .

We use (64) to estimate $\sum_{km} |U_{km}|$. Again we bring in the energy intervals $J(E)$. The length of $J(E)$ is $\sim (\phi'(E))^{-1} \leq C_\# \frac{S(x_0)}{\lambda(x_0)}$ by (14). It follows that $1 + \tau^{1/2} \lambda(x_0) \frac{|E_k - E_m|}{S(x_0)} \sim 1 + \tau^{1/2} \lambda(x_0) \frac{|E - E'|}{S(x_0)}$ if $E_k \in J(E)$ and $E_m \in J(E')$. Also,

$$
[\phi'(E_k)]^{-1} \sim \int_{E \in [E_{\min}, E_{\max}]} \chi_{E_k \in J(E)} \, dE \quad \text{and}
$$

$$
[\phi'(E_m)]^{-1} \sim \int_{E' \in [E_{\min}, E_{\max}]} \chi_{E_m \in J(E')} \, dE' \ .
$$

Hence (64) implies

$$\sum_{km} |U_{km}| \le C_*^{\overline{N}} \int_{E,E' \in [E_{\min}, E_{\max}]} \{\text{No. of } E_k \in J(E)\}$$
$$\{\text{No. of } E_m \in J(E')\} \frac{B(x_0)}{\tau S^2(x_0)} \Big(1 + \tau^{1/2}\lambda(x_0)\frac{|E-E'|}{S(x_0)}\Big)^{-\overline{N}} \, dEdE' \ .$$

Since $J(E)$, $J(E')$ contain at most $C_\#$ of the E_j, this simplifies to

$$\sum_{k,m} |U_{km}| \le \int_{E_{\min}}^{E_{\max}} \int_{E_{\min}}^{E_{\max}} C_*^{\overline{N}} \frac{B(x_0)}{\tau S^2(x_0)} \Big(1 + \tau^{1/2}\lambda(x_0)\frac{|E-E'|}{S(x_0)}\Big)^{-\overline{N}} \, dEdE'$$
$$\le C_*[E_{\max} - E_{\min}] \Big[\frac{S(x_0)}{\tau^{1/2}\lambda(x_0)}\Big] \cdot \frac{B(x_0)}{\tau S^2(x_0)} \quad \text{if we take } \overline{N} = 10 \ .$$

Since $[E_{\max} - E_{\min}] \le C_\# S(x_0)$, this implies

$$\sum_{k,m} |U_{km}| \le \frac{C_* B(x_0)}{\tau^{3/2}\lambda(x_0)} \ .$$

By definition of U_{km}, it follows that

$$\text{(65)} \qquad \int_{|x-x_0|<c_\# B(x_0)} \Big| \sum_k \sigma_k(x) e^{2i\eta(x,E_k)} \Big|^2 \, dx \le \frac{C_* B(x_0)}{\tau^{3/2}\lambda(x_0)} \ .$$

Estimates (59), (65) and equation (60) give

$$\text{(66)} \qquad \rho_+(x,g) = \frac{d}{dx}\mathcal{H}_+(x,g) \quad \text{with}$$

(67)

$$\int_{|x-x_0|<c_\# B(x_0)} |\mathcal{H}_+(x,g)|^2 \, dx \le C_* \frac{B(x_0)}{\tau^{3/2}\lambda(x_0)} + C_* \Lambda_{\min}^{-2N} \lambda^2(x_0) B(x_0) \ .$$

A completely analogous discussion of $\rho_-(x,g)$ shows that

$$\text{(68)} \qquad \rho_-(x,g) = \frac{d}{dx}\mathcal{H}_-(x,g) \quad \text{with}$$

$$\text{(69)} \qquad \int_{|x-x_0|<c_\# B(x_0)} |\mathcal{H}_-(x,g)|^2 \, dx \le C_* \frac{B(x_0)}{\tau^{3/2}\lambda(x_0)} + C_* \Lambda_{\min}^{-2N} \lambda^2(x_0) B(x_0) \ .$$

Set $\mathcal{H}_0(x,g) = \int_{x_0}^{x} [\rho_0(x',g) - \rho_{sc}(x',g)]dx'$, so that

$$\rho_0(x,g) = \rho_{sc}(x,g) + \frac{d}{dx}\mathcal{H}_0(x,g) \; ; \tag{70}$$

and by (48), (49), we have

$$\int_{|x-x_0|<c_\# B(x_0)} |\mathcal{H}_0(x,g)|^2 \; dx \le C_* \Lambda_{\min}^{8\varepsilon-4} \tau^{-5} \lambda^2(x_0) B(x_0) \tag{71}$$

$$\text{if} \quad \min_{k\in\mathbb{Z}} |\phi(0) - \pi(k+\frac{1}{2})| \ge \frac{\overline{C}_\#}{\Lambda_{\min}} \; ,$$

(72)

$$\int_{|x-x_0|<c_\# B(x_0)} |\mathcal{H}_0(x,g)|^2 \; dx \le C_* \Lambda_{\min}^{8\varepsilon-4} \tau^{-5} \lambda^2(x_0) B(x_0) + C_* \tau B(x_0)$$

$$\text{if} \quad \min_{k\in\mathbb{Z}} |\phi(0) - \pi(k+1/2)| \le \frac{\overline{C}_\#}{\Lambda_{\min}} \; .$$

Finally, set $\mathcal{H}_{\text{extra}}(x,g) = \int_{x_0}^{x} [\rho(x',g) - \rho_0(x',g) - \sum_{\pm} \rho_\pm(x',g)] \; dx'$, so that

$$\rho(x,g) = \rho_0(x,g) + \rho_+(x,g) + \rho_-(x,g) + \frac{d}{dx}\mathcal{H}_{\text{extra}}(x,g) \; ; \tag{73}$$

and by (16) we have

$$\int_{|x-x_0|<c_\# B(x_0)} |\mathcal{H}_{\text{extra}}(x,g)|^2 \; dx \le C_* \Lambda_{\min}^{-N''} (\phi(0)+1)^2 B(x_0) \; . \tag{74}$$

Define $\mathcal{H}(x,g) = \mathcal{H}_+(x,g) + \mathcal{H}_-(x,g) + \mathcal{H}_0(x,g) + \mathcal{H}_{\text{extra}}(x,g)$, so that

$$\rho(x,g) = \rho_{sc}(x,g) + \frac{d}{dx}\mathcal{H}(x,g) \; , \quad \text{by (66), (68), (70), (73)} \; . \tag{75}$$

Combining our estimates (67), (69), (71), (72), (74), we obtain the following lemma, the main result of this section.

Oscillatory Density Lemma. *Suppose the potential $V(x)$, the weight functions $S(x)$, $B(x)$, and the cutoff function $g(x,E)$ satisfy assumptions (A1)...(A6). Set*

$\phi(0) = \int_I (-V(x))_+^{1/2} dx$. *Then the microlocalized density* $\rho(x,g)$ *is given by* $\rho(x,g) = \rho_{sc}(x,g) + \frac{d}{dx}H(x)$, *where* $\rho_{sc}(x,g)$ *is the semiclassical approximation, and* $H(x)$ *satisfies the following estimates on* $I_{x_0} = \{|x - x_0| < c_\# B(x_0)\}$.

(a) *If* $\min_{k\in\mathbb{Z}} |\phi(0) - \pi(k+1/2)| \geq \overline{C}_\# \Lambda_{\min}^{-1}$, *then*

$$\frac{1}{B(x_0)} \int\limits_{I_{x_0}} |H(x)|^2 \, dx \leq C_* \tau^{-3/2} \lambda^{-1}(x_0) + C_* \Lambda_{\min}^{8\varepsilon - 4} \tau^{-5} \lambda^2(x_0) + C_* \Lambda_{\min}^{-N''} (\phi(0)+1)^2 \ .$$

(b) *If* $\min_{k\in\mathbb{Z}} |\phi(0) - \pi(k+\frac{1}{2})| \leq \overline{C}_\# \Lambda_{\min}^{-1}$, *then*

$$\frac{1}{B(x_0)} \int\limits_{I_{x_0}} |H(x)|^2 \, dx \leq C_* \tau + C_* \tau^{-3/2} \lambda^{-1}(x_0) + C_* \Lambda_{\min}^{8\varepsilon - 4} \tau^{-5} \lambda^2(x_0) + C_* \Lambda_{\min}^{-N''} (\phi(0)+1)^2 \ .$$

Here, the constants $c_\#$, $\overline{C}_\#$, *depend only on* ε, K, N, c, C, c_1, c_2, c_3, C_α *in the assumptions (A1), (A2). The constant* C_* *depends on these quantities, and also on* $\hat{c}$, $\hat{c}_1$, $\hat{C}$, $\hat{C}_{\alpha\beta}$ *in assumptions (A3), (A5).*

THE MICROLOCALIZED DENSITY NEAR THE MINIMUM OF THE POTENTIAL

Assume the hypotheses of the WKB Theorem on low eigenvalues, with $E_\infty = 0$. Take a function $g(E)$ satisfying

$$\text{supp } g(E) \subset \{|E - V(x_0)| < \tau^2 S\} \quad \text{with} \quad \lambda^{2\varepsilon - \frac{1}{2}} < \tau < \lambda^{-2\varepsilon}, \quad \text{and}$$
$$\Big|\big(\frac{d}{dE}\big)^\beta g(E)\Big| \le \hat{C}_\beta (\tau^2 S)^{-\beta} \ .$$

Regard g as a function of (x, E) which happens to be independent of x, and form the microlocalized density $\rho(x, g)$. Let $\rho_{sc}(x, g)$ be the corresponding semiclassical approximation. Our goal is to give crude estimates for an indefinite integral of $\rho(x, g) - \rho_{sc}(x, g)$.

We write $c_\#$, $C_\#$ etc. for constants that depend only on the constants in the hypotheses of the WKB Theorem on low eigenvalues. We write C_*, c_* etc. for constants that depend also on the $\hat{C}_\beta$ above.

The WKB Theorem on low eigenvalues implies that the eigenvalues in $\{|E - V(x_0)| \le \tau^2 S\} \cap (-\infty, 0]$ may be written as $E_0, E_1, \dots, E_{k_{hi}}$, with

$$(1) \qquad \Big|\frac{1}{\pi}\phi(E_k) - \frac{1}{2} - k\Big| \le C_\# \lambda^{-1} \quad \text{for } 0 \le k \le k_{hi}$$
$$\{k \in \mathbb{Z} \mid 0 \le k \le k_{hi}\} = \mathbb{Z} \cap [-\frac{1}{2}, b], \text{ with}$$
$$b = \frac{1}{\pi}\phi(E_{hi}) - \frac{1}{2} + \omega_{hi}, \quad |\omega_{hi}| \le C_\# \lambda^{-1}, E_{hi} = \min(0, V(x_0) + \tau^2 S) \ .$$

Details (for once) are left to the reader.

For convenience, set $b_0 = \frac{1}{\pi}\phi(E_{hi}) - \frac{1}{2}$, $E_{\max} = V(x_0) + 2\tau^2 S$. The phase $\phi(E)$ satisfies the following, by the results in the section on WKB for low eigenvalues.

$$\phi(V(x_0)) = 0$$
$$\Big|\big(\frac{d}{dE}\big)^\beta \phi(E)\Big| \le C_\#^\beta \lambda S^{-\beta} \quad \text{for} \quad E \in [V(x_0), E_{\max}]$$
$$\frac{d\phi(E)}{dE} \ge c_\# \lambda S^{-1} \quad \text{for} \quad E \in [V(x_0), E_{\max}] \ .$$

For $t \in [-\frac{1}{2}, \frac{1}{\pi}\phi(E_{\max}) - \frac{1}{2}] = \mathcal{J}$, we can therefore solve the equation $\frac{1}{\pi}\phi(E) - \frac{1}{2} = t$, obtaining a solution $E = E(t) \in [V(x_0), E_{\max}]$, with

$$\left|\left(\frac{d}{dt}\right)^m E(t)\right| \le C_\#^m S\lambda^{-m} (m \ge 1) \quad \text{for } t \in \mathcal{J}\ , \quad \text{and}$$

$$\frac{d}{dt}E(t) \ge c_\# S\lambda^{-1} \quad \text{for} \quad t \in \mathcal{J}\ .$$

Also, $E(-\frac{1}{2}) = V(x_0)$. For $0 \le k \le k_{hi}$ we have $k \in \mathcal{J}$, so $E(k)$ is well-defined and lies in $[V(x_0), E_{\max}]$. Also, $E_k \in [V(x_0), E_{\max}]$, so $E_k = E(t_k)$ with $t_k = \frac{1}{\pi}\phi(E_k) - \frac{1}{2} \in \mathcal{J}$. We have $|t_k - k| \le C_\#\lambda^{-1}$ by (1), so $|E_k - E(k)| = |E(t_k) - E(k)| \le C_\#\lambda^{-1}\max_{t\in\mathcal{J}}\left|\frac{dE(t)}{dt}\right| \le C_\# S\lambda^{-2}$. Note that $b_0 \le \frac{1}{\pi}\phi(E_{\max}) = \frac{1}{\pi}\phi(E_{\max}) - \frac{1}{\pi}\phi(V(x_0)) \le |E_{\max} - V(x_0)| \cdot \max_{E\in[V(x_0),E_{\max}]}\left(\frac{d\phi(E)}{dE}\right) \le 2\tau^2 S \cdot C_\#\lambda S^{-1} = C'_\#\tau^2\lambda$. Since k_{hi} is the greatest integer in b and $|b - b_0| \le C_\#\lambda^{-1}$, it follows that $k_{hi} + 1 \le C'_\#\tau^2\lambda + C_\# \le C''_\#\tau^2\lambda$ (since $\tau > \lambda^{\varepsilon - \frac{1}{2}}$). We apply these observations to study the sum $\sum_{k=0}^{k_{hi}} g(E_k)$. In fact,

$$|g(E_k) - g(E(k))| \le |E(k) - E_k| \cdot \max|g'| \le C_\# S\lambda^{-2} \cdot C_*(\tau^2 S)^{-1} = C_*(\tau^2\lambda^2)^{-1}\ .$$

Hence,

$$\text{(2)} \qquad \left|\sum_{k=0}^{k_{hi}} g(E_k) - \sum_{k=0}^{k_{hi}} g(E(k))\right| \le (k_{hi} + 1)\cdot C_*(\tau^2\lambda^2)^{-1}$$

$$\le C_\#\tau^2\lambda \cdot C_*(\tau^2\lambda^2)^{-1} \le C_*\lambda^{-1}\ .$$

We want to compute $\sum_{k=0}^{k_{hi}} g(E(k))$ using the lemma on Riemann sums. This requires bounds for $\left(\frac{d}{dt}\right)^m g(E(t))$, which is a sum of terms of the form

$$\left(\frac{d}{dE}\right)^\beta g(E)\ |_{E=E(t)} \cdot \prod_{\nu=1}^{\beta}\left\{\left(\frac{d}{dt}\right)^{m_\nu} E(t)\right\} \quad \text{with} \quad m_\nu \ge 1, m_1 + \ldots + m_\beta = m\ ,$$

$$\text{hence} \quad 0 \le \beta \le m\ .$$

This term is dominated by $C_*^m(\tau^2 S)^{-\beta} \cdot S^\beta\lambda^{-m} = C_*^m(\tau^2)^{-\beta}\lambda^{-m} \le C_*^m(\tau^2\lambda)^{-m}$. Therefore,

$$\left|\left(\frac{d}{dt}\right)^m g(E(t))\right| \le C_*^m(\tau^2\lambda)^{-m} \quad \text{for} \quad t \in \mathcal{J}, m \ge 0\ .$$

Note that $\tau^2\lambda \geq 10$, as required in the hypothesis of the lemma on Riemann sums. That lemma now implies

$$\sum_{k=0}^{k_{hi}} g(E(k)) = \sum_{k\in\mathbb{Z}\cap[-\frac{1}{2},b]} g(E(k)) = \int_{-\frac{1}{2}}^{b} g(E(t))dt - g(E(b))\chi_-(b) + \text{ Err}_0 \ ,$$

with

$$|\text{Err}_0| \leq C_*(\tau^2\lambda)^{-1} + \int_{-\frac{1}{2}}^{b} C_*^{\overline{N}}(\tau^2\lambda)^{-\overline{N}}\ dt \leq C_*(\tau^2\lambda)^{-1}, \quad \text{since} \quad b \leq C_\#\tau^2\lambda \ .$$

Combining this with (2), we get

$$\sum_{k=0}^{k_{hi}} g(E_k) = \int_{-\frac{1}{2}}^{b} g(E(t))\ dt - g(E(b))\chi_-(b) + \text{ Err}_1 \ , \quad \text{with} \tag{3}$$

$$|\text{Err}_1| \leq C_*(\tau^2\lambda)^{-1} \ .$$

In (3), we want to use b_0 in place of b. Since $|b_0 - b| \leq C_\#\lambda^{-1}$, $|g(E(t))| \leq C_*$, $|\frac{d}{dt}g(E(t))| \leq C_*(\tau^2\lambda)^{-1}$, we have

$$\Big|\int_{b}^{b_0} g(E(t))dt\Big| \leq C_*\lambda^{-1} \ ,$$

and

$$|g(E(b)) - g(E(b_0))| \leq C_\#\lambda^{-1}\cdot C_*(\tau^2\lambda)^{-1} \ .$$

Putting these estimates into (3), we find that

$$\sum_{k=0}^{k_{hi}} g(E_k) = \int_{-\frac{1}{2}}^{b_0} g(E(t))dt - g(E(b_0))\chi_-(b) + \text{ Err}_2 \ , \quad \text{with} \tag{4}$$

$$|\text{Err}_2| \leq C_*(\tau^2\lambda)^{-1} \ .$$

We have $E(b_0) = E_{hi}$ by definition of b_0 and $E(t)$. Either $g(E_{hi}) = g(0) = 0$ or else $E_{hi} = 0$ and $b = \frac{1}{\pi}\phi(0) - \frac{1}{2} + \omega_{hi}$. In either case, we have

$$-g(E(b_0))\chi_-(b) = -g(0)\chi_-\big(\frac{1}{\pi}\phi(0) - \frac{1}{2} + \omega_{hi}\big) \ . \tag{5}$$

Also,

$$\int_{-1/2}^{b_0} g(E(t))dt = \int_{V(x_0)}^{E_{hi}} g(E) \cdot \frac{1}{\pi} \frac{d\phi(E)}{dE}\ dE$$
$$= \int_{E\in[V(x_0),E_{hi}]} g(E) \Big\{ \frac{1}{2\pi} \int_{x\in I_{\mathrm{BVP}}} (E - V(x))_+^{-\frac{1}{2}}\ dx \Big\}\ dE$$
$$= \int_{x\in I_{\mathrm{BVP}}} \Big\{ \frac{1}{2\pi} \int_{-\infty}^{0} g(E) \cdot (E - V(x))_+^{-\frac{1}{2}}\ dE \Big\}\ dx\ ,$$

in view of our assumption on the support of g. Putting this and (5) into (4), and writing $-g(0)[\phi'(0)]^{-1} \cdot \int_{x\in I_{\mathrm{BVP}}} \frac{1}{2}(-V(x))_+^{-\frac{1}{2}}\ dx \chi_-(\frac{1}{\pi}\phi(0) - \frac{1}{2} + \omega_{hi})$ for the right-hand side of (5),we get

(6)
$$\sum_{k=0}^{k_{hi}} g(E_k) = \int_{x\in I_{\mathrm{BVP}}} dx \Big\{ \frac{1}{2\pi} \int_{-\infty}^{0} g(E) \cdot (E - V(x))_+^{-\frac{1}{2}}\ dx - g(0)[\phi'(0)]^{-1}$$
$$\cdot \frac{1}{2}(-V(x))_+^{-\frac{1}{2}} \chi_-(\frac{1}{\pi}\phi(0) - \frac{1}{2} + \omega_{hi}) \Big\} + \ \mathrm{Err}_3\ , \quad \text{with} \quad |\mathrm{Err}_3| \le C_*(\tau^2\lambda)^{-1}\ .$$

Since $|\omega_{hi}| \le C_\# \lambda^{-1}$, we have

$$\Big|\chi_-(\frac{1}{\pi}\phi(0) - \frac{1}{2} + \omega_{hi}) - \chi_-\big(\frac{1}{\pi}\phi(0) - \frac{1}{2}\big)\Big|$$
$$\le \begin{cases} C_\# \lambda^{-1} & \text{if } \min_{k\in\mathbb{Z}} |\ \phi(0) - \pi(k + 1/2)| \ge \overline{C}_\# \lambda^{-1} \\ C_\# & \text{if } \min_{k\in\mathbb{Z}} |\ \phi(0) - \pi(k + 1/2)| \le \overline{C}_\# \lambda^{-1}\ , \end{cases}$$

So (6) implies

(7)
$$\sum_{k=0}^{k_{hi}} g(E_k) = \int_{x\in I_{\mathrm{BVP}}} \rho_{sc}(x, g)dx + \ \mathrm{Err}_4\ , \qquad \text{with}$$

(8)
$$|\mathrm{Err}_4| \le C_*(\tau^2\lambda)^{-1} \quad \text{if} \quad \min_{k\in\mathbb{Z}} |\phi(0) - \pi(k + 1/2)| \ge \overline{C}_\# \lambda^{-1}$$

(9)
$$|\mathrm{Err}_4| \le C_* \quad \text{if} \quad \min_{k\in\mathbb{Z}} |\phi(0) - \pi(k + 1/2)| \le \overline{C}_\# \lambda^{-1}\ .$$

Now let $u_k(x)$ be the real, normalized eigenfunction corresponding to E_k. Thus,

$$\rho(x,g) = \sum_{k=0}^{k_{hi}} g(E_k) u_k^2(x) \ . \tag{10}$$

We know that

$$\int_{|x-x_0|>\lambda^\varepsilon \tau B} |u_k(x)|^2 \ dx \leq C_\# \lambda^{-N'} \ . \tag{11}$$

This follows by applying lemma 2 in the section on the WKB Theorem for low eigenvalues, to the potential $\tilde{V}(x) = V(x) - \min(0, V(x_0) + \tilde{S})$, with $\tilde{S} = \lambda^{2\varepsilon}\tau^2 S$, $\tilde{B} = \lambda^\varepsilon \tau B$ in place of S, B.

Let

$$\begin{aligned} H_-(x) &= \int_{x' \in I_{\mathrm{BVP}} \cap (-\infty, x]} [\rho(x',g) - \rho_{sc}(x',g)] \ dx' \\ H_+(x) &= \int_{x' \in I_{\mathrm{BVP}} \cap [x, \infty)} [\rho(x',g) - \rho_{sc}(x',g)] \ dx' \ . \end{aligned}$$

Note that for $|x' - x_0| > \lambda^\varepsilon \tau B$ we have $E - V(x') < 0$ for $E \in \mathrm{supp}\ g \ \cap (-\infty, 0]$. Hence $\rho_{sc}(x', g)$ is supported in $\{|x' - x_0| < \lambda^\varepsilon \tau B\}$. Hence for $x \in I_{\mathrm{BVP}}$, $x < x_0 - \lambda^\varepsilon \tau B$ we have

$$\begin{aligned} |H_-(x)| &= \Big| \int_{x' \in I_{\mathrm{BVP}} \cap (-\infty, x]} \rho(x', g) \ dx' \Big| = \Big| \sum_{k=0}^{k_{hi}} g(E_k) \int_{x' \in I_{\mathrm{BVP}} \cap (-\infty, x]} u_k^2(x') \ dx' \Big| \\ &\leq C_* \sum_{k=0}^{k_{hi}} \int_{x' \in I_{\mathrm{BVP}} \setminus \{|x'-x_0| < \lambda^\varepsilon \tau B\}} |u_k(x')|^2 \ dx' \leq C_* (k_{hi} + 1) \lambda^{-N'} \\ &\leq C_* \lambda^{1-N'} \ , \text{ by (11) and our estimate for } k_{hi} \ . \end{aligned} \tag{12}$$

Similarly, for $x \in I_{\mathrm{BVP}}$, $x > x_0 + \lambda^\varepsilon \tau B$ we have

$$|H_+(x)| \leq C_* \lambda^{1-N'} \ . \tag{13}$$

Also, $H_-(x) + H_+(x) = \int_{x' \in I_{\rm BVP}} [\rho(x', g) - \rho_{sc}(x', g)]\ dx' = \sum_{k=0}^{k_{hi}} g(E_k) - \int_{I_{\rm BVP}} \rho_{sc}(x', g)\ dx'$, so (7), (8), (9), (13) yield the estimates:

$$(14) \quad |H_-(x)| \le C_*(\tau^2\lambda)^{-1} \quad \text{for} \quad x > x_0 + \lambda^\varepsilon \tau B,$$

$$\text{if} \quad \min_{k\in\mathbb{Z}} |\phi(0) - \pi(k+1/2)| \ge \overline{C}_\# \lambda^{-1}$$

$$(15) \quad |H_-(x)| \le C_* \quad \text{for} \quad x > x_0 + \lambda^\varepsilon \tau B\ ,$$

$$\text{if} \quad \min_{k\in\mathbb{Z}} |\phi(0) - \pi(k+1/2)| \le \overline{C}_\# \lambda^{-1}\ .$$

We estimate $H_-(x)$ in $\{|x - x_0| < \lambda^\varepsilon \tau B\}$, contenting ourselves with the crudest result. Trivially,

$$(16) \quad \Big| \int_{x' \in I_{\rm BVP} \cap (-\infty, x]} \rho(x', g)\ dx' \Big| \le \sum_{k=0}^{k_{hi}} |g(E_k)| \int_{x' \in I_{\rm BVP}} |u_k(x')|^2\ dx'$$

$$\le C_*(k_{hi} + 1) \le C_* \tau^2 \lambda\ , \quad \text{by our estimate for } k_{hi}\ .$$

Also,

$$\Big| \int_{x' \in I_{\rm BVP} \cap (-\infty, x]} \Big\{ \int_{-\infty}^0 g(E)(E - V(x'))_+^{-1/2}\ dE \Big\}\ dx' \Big| \le$$

$$\int_{\substack{V(x') < E < 0 \\ |E - V(x_0)| < \tau^2 S}} \int C_*(E - V(x'))_+^{-1/2}\ dE dx' \le$$

$$C_* \int_{\{V(x') \le \min(0, V(x_0) + \tau^2 S)\}} \Big\{ \int_{V(x')}^{V(x_0)+\tau^2 S} (E - V(x'))^{-1/2}\ dE \Big\}\ dx' \le$$

$$C_* \int_{\{V(x') \le \min(0, V(x_0) + \tau^2 S)\}} \Big\{ \int_{V(x')}^{V(x')+\tau^2 S} (E - V(x'))^{-1/2} dE \Big\}\ dx' \le$$

$$C_* \tau S^{1/2} \int_{\{V(x') \le \min(0, V(x_0) + \tau^2 S)\}} dx' \le C_* \tau S^{1/2} \cdot \tau B = C_* \tau^2 \lambda\ ,$$

and

$$\Big| \int_{x' \in I_{\rm BVP} \cap (-\infty, x]} g(0) \cdot \frac{1}{2}(-V(x'))_+^{-1/2} [\phi'(0)]^{-1} \chi_-(\frac{1}{\pi}\phi(0) - \frac{1}{2})\ dx' \Big|$$

$$\le |g(0)|\ |\chi_-(\frac{1}{\pi}\phi(0) - \frac{1}{2})| \le C_*\ .$$

Putting these estimates together, we get

$$\Big| \int\limits_{x' \in I_{\mathrm{BVP}} \cap (-\infty, x]} \rho_{sc}(x', g)\; dx' \Big| \le C_*(\tau^2 \lambda) \;.$$

This and (16) give

$$|H_-(x)| \le C_*(\tau^2\lambda) \quad \text{for} \quad |x - x_0| \le \lambda^{\varepsilon} \tau B \;. \tag{17}$$

Estimates (12), (14), (15), (17) are the main results of this section. We summarize them in the following lemma.

Lemma. *Assume the hypotheses of the WKB Theorem on low eigenvalues, with* $E_\infty = 0$. *Suppose* $g(E)$ *satisfies:*

$$\textit{supp}\; g(E) \subset \{|E - V(x_0)| < \tau^2 S\} \quad \textit{with} \quad \lambda^{2\varepsilon - 1/2} < \tau < \lambda^{-2\varepsilon} \;; \quad \textit{and}$$

$$\Big| \Big(\frac{d}{dE}\Big)^{\beta} g(E) \Big| \le \hat{C}_\beta (\tau^2 S)^{-\beta} \;.$$

Let $\rho(x, g)$, $\rho_{sc}(x, g)$ *be the microlocalized density for* g, *and its semiclassical approximation.*

Define

$$H(x) = \int\limits_{x' \in I_{\mathrm{BVP}} \cap (-\infty, x]} [\rho(x', g) - \rho_{sc}(x', g)]\; dx' \;.$$

Then

(a) $$|H(x)| \le C_* \lambda^{1 - N'} \quad \textit{if} \quad x \in I_{\mathrm{BVP}} \;,\; x < x_0 - \lambda^{\varepsilon} \tau B \;.$$

(b) $$|H(x)| \le C_*(\tau^2 \lambda) \quad \textit{if} \quad |x - x_0| \le \lambda^{\varepsilon} \tau B \;.$$

(c) $$|H(x)| \le C_*(\tau^2\lambda)^{-1} \quad \textit{if} \quad x \in I_{\mathrm{BVP}} \;,\; x > x_0 + \lambda^{\varepsilon}\tau B \;,$$
$$\min_{k \in \mathbb{Z}} |\phi(0) - \pi(k + 1/2)| \ge \overline{C}_{\#} \lambda^{-1}$$

(d) $$|H(x)| \le C_* \quad \textit{if} \quad x \in I_{\mathrm{BVP}} \;,\; x > x_0 + \lambda^{\varepsilon}\tau B \;,$$
$$\min_{k \in \mathbb{Z}} |\phi(0) - \pi(k + 1/2)| \le \overline{C}_{\#} \lambda^{-1} \;.$$

The constant $\overline{C}_{\#}$ *depends only on the constants in the hypotheses of the WKB Theorem on low eigenvalues. The constant* C_* *depends also on the* $\hat{C}_\beta$ *above.*

COMBINING THE MICROLOCALIZED RESULTS

In this section, we combine our previous results on microlocalized densities by using a partition of unity. Our goal is to control $\rho(x,\varphi)$ where $\varphi(E)$ is a function of energy alone, that vanishes near the minimum of the potential. In the next section, we will remove the assumption that φ vanishes near the minimum of the potential, and take $\varphi \equiv 1$ to control the density $\rho(x)$. Our present set-up is as follows.

We are given a potential $V(x)$ defined on a (possibly unbounded) interval I_{BVP}. On a subinterval $I \subset I_{\mathrm{BVP}}$ we are given positive functions $S(x)$, $B(x)$. We are given positive numbers K, ε, N with $K > 100$, $\varepsilon < \frac{1}{1000K}$, $N > \frac{K}{\varepsilon^{50}}$. We set $N' = [\varepsilon N/500]$ and $N'' = \frac{3}{2}\varepsilon N' - 30000K - 33$. We are given a point $x_0 \in I$.

Define $H = -\frac{d^2}{dx^2} + V(x)$ on I_{BVP}, with Dirichlet or Neumann boundary conditions. When we speak of eigenvalues or eigenfunctions, we mean those of H. We are given a function $\varphi(E)$ and a constant $\hat{c} > 0$. We make the following hypotheses.

Assumptions concerning $V(x)$, $S(x)$, $B(x)$ on I.

(Y0) If $x, y \in I$ and $|x-y| < cB(x)$, then $c < \frac{B(y)}{B(x)} < C$ and $c < \frac{S(y)}{S(x)} < C$.

(Y1) If $x \in I$ and $\alpha \geq 0$ then $|(\frac{d}{dx})^\alpha V(x)| \leq C_\alpha S(x) B^{-\alpha}(x)$.

(Y2) The set $\{x \in I \mid V(x) < 0\}$ is a non-empty interval $(x_{\mathrm{left}}(0), x_{\mathrm{rt}}(0))$, with $\mathrm{dist}(x_{\mathrm{left}}(0), \partial I) > cB(x_{\mathrm{left}}(0))$, $\mathrm{dist}(x_{\mathrm{rt}}(0), \partial I) > cB(x_{\mathrm{rt}}(0))$.

(Y3) We have $V(x_0) < -c_2 S(x_0)$, $V'(x_0) = 0$; and for $|x - x_0| \leq c_1 B(x_0)$ we have $V''(x) \geq cS(x_0)B^{-2}(x_0)$.

(Y4) For $x_{\mathrm{left}}(0) \leq x \leq x_0 - c_1 B(x_0)$ we have $-V'(x) \geq cS(x)B^{-1}(x)$, and for $x_0 + c_1 B(x_0) \leq x \leq x_{\mathrm{rt}}(0)$ we have $+V'(x) \geq cS(x)B^{-1}(x)$.

As usual, define $\lambda(x) = S^{1/2}(x)B(x)$. Set $\Lambda = \left(\int_{x_{\mathrm{left}}(0)}^{x_{\mathrm{rt}}(0)} \frac{dx}{\lambda(x)B(x)}\right)^{-1}$.

Assumptions concerning $V(x)$ on all of I_{BVP}.

(Y5) For all $x \in I_{\mathrm{BVP}} \backslash [x_{\mathrm{left}}(0), x_{\mathrm{rt}}(0)]$ we have $V(x) > 0$.

(Y6) For all $x \in I_{\mathrm{BVP}}$ with $x < x_{\mathrm{left}}(0) - \Lambda^K B(x_{\mathrm{left}}(0))$, we have $V(x) \geq \frac{1000}{|x-x_{\mathrm{left}}(0)|^2}$; and for all $x \in I_{\mathrm{BVP}}$ with $x > x_{\mathrm{rt}}(0) + \Lambda^K B(x_{\mathrm{rt}}(0))$, we have $V(x) \geq \frac{1000}{|x-x_{\mathrm{rt}}(0)|^2}$.

Polynomial growth conditions on $S(x)$, $B(x)$.

(Y7) We have $\max_{x\in I} B(x) < \Lambda^K \cdot \min_{x\in I} B(x)$, $\max_{x\in I} S(x) < \Lambda^K \cdot \min_{x\in I} S(x)$, and $|I| < \Lambda^K \cdot \min_{x\in I} B(x)$.

Assumptions on the function $\varphi(E)$ and the constant $\hat{c}$.

(Y8) We have $|(\frac{d}{dE})^\beta \varphi(E)| \leq \hat{C}_\beta (S(x_0))^{-\beta}$ for all E.

(Y9) For $-\infty < E \leq V(x_0) + c_2 S(x_0)$ we have $\varphi(E) = 0$.

(Y10) The constant $\hat{c}$ is bounded above by a certain small, positive number determined by ε, K, N, c, C, c_1, C_α, c_2 above.

The WKB Hypothesis.

(Y11) Λ is bounded below by a certain large, positive number determined by ε, K, N, c, C, c_1, C_α ,c_2, $\hat{c}$, $\hat{C}_\beta$ above.

Let $u_k(x)$ and E_k denote the (real, normalized) eigenfunctions and eigenvalues of H, with $E_k \leq 0$.

We denote by $c_\#$, $C_\#$ etc. constants that depend only on ε, K, N, c, C, c_1, C_α, c_2 in (Y0)...(Y11); while c_*, C_* etc. denote constants that depend only on ε, K, N, c, C, c_1, C_α, c_2, $\hat{c}$, $\hat{C}_\beta$ in (Y0)...(Y11).

Our goal in this section is to compare $\rho(x, \varphi)$ with its semiclassical approximation. To carry this out, we construct a suitable partition of unity. More precisely, we will construct functions $g_{\nu k}(x, E)$ with the following properties:

(i) Each $\rho(x, g_{\nu k})$ can be compared with its semiclassical approximation, either by the Oscillatory density lemma or by the Airey density lemma.

(ii) $\sum_{\nu k} g_{\nu k}(x,E) = \varphi(E)$, except on a set where microlocalized densities are negligibly small because $E < V(x)$.

We then apply (i) and (ii) to show that

$$\rho(x,\varphi) = \sum_{\nu k} \rho(x, g_{\nu k}) + \rho(x, \varphi - \sum_{\nu k} g_{\nu k}) \approx \sum_{\nu k} \rho_{sc}(x, g_{\nu k}) \approx \rho_{sc}(x,\varphi)$$

as desired. We begin with the construction of the $g_{\nu k}$, which proceeds as follows. Introduce an interval $\tilde{I} = [x_{\text{left}}(0) - c^1_\# B(x_{\text{left}}(0)), x_{\text{rt}}(0) + c^1_\# B(x_{\text{rt}}(0))] \subset I$, for which we have:

$$S(x) \le C_\# S(x_{\text{left}}(0)) \quad \text{and} \quad -V'(x) > c_\# S(x) B^{-1}(x)$$
$$\text{for} \quad x \in \tilde{I} \cap (-\infty, x_{\text{left}}(0)] \; ; \quad \text{and}$$

$$S(x) \le C_\# S(x_{\text{rt}}(0)) \quad \text{and} \quad +V'(x) > c_\# S(x) B^{-1}(x)$$
$$\text{for} \quad x \in \tilde{I} \cap [x_{\text{rt}}(0), +\infty) \; .$$

Then set

$$\check{I} = [x_{\text{left}}(0) - \Lambda^{-\varepsilon} B(x_{\text{left}}(0)), x_{\text{rt}}(0) + \Lambda^{-\varepsilon} B(x_{\text{rt}}(0))] \subset \tilde{I} \; .$$

We can produce a partition of unity $\{\theta_\nu(x)\}$ with the following properties.

(1) $$\sum_\nu \theta_\nu(x) = 1 \quad \text{for} \quad x \in \check{I}$$

(2) $$\text{supp}\ \theta_\nu(x) \subset \{|x - x_\nu| < \hat{c} B(x_\nu)\} \subset \tilde{I}$$

(3) $$\left| \left(\frac{d}{dx}\right)^\alpha \theta_\nu(x) \right| \le C^\alpha_* (B(x_\nu))^{-\alpha}$$

(4) Each point $\tilde{x}$ belongs to at most $C_\#$ of the intervals $\{|x - x_\nu| < 10\,\hat{c} B(x_\nu)\}$

$$(5) \qquad \theta_\nu \geq 0 \text{ everywhere, and } \theta_\nu(x_\nu) \geq c_\# > 0 \ .$$

We can also achieve

(5a) supp θ_ν meets $\check{I}$ for each ν,

simply by deleting the θ_ν whose supports fail to meet $\check{I}$.

(To avoid doing violence to the notation, we may suppose x_0 in (Y0)...(Y11) is equal to our present x_ν with $\nu = 0$.)

Next, pick $k_{\max}$ so that $\frac{1}{2}\Lambda^{-2/7} < 2^{-k_{\max}} \leq \Lambda^{-2/7}$, and take functions $\chi_0(t), \chi_1(t), \ldots, \chi_{k_{\max}}(t)$ with the following properties.

$$(6) \qquad \chi_k(t) \text{ is supported in } \{2^{-k} \leq t \leq C_* 2^{-k}\} \text{ for } 0 \leq k < k_{\max} \ .$$

$$(7) \qquad \chi_{k_{\max}}(t) \text{ is supported in } \{|t| \leq C_* 2^{-k_{\max}}\}$$

$$(8) \qquad \left|\left(\frac{d}{dt}\right)^m \chi_k(t)\right| \leq C_*^m (2^{-k})^{-m} \text{ for } 0 \leq k \leq k_{\max} \text{ and all } t, m \ .$$

$$(9) \qquad \sum_{0 \leq k \leq k_{\max}} \chi_k(t) = 1 \quad \text{for} \quad -\Lambda^{-2/7} \leq t \leq 1/\hat{c} \ .$$

Then define $h_{\nu k}(x, E) = \theta_\nu(x)\chi_k\left(\frac{E - V(x)}{S(x_\nu)}\right)$, and set $g_{\nu k}(x, E) = \varphi(E) \cdot h_{\nu k}(x, E)$. Thus, we have defined $g_{\nu k}$. We want to check properties (i) and (ii) above. Let us begin with (ii). Equation (9) shows that

$$(10) \qquad \sum_{0 \leq k \leq k_{\max}} h_{\nu k}(x, E) = \theta_\nu(x) \text{ if } E \in [V(x) - \Lambda^{-2/7} S(x_\nu), V(x) + \frac{S(x_\nu)}{\hat{c}}] \ .$$

If $x \in$ supp θ_ν, then $c_\# S(x_\nu) < S(x) < C_\# S(x_\nu)$ and $|V(x)| \leq C_\# S(x)$. Hence the energy interval in (10) contains $[V(x) - c_\# \Lambda^{-2/7} S(x), 0]$ by virtue of (Y10). So $\sum_{0 \leq k \leq k_{\max}} h_{\nu k}(x, E) = \theta_\nu(x)$ if $V(x) - c_\# \Lambda^{-2/7} S(x) \leq E \leq 0$ and $x \in$ supp θ_ν. The assumption $x \in \operatorname{supp} \theta_\nu$ is unnecessary, since $h_{\nu k}(x,E) = \theta_\nu(x) = 0$ for $x \notin \operatorname{supp} \theta_\nu$. Summing on ν and using (1), (4), we obtain

$$\sum_{\nu k} h_{\nu k}(x,E) = 1 \quad \text{if} \quad x \in \check{I} \quad \text{and} \quad V(x) - c_\# \Lambda^{-2/7} S(x) \le E \le 0, \text{ and} \tag{11}$$

$$\Big| \sum_{\nu k} h_{\nu k}(x,E) \Big| \le C_* \quad \text{for all } x, E \ . \tag{12}$$

Hypotheses (Y3), (Y4) show that $V(x)$ is decreasing in $[x_{\text{left}}(0), x_0]$ and increasing in $[x_0, x_{\text{rt}}(0)]$. Hence for $V(x_0) < E \le 0$, the set $\{x \in [x_{\text{left}}(0), x_{\text{rt}}(0)] | V(x) < E\}$ is an interval $(x_{\text{left}}(E), x_{\text{rt}}(E))$, with $x_{\text{left}}(0) \le x_{\text{left}}(E) < x_0 < x_{\text{rt}}(E) \le x_{\text{rt}}(0)$, $V(x_{\text{left}}(E)) = V(x_{\text{rt}}(E)) = E$. Also $V(x) > E$ in $[x_{\text{left}}(0), x_{\text{rt}}(0)] \backslash [x_{\text{left}}(E), x_{\text{rt}}(E)]$. Hypotheses (Y2), (Y5) show that in fact $\{x \in I_{\text{BVP}} \mid V(x) < E\} =$ $(x_{\text{left}}(E), x_{\text{rt}}(E))$ for $V(x_0) < E \le 0$, and $V(x) > E$ for $x \in I_{\text{BVP}} \backslash [x_{\text{left}}(E), x_{\text{rt}}(E)]$.

If $V(x_0) < E \le 0$ and $x \in [x_{\text{left}}(E) - B(x_{\text{left}}(E))\Lambda^{-3/7}, x_{\text{rt}}(E) + B(x_{\text{rt}}(E))\Lambda^{-3/7}]$, then we will check that

$$V(x) - c_\# \Lambda^{-2/7} S(x) \le E \ , \quad \text{with} \quad c_\# \quad \text{as in (11)} \ . \tag{13}$$

In fact, this is obvious for $x \in [x_{\text{left}}(E), x_{\text{rt}}(E)]$, since then already $V(x) \le E$.

For $x \in [x_{\text{left}}(E) - \Lambda^{-3/7} B(x_{\text{left}}(E)), x_{\text{left}}(E)]$, we have $V(x) \le V(x_{\text{left}}(E)) + C_\# S(x) B^{-1}(x) |x - x_{\text{left}}(E)| \le V(x_{\text{left}}(E)) + C'_\# \Lambda^{-3/7} S(x) = E + C'_\# \Lambda^{-3/7} S(x)$, which implies (13) by the WKB hypothesis (Y11). Similarly, (13) holds for $x \in [x_{\text{rt}}(E), x_{\text{rt}}(E) + \Lambda^{-3/7} B(x_{\text{rt}}(E))]$. This completes the verification of (13).

Combining (11) and (13), we get

$$\sum_{\nu k} h_{\nu k}(x,E) = 1 \quad \text{and} \quad \sum_{\nu k} g_{\nu k}(x,E) = \varphi(E) \text{ for } V(x_0) < E \le 0 \text{ and} \tag{14}$$
$$x \in [x_{\text{left}}(E) - \Lambda^{-3/7} B(x_{\text{left}}(E)), x_{\text{rt}}(E) + \Lambda^{-3/7} B(x_{\text{rt}}(E))] \ .$$

(Note that the x-interval in (14) is contained in $\check{I}$.)

Equation (14) is the precise form of property (ii) above.

Next, we prepare to check property (i), by proving the basic estimates for the derivatives of $h_{\nu k}(x,E)$ and $g_{\nu k}(x,E)$. The derivative $\partial_x^\alpha \partial_E^\beta \chi_k\left(\frac{E-V(x)}{S(x_\nu)}\right)$ is a sum of terms

$$(15)\quad \left[\left(\frac{d}{dt}\right)^m \chi_k(t)\,\Big|_{t=\frac{E-V(x)}{S(x_\nu)}}\right]\cdot\prod_{\mu=1}^m\left[\partial_x^{\alpha_\mu}\partial_E^{\beta_\mu}\left(\frac{E-V(x)}{S(x_\nu)}\right)\right],$$

$$\text{with}\quad \alpha_\mu+\beta_\mu\ge 1, \alpha_1+\ldots+\alpha_m=\alpha, \beta_1+\ldots+\beta_m=\beta.$$

$$\text{In particular, } 0\le m\le \alpha+\beta\,.$$

If $\beta_\mu\ge 2$ then $\partial_x^{\alpha_\mu}\partial_E^{\beta_\mu}\left(\frac{E-V(x)}{S(x_\nu)}\right)=0$.
If $\beta_\mu=1$ and $\alpha_\mu\ge 1$, then again $\partial_x^{\alpha_\mu}\partial_E^{\beta_\mu}\left(\frac{E-V(x)}{S(x_\nu)}\right)=0$.
If $\beta_\mu=1$ and $\alpha_\mu=0$, then $\partial_x^{\alpha_\mu}\partial_E^{\beta_\mu}\left(\frac{E-V(x)}{S(x_\nu)}\right)=S^{-1}(x_\nu)$.
If $\beta_\mu=0$ and $\alpha_\mu\ge 1$ then $\left|\partial_x^{\alpha_\mu}\partial_E^{\beta_\mu}\left(\frac{E-V(x)}{S(x_\nu)}\right)\right|\le C_\#^{\alpha_\mu}B^{-\alpha_\mu}(x_\nu)$ for $x\in$ supp θ_ν.
Hence $\left|\partial_x^{\alpha_\mu}\partial_E^{\beta_\mu}\left(\frac{E-V(x)}{S(x_\nu)}\right)\right|\le C_\#^{\alpha_\mu\beta_\mu}B^{-\alpha_\mu}(x_\nu)S^{-\beta_\mu}(x_\nu)$ for $\alpha_\mu+\beta_\mu\ge 1$, $x\in$ supp θ_ν.
This and (8) show that the term (15) is dominated by
$C_\#^{\alpha\beta}(2^{-k})^{-m}B^{-\alpha}(x_\nu)S^{-\beta}(x_\nu)$, which in turn is dominated by
$C_\#^{\alpha\beta}(2^{-k}B(x_\nu))^{-\alpha}(2^{-k}S(x_\nu))^{-\beta}$, since $0\le m\le\alpha+\beta$. Hence,

$$(16)\quad \left|\partial_x^\alpha\partial_E^\beta\chi_k\left(\frac{E-V(x)}{S(x_\nu)}\right)\right|\le C_\#^{\alpha\beta}(2^{-k}B(x_\nu))^{-\alpha}(2^{-k}S(x_\nu))^{-\beta}\quad\text{for } x\in \text{ supp }\theta_\nu\,.$$

Together with (3), this yields

$$(17)\quad \left|\partial_x^\alpha\partial_E^\beta h_{\nu k}\left(\frac{E-V(x)}{S(x_\nu)}\right)\right|\le C_*^{\alpha\beta}(2^{-k}B(x_\nu))^{-\alpha}(2^{-k}S(x_\nu))^{-\beta}\,,\quad\text{all}\quad (x,E)\,.$$

To prove analogous estimates for $g_{\nu k}(x,E)$, we need to compare $S(x_\nu)$ with $S(x_0)$. Hence make the following argument. For $x\in[x_{\text{left}}(0),x_{\text{rt}}(0)]$, define $I_x=[x_{\text{left}}(0),x_{\text{rt}}(0)]\cap[x-c_\#^2B(x),x+c_\#^2B(x)]$. This interval has length at least $c_\#B(x)$, and hypotheses (Y3), (Y4) show that $V(\tilde{x})$ varies by at least $c_\#S(x)$ as $\tilde{x}$ varies in I_x. Hence, $c_\#S(x)\le\max_{\tilde{x}\in I_x}|V(\tilde{x})|\le|V(x_0)|$ (since $V(x_0)\le V(\tilde{x})\le 0$ in I_x) $\le C_\#S(x_0)$. Thus, $S(x)\le C_\#S(x_0)$ for $x\in[x_{\text{left}}(0),x_{\text{rt}}(0)]$. The defining properties of $\tilde{I}$ show that for every $x \in \tilde{I}$ there is an $x' \in [x_{\rm left}(0), x_{\rm rt}(0)]$ with $S(x) \le C_\# S(x')$. (In fact, set $x' =$ either $x, x_{\rm left}(0)$, or $x_{\rm rt}(0)$.) We conclude that $S(x) \le C_\# S(x_0)$ for all $x \in \tilde{I}$, hence

$$S(x_\nu) \le C_\# S(x_0), \text{ all } \nu \ . \tag{18}$$

From (17), (18), and hypothesis (Y8) we get

$$|\partial_x^\alpha \partial_E^\beta g_{\nu k}(x,E)| \le C_*^{\alpha\beta}(2^{-k}B(x_\nu))^{-\alpha}(2^{-k}S(x_\nu))^{-\beta} \text{ for all } (x,E) \ . \tag{19}$$

The basic results needed to establish property (i) for $g_{\nu k}$ are as follows.

Lemma 1. *Suppose $V(x_0) + \frac{1}{10}c_2 S(x_0) \le E_0 \le 0$, with c_2 as in hypothesis (Y9). Then the hypotheses (Hyp 0)... (Hyp 10) of the WKB Theorems are satisfied, with $E_\infty = 0$, and with $300K$ in place of K. The constants called c, C, c_1, c_2, C_α in (Hyp 0) ... (Hyp 10) depend only on the constants ε, K, N, c, C, c_1, c_2, C_α in our present hypotheses (Y0)... (Y11). The number called Λ in (Hyp 0)... (Hyp 10) is greater than or equal to the number Λ appearing in (Y0)... (Y11).*

Lemma 2. *For each ν, k with $0 \le k < k_{\max}$, one of the following alternatives holds.*

(A) $g_{\nu k}(x,E) = 0$ for all $x \in I_{\rm BVP}$, $E \in (-\infty, 0]$.

(B) For suitable $\tilde{x}_0$, $E_{\ell o}$, E_{hi}, the functions $V(x)$, $S(x)$, $B(x)$, $g_{\nu k}(x,E)$ satisfy the hypotheses (A1)... (A6) of the Oscillatory Density Lemma, with x_ν in place of x_0, with $300K$ in place of K, and with $\tau = 2^{-k}$. The constants called ε, K, N, c, C, c_1, c_2, C_α in (A1), and the constant called c_3 in (A2) depend only on the constants ε, K, N, c, C, c_1, c_2, C_α in our present hypotheses (Y0)... (Y11). The constants called $\hat{c}$, $\hat{c}_1$, $\hat{C}$, $\hat{C}_{\alpha\beta}$, in (A2)... (A5) depend only on the constants ε, K, N, c, C, c_1, c_2, C_α, $\hat{c}$, $\hat{C}_\beta$ in (Y0)... (Y11). The number called $\Lambda_{\min}$ in (A1)... (A6) is greater than or equal to the number Λ appearing in (Y0)... (Y11).

Lemma 3. *For each ν, one of the following alternatives holds.*

(A) $g_{\nu k_{\max}}(x,E) = 0$ *whenever* $E \le 0$.

(B) $V(x)$, $S(x)$, $B(x)$, $g_{\nu k_{\max}}(x,E)$ *satisfy the hypotheses (X0)... (X12) of the Airey density lemma, with* $E_0 = \min(V(x_\nu), 0)$, *with suitable* δE, *with* $c_*\Lambda^{-2/7} \le \frac{\delta x}{B(x_\nu)} < C_*\Lambda^{-2/7}$, *and with* $300K$ *in place of* K. *The constants called* ε, K, N, c, C, c_1, C_α *in (X0)... (X12) depend only on the constants* ε, K, N, c, C, c_1, c_2, C_α *in our present hypotheses (Y0)... (Y11). The constants called* $\hat{c}$, $\hat{C}_{\alpha\beta}$ *in (X0)... (X12) depend only on the constants* ε, K, N, c, C, c_1, c_2, C_α $\hat{c}$, $\hat{C}_\beta$ *in (Y0)... (Y11). The number* $\Lambda_{\min}$ *in (X0)... (X12) is greater than or equal to the number* Λ *in (Y0)... (Y11). Also,* $|x_\nu - x_{\text{left}}(E_0)| \le 2\hat{c}B(x_\nu)$.

(C) The same conclusions as (B) hold, with the rôles of "left" and "right" interchanged. (In particular, in place of the Airey density lemma, we use its analogue with the rôles of "left" and "rt" interchanged.)

The proofs of Lemmas 1, 2, 3 are as follows.

Proof of Lemma 1. We have to check (Hyp 0)... (Hyp 10). First of all, (Hyp 0) and (Hyp 1) are our present hypotheses (Y0), (Y1). To check (Hyp 2), let $E \in (V(x_0), 0]$. We know that $V(x_{\text{left}}(E)) = V(x_{\text{rt}}(E)) = E$ with $x_{\text{left}}(0) \le x_{\text{left}}(E) < x_{\text{rt}}(E) \le x_{\text{rt}}(0)$, that

$$V(x) < E \quad \text{for} \quad x \in (x_{\text{left}}(E), x_{\text{rt}}(E)) \ , \quad \text{and that}$$
$$V(x) > E \quad \text{for} \quad x \in I_{\text{BVP}} \backslash [x_{\text{left}}(E), x_{\text{rt}}(E)] \ .$$

In particular, $\text{dist}(x_{\text{left}}(E), \partial I)$
$> c_\# B(x_{\text{left}}(E))$ and $\text{dist}(x_{\text{rt}}(E), \partial I) > c_\# B(x_{\text{rt}}(E))$ by virtue of (Y2). Taking $E = E_0$, we get (Hyp 2). To check (Hyp 3) we note that $x_{\text{left}}(E_0) \notin [x_0 - c_\# B(x_0), x_0 + c_\# B(x_0)]$. (Otherwise, we could not have $E_0 = V(x_{\text{left}}(E_0)) \ge V(x_0) + \frac{1}{10}c_2 S(x_0)$.) Hypotheses (Y3), (Y4) then imply $-V'(x_{\text{left}}(E_0)) > c'_\# S(x_{\text{left}}(E_0))B^{-1}(x_{\text{left}}(E_0))$, since $x_{\text{left}}(E_0) < x_0$. Similarly, $+V'(x_{\text{rt}}(E_0)) > c'_\# S(x_{\text{rt}}(E_0))B^{-1}(x_{\text{rt}}(E_0))$. Our

estimates on $|V''|$ imply therefore

$$-V'(x) > c''_{\#} S(x_{\rm left}(E_0)) B^{-1}(x_{\rm left}(E_0)) \qquad \text{for} \quad x_{\rm left}(E_0) \le x \le x_{\rm left}(E_0) + 2c''_{\#} B(x_{\rm left}(E_0))$$

and

$$+V'(x) > c''_{\#} S(x_{\rm rt}(E_0)) B^{-1}(x_{\rm rt}(E_0)) \qquad \text{for} \quad x_{\rm rt}(E_0) - 2c''_{\#} B(x_{\rm rt}(E_0)) \le x \le x_{\rm rt}(E_0)\ .$$

These estimates imply (Hyp 3), with $c''_{\#}$ in place of c_1. To check (Hyp 4), we first note that $E_0 \le 0$, so $E_0 - V(x) \le -V(x) \le |V(x)| \le C_{\#} S(x)$. Hence to establish (Hyp 4), we need only check that $E_0 - V(x) \ge c'''_{\#} S(x)$ for $x \in [x_{\rm left}(E_0) + c''_{\#} B(x_{\rm left}(E_0)),\ x_{\rm rt}(E_0) - c''_{\#} B(x_{\rm rt}(E_0))]$. We verify this as follows.

First of all, $V(x) \le V(x_0) + \frac{1}{20} c_2 S(x_0)$ if $|x - x_0| < c_{\#} B(x_0)$. Since $E_0 \ge V(x_0) + \frac{1}{10} c_2 S(x_0)$, it follows that $E_0 - V(x) \ge \frac{1}{20} c_2 S(x_0) > \tilde{c}_{\#} S(x)$ if $|x - x_0| < c_{\#} B(x_0)$. So it is enough to look at $x \in [x_{\rm left}(E_0) + c''_{\#} B(x_{\rm left}(E_0)),\ x_{\rm rt}(E_0) - c''_{\#} B(x_{\rm rt}(E_0))] \backslash \{|x - x_0| < c_{\#} B(x_0)\}$. For such x, either

$$x_{\rm left}(E_0) < x - \tilde{c}'_{\#} B(x) < x < x_0 - c_{\#} B(x_0)$$

or

$$x_{\rm rt}(E_0) > x + \tilde{c}'_{\#} B(x) > x > x_0 + c_{\#} B(x_0)\ .$$

Here we discuss only the first case since the second is analogous. For $x_{\rm left}(E_0) < x' < x$ we have $-V'(x') \ge \tilde{c}''_{\#} S(x') B^{-1}(x') > 0$, so $E_0 - V(x) = V(x_{\rm left}(E_0)) - V(x) = \int_{x_{\rm left}(E_0)}^{x} (-V'(x'))dx' \ge \int_{x - \tilde{c}'_{\#} B(x)}^{x} \tilde{c}''_{\#} S(x') B^{-1}(x') dx' \ge c'''_{\#} S(x)$ as needed. The proof of (Hyp 4) is complete.

To check (Hyp 5), we take $10^{-5} c_2$ in place of c_2. Since $S_{\rm min} = \min_{x \in [x_{\rm left}(E_0), x_{\rm rt}(E_0)]} S(x) \le S(x_0)$ and $E_0 \ge V(x_0) + \frac{1}{10} c_2 S(x_0)$, the assumptions on E in (Hyp 5) imply $0 \geq E \geq E_0 - 10^{-5} c_2 S_{\min} \geq (V(x_0) + \frac{1}{10} c_2 S(x_0)) - 10^{-5} c_2 S(x_0) > V(x_0)$. As we saw before and noted in the discussion of Hyp(2), $0 \geq E > V(x_0)$ implies $V(x) > E$ for all $x \in I_{\mathrm{BVP}} \backslash [x_{\mathrm{left}}(E), x_{\mathrm{rt}}(E)]$. This proves (Hyp 5).

Next we note that the quantity called Λ in (Hyp 7)...(Hyp 10) is in fact $\Lambda(E_0) = \left(\int_{x_{\mathrm{left}}(E_0)}^{x_{\mathrm{rt}}(E_0)} \frac{dx}{\lambda(x) B(x)}\right)^{-1} \geq \left(\int_{x_{\mathrm{left}}(0)}^{x_{\mathrm{rt}}(0)} \frac{dx}{\lambda(x) B(x)}\right)^{-1} = \Lambda$. So the quantity called Λ in (Hyp 7)...(Hyp 10) is greater than or equal to our present Λ from (Y0)...(Y11). Hence in proving (Hyp 7)...(Hyp 10), it is enough to use our present Λ in place of $\Lambda(E_0)$.

To check (Hyp 6) with $300K$ in place of K, suppose $x \in I_{\mathrm{BVP}}$, with $x < x_{\mathrm{left}}(E_0) - \frac{1}{2}(\lambda(x_{\mathrm{left}}(E_0)))^{300K} B(x_{\mathrm{left}}(E_0))$. We have $\lambda(x_{\mathrm{left}}(E_0)) \geq c_{\#}\Lambda$, and $|x_{\mathrm{left}}(E_0) - x_{\mathrm{left}}(0)| \leq |I| \leq \Lambda^K B(x_{\mathrm{left}}(E_0))$ by (Y7), so

$$\begin{aligned} x &< [x_{\mathrm{left}}(0) + \Lambda^K B(x_{\mathrm{left}}(E_0))] - \frac{1}{2} c_{\#} \Lambda^{300K} B(x_{\mathrm{left}}(E_0)) \\ &< x_{\mathrm{left}}(0) - \Lambda^{200K} B(x_{\mathrm{left}}(E_0)) \end{aligned}$$

(by the WKB hypothesis (Y11)) $< x_{\mathrm{left}}(0) - \Lambda^{200K} \cdot \Lambda^{-K} B(x_{\mathrm{left}}(0))$ (by another application of (Y7)).

Hence $x < x_{\mathrm{left}}(0) - \Lambda^K B(x_{\mathrm{left}}(0))$, so (Y6) yields $V(x) \geq \frac{1000}{|x - x_{\mathrm{left}}(0)|^2} \geq \frac{1000}{|x - x_{\mathrm{left}}(E_0)|^2}$, since $x < x_{\mathrm{left}}(0) \leq x_{\mathrm{left}}(E_0)$. Thus, $V(x) \geq \frac{1000}{|x - x_{\mathrm{left}}(E_0)|^2}$ if $x \in I_{\mathrm{BVP}}$ and $x < x_{\mathrm{left}}(E_0) - \frac{1}{2}(\lambda(x_{\mathrm{left}}(E_0)))^{300K} B(x_{\mathrm{left}}(E_0))$. Similarly, $V(x) \geq \frac{1000}{|x - x_{\mathrm{rt}}(E_0)|^2}$ if $x \in I_{\mathrm{BVP}}$ and $x > x_{\mathrm{rt}}(E_0) + \frac{1}{2}(\lambda(x_{\mathrm{rt}}(E_0)))^{300K} B(x_{\mathrm{rt}}(E_0))$. This completes the proof of (Hyp 6).

(Hyp 7) with $300K$ in place of K follows trivially from (Y7) and the fact that $\lambda(x_{\mathrm{left}}(E_0)), \lambda(x_{\mathrm{rt}}(E_0)) \geq c_{\#}\Lambda$.

To check (Hyp 8) with $300K$ in place of K let $\tilde{x} \in I$ be given, and note that $S(x) \geq \Lambda^{-K} S(\tilde{x})$ for $x \in I$ by (Y7). Hence $\int_I \frac{dx}{S^{1/2}(x)} \leq \Lambda^{\frac{1}{2}K} S^{-1/2}(\tilde{x}) |I| \leq \Lambda^{\frac{1}{2}K} S^{-1/2}(\tilde{x}) \cdot \Lambda^K B(\tilde{x})$, again by (Y7) . Taking $\tilde{x} = x_{\mathrm{left}}(E_0)$, $x_{\mathrm{rt}}(E_0)$, we obtain (Hyp 8).

To check (Hyp 9) with $300K$ in place of K we again fix $\tilde{x} \in I$ and apply (Y7), to deduce

$$\begin{aligned}
&\Big[\int_I \frac{dx}{S^{1/2}(x)B^4(x)}\Big]\Big[\int_I \frac{dx}{S^{1/2}(x)}\Big] \\
&\qquad \leq \Big[\Lambda^{1/2K}S^{-1/2}(\tilde{x})\Lambda^{4K}B^{-4}(\tilde{x})|I|\Big]\Big[\Lambda^{1/2K}S^{-1/2}(\tilde{x})|I|\Big] \\
&\qquad = \Lambda^{5K}S^{-1}(\tilde{x})B^{-4}(\tilde{x})|I|^2 \leq \Lambda^{5K}S^{-1}(\tilde{x})B^{-4}(\tilde{x})[\Lambda^K B(\tilde{x})]^2 \\
&\qquad\qquad = \frac{\Lambda^{7K}}{S(\tilde{x})B^2(\tilde{x})} = \frac{\Lambda^{7K}}{\lambda^2(\tilde{x})} \leq C_\# \Lambda^{7K-2} \ .
\end{aligned}$$

This trivially implies (Hyp 9).

By reviewing our proofs of (Hyp 0)...(Hyp 9), the reader will see that the constants called ε, K, N, c, C, c_1, c_2, C_α in (Hyp 0)...(Hyp 9) depend only on ε, K, c, C, c_1, c_2, C_α in (Y0)...(Y11). Hence, (Hyp 10) follows from the WKB hypothesis (Y11).

The verification of (Hyp 0)...(Hyp 10) is complete. We already checked that the quantity called Λ in (Hyp 0)...(Hyp 10) is greater than or equal to the number Λ in (Y0)...(Y11). Thus, we have verified all the statements in Lemma 1. ■

Proof of Lemma 2. We may assume $g_{\nu k}(x,E) \neq 0$ for some $(x,E) \in I_{\mathrm{BVP}} \times (-\infty, 0]$. Our first step is to pick $\tilde{x}_0$, $E_{\ell o}$, E_{hi}. We distinguish two cases:

(i) $V(x_\nu) \leq V(x_0) + \frac{1}{2}c_2 S(x_0)$.

(ii) $V(x_\nu) > V(x_0) + \frac{1}{2}c_2 S(x_0)$.

In case (i), we take $\tilde{x}_0 = x_\nu$, $E_{\ell o} = V(x_0) + \frac{3}{4}c_2 S(x_0)$, $E_{hi} = 0$, and note that $E_{\ell o} < 0$. In case (ii), we first note that $|x_\nu - x_0| > c_\# B(x_0)$. Hence, with (sgn) $= \pm 1$, we have

$$(20.1) \quad c_\# S(x_\nu)B^{-1}(x_\nu) < (\mathrm{sgn}) \cdot V'(x) < C_\# S(x_\nu)B^{-1}(x_\nu) \qquad \text{for} \quad |x - x_\nu| < c'_\# B(x_\nu) \ .$$

(This follows from (Y3), (Y4) and the defining properties of $\tilde{I}$. Recall that $x_\nu \in \tilde{I}$.)

For a small, positive constant b to be picked later, we define

$$\tilde{x}_0 = x_\nu - b(\text{sgn})B(x_\nu) \ . \tag{20.2}$$

If b satisfies

$$b < c'_\# \quad \text{with} \quad c'_\# \quad \text{as in (20.1), and} \tag{20.3}$$

$$\hat{c} < \frac{1}{2}b \ , \tag{20.4}$$

then from (20.1) we get

$$c''_\# bS(x_\nu) < V(x) - V(\tilde{x}_0) < C''_\# bS(x_\nu) \qquad \text{for} \quad |x - x_\nu| < \hat{c}B(x_\nu) \ . \tag{20.5}$$

We set $E_{hi} = 0$ and $E_{\ell o} = V(\tilde{x}_0) + \frac{1}{2}c''_\# bS(x_\nu)$, with $c''_\#$ as in (20.5). Note that $E_{\ell o} < 0$. In fact, if $E_{\ell o} \geq 0$, then (20.5) would show that $V(x) > 0$ for $|x - x_\nu| < \hat{c}B(x_\nu)$, which implies easily that $g_{\nu k}(x, E) = 0$ for all $(x, E) \in I_{\text{BVP}} \times (-\infty, 0]$, in contradiction to our initial assumption. Hence $[E_{\ell o}, E_{hi}]$ is a non-empty subinterval of $(-\infty, 0]$ in case (ii). This completes our selection of $\tilde{x}_0$, $E_{\ell o}$, E_{hi}.

The next step is to show that

$$E_{\ell o} \geq V(x_0) + \frac{1}{10}c_2 S(x_0) \ . \tag{20.6}$$

This is obvious in case (i). In case (ii) we use (20.5), (ii) and (18) to write

$$\begin{aligned} E_{\ell o} = V(\tilde{x}_0) + \frac{1}{2}c''_\# bS(x_\nu) > V(\tilde{x}_0) > V(x_\nu) - C''_\# bS(x_\nu) \\ > [V(x_0) + \frac{1}{2}c_2 S(x_0)] - C''_\# bS(x_\nu) \\ > V(x_0) + \frac{1}{2}c_2 S(x_0) - C'''_\# bS(x_0) \ , \end{aligned} \tag{20.7}$$

which yields (20.6) provided we take

$$b < \frac{1}{10}c_2 (C'''_\#)^{-1} \ , \quad \text{with} \quad C'''_\# \quad \text{as in (20.7)} \ . \tag{20.8}$$

The proof of (20.6) is complete.

Applying (20.6) and Lemma 1, we draw the following conclusions. The hypotheses (Hyp 0)...(Hyp 10) hold for any E_0 in $[E_{\ell o}, E_{hi}]$, with $E_\infty = 0$ and with $300K$ in place of K. The constants called c, C, c_1, c_2, C_α in (Hyp 0)...(Hyp 10) may all be taken to have the form $c_\#$. In particular, let $c_\#^{\mathrm{HYP}}$ denote the constant called c in (Hyp 0)...(Hyp 10). Finally, $\Lambda(E_0) \geq \Lambda$ for $E_0 \in [E_{\ell o}, E_{hi}]$. These conclusions imply hypothesis (A1) of the oscillatory density lemma.

Next we check (A2), with x_ν in place of x_0, and with $c_\#^{\mathrm{HYP}}$ in place of c. In case (i) we have $|x_\nu - \tilde{x}_0| < c_\#^{\mathrm{HYP}} B(x_\nu)$ since $\tilde{x}_0 = x_\nu$; and

$$E_{\ell o} = V(x_0) + \frac{3}{4} c_2 S(x_0) \geq V(x_\nu) + \frac{1}{4} c_2 S(x_0)$$

$$\text{(since we are in case (i))} \ \geq V(x_\nu) + c_\# S(x_\nu)$$

(by (18)) $= V(\tilde{x}_0) + c_\# S(\tilde{x}_0)$. This proves (A2) in case (i), with c_3 of the form $c_\#$. In case (ii), the definition (20.2) yields $|x_\nu - \tilde{x}_0| < c_\#^{\mathrm{HYP}} B(x_\nu)$, provided we take

(20.9) $$b < c_\#^{\mathrm{HYP}} \ .$$

Also from (20.2) we get

(20.10) $$S(x_\nu) \geq c_\# S(\tilde{x}_0) \ ,$$

provided we take

(20.11) $$b < c_\# \quad \text{for a suitable small constant } c_\# \ .$$

Hence, $E_{\ell o} = V(\tilde{x}_0) + \frac{1}{2} c''_\# b S(x_\nu) \geq V(\tilde{x}_0) + [\frac{1}{2} c''_\# b c_\#] S(\tilde{x}_0)$. Thus, (A2) holds in both cases (i) and (ii), with x_ν in place of x_0, and with $c_\#^{\mathrm{HYP}}$ in place of c. The quantity called c_3 in (A2) either has the form $c_\#$ or $c_\# b$.

Next we check (A3), with x_ν in place of x_0, and with $\tau = 2^{-k}$. Thus, suppose we are given $(x, E) \in I_{\mathrm{BVP}} \times \mathbb{R}^1$ with $g_{\nu k}(x, E) \neq 0$. We must have $\theta_\nu(x) \neq 0$ and $\chi_k\left(\frac{E - V(x)}{S(x_\nu)}\right) \neq 0$. Therefore,

(20.12) $$|x - x_\nu| < \hat{c} B(x_\nu) \quad \text{and} \quad 2^{-k} S(x_\nu) < E - V(x) < C_* 2^{-k} S(x_\nu) \ .$$

To complete the proof of (A3), it remains only to check that $E_{\ell o} + \hat{c}_1 S_{\min}(E_{\ell o}) \leq E \leq E_{hi}$ or else $E > 0$. That is, we must show that

$$E > E_{\ell o} + \hat{c}_1 S_{\min}(E_{\ell o}) \ . \tag{20.13}$$

We check (20.13) in case (i). Since $E_{\ell o} < 0$ and $V(x_0) < E_{\ell o}$, we have $S_{\min}(E_{\ell o}) = \inf_{V(x)<E_{\ell o}} S(x) \leq S(x_0)$. Since $g_{\nu k}(x, E) \neq 0$, we have $\varphi(E) \neq 0$ and hence $E \geq V(x_0) + c_2 S(x_0) = E_{\ell o} + \frac{1}{4} c_2 S(x_0) \geq E_{\ell o} + \frac{1}{4} c_2 S_{\min}(E_{\ell o})$, proving (20.13) with $\hat{c}_1 = \frac{1}{8} c_2$.

To check (20.13) in case (ii), we use (20.5), (20.10), (20.12) to write

$$\begin{aligned} E > V(x) > V(\tilde{x}_0) + c''_{\#} b S(x_\nu) &= E_{\ell o} + \frac{1}{2} c''_{\#} b S(x_\nu) \\ &\geq E_{\ell o} + \frac{1}{2} c''_{\#} c_{\#} b S(\tilde{x}_0) \ . \end{aligned} \tag{20.14}$$

Also $V(\tilde{x}_0) < E_{\ell o} < 0$, so $S_{\min}(E_{\ell o}) = \inf_{V(x)<E_{\ell o}} S(x) \leq S(\tilde{x}_0)$. Hence (20.14) implies $E \geq E_{\ell o} + (\frac{1}{2} c''_{\#} c_{\#} b) S_{\min}(E_{\ell o})$, which proves (20.13) with $\hat{c}_1 = \frac{1}{4} c''_{\#} c_{\#} b$. The proof of (A3) is complete.

To prove (A4), we recall that $\Lambda_{\min} = \inf_{E \in [E_{\ell o}, E_{hi}]} \Lambda(E) \geq \Lambda$ and that $\tau = 2^{-k}$ with $\frac{1}{2} \Lambda^{-2/7} < \tau \leq 1$. Hence $\Lambda_{\min}^{\varepsilon - 2/3} \leq \Lambda^{\varepsilon - 2/3} \leq \frac{1}{2} \Lambda^{-2/7} < \tau \leq 1$, which yields (A4).

Since $\tau = 2^{-k}$, hypothesis (A5) is merely our estimate (19).

Now we pick the constant b. We merely take $b = c'''_{\#}$ small enough to satisfy (20.3), (20.8), (20.9), and (20.11). From (Y10) we see that (20.4) holds also. Thus, we have proven (A1)...(A5) with constants c, C, c_1, c_2, c_3, $\hat{c}_1$, C_α of the form $C_{\#}$, and with constants $\hat{c}$, $\hat{C}$, $\hat{C}_{\alpha\beta}$ of the form C_*. Since $\Lambda_{\min} \geq \Lambda$, hypothesis (A6) follows from (Y11). We have proven everything asserted in Lemma 2.

Proof of Lemma 3. We have to establish one of the alternatives (A), (B), (C) for a given ν. We distinguish several cases.

First of all, if $V(x_\nu) \le V(x_0) + \frac{1}{2}c_2 S(x_0)$, then alternative (A) holds. To see this, note that $\theta_\nu(x)\chi_{k_{\max}}\big(\frac{E-V(x)}{S(x_\nu)}\big)$ is supported in $\{|E - V(x)| \le C_* \Lambda^{-2/7} S(x_\nu)$, $|x - x_\nu| < \hat{c}B(x_\nu)\}$, which is contained in $\{|E - V(x)| \le C_* \Lambda^{-2/7} S(x_\nu), |V(x) - V(x_\nu)| \le C_\# \hat{c} S(x_\nu)\}$. This in turn is contained in $\{|E - V(x_\nu)| \le \frac{1}{10} c_2 S(x_0)\}$ by (18) and hypotheses (Y10), (Y11). Consequently, if $V(x_\nu) \le V(x_0) + \frac{1}{2}c_2 S(x_0)$, then $h_{\nu k}(x, E)$ is supported in $\{E \le V(x_0) + \frac{2}{3} c_2 S(x_0)\}$. Since $\varphi(E) = 0$ for $E \le V(x_0) + c_2 S(x_0)$, it follows that $g_{\nu k_{\max}}(x, E) \equiv 0$. Hence we are in alternative (A).

For the rest of the proof of Lemma 3, we may therefore assume that

$$V(x_\nu) > V(x_0) + \frac{1}{2} c_2 S(x_0) \ , \tag{21}$$

which implies $|x_\nu - x_0| > c_\# B(x_0)$. Hence either $x_\nu < x_0 - c_\# B(x_0)$ or $x_\nu > x_0 + c_\# B(x_0)$. We assume we are in the first case, and show we are in alternative (A) or (B). An analogous argument (which we omit) applies to the second case, to derive alternative (A) or (C). So for the rest of the proof of Lemma 3, we may assume

$$x_\nu \in \tilde{I} \cap (-\infty, x_0 - c_\# B(x_0)] \ . \tag{22}$$

(We know that $x_\nu \in \tilde{I}$ by (2).)

The defining properties of $\tilde{I}$, together with hypotheses (Y3), (Y4), show that (22) implies

$$-V'(x_\nu) \ge c_\# S(x_\nu) B^{-1}(x_\nu) \ . \tag{23}$$

Assume for a moment that $V(x) \ge \Lambda^{\varepsilon - 2/7} S(x_\nu)$ for all $x \in$ supp θ_ν. Then the function $\theta_\nu(x)\chi_{k_{\max}}\big(\frac{E-V(x)}{S(x_\nu)}\big)$ would be supported in the set $\{V(x) \ge \Lambda^{\varepsilon-2/7} S(x_\nu)$, $|E - V(x)| \le C_* \Lambda^{-2/7} S(x_\nu)\} \subset \{E > 0\}$. Therefore, $g_{\nu k_{\max}}(x, E) = 0$ whenever $E \le 0$, and we are in alternative (A).

So for the rest of the proof of Lemma 3, we may assume

(24) $$V(x'_\nu) \le \Lambda^{\varepsilon-2/7} S(x_\nu) \text{ for some } x'_\nu \in \text{ supp } \theta_\nu \subset \{|x - x_\nu| < \hat{c}B(x_\nu)\} \ .$$

By (23), (24), and the estimates (Y1) for $|V''(x)|$, we conclude that

(25) $$V(x''_\nu) < 0 \text{ for some } x''_\nu \text{ with } |x''_\nu - x_\nu| < 2\hat{c}B(x_\nu) \ .$$

(To deduce (25) we used also the WKB hypothesis (Y11).)

Now recall $E_0 = \min(V(x_\nu), 0)$. If $V(x_\nu) > 0$, then by (25) there is a point $\overline{x}_\nu$ satisfying

(26) $$V(\overline{x}_\nu) = E_0 \quad \text{and} \quad |\overline{x}_\nu - x_\nu| < 2\hat{c}B(x_\nu) \ .$$

If instead $V(x_\nu) \le 0$, then we take $\overline{x}_\nu = x_\nu$, and (26) still holds. Hence, we can always take $\overline{x}_\nu$ to satisfy (26). Note that $V'(\overline{x}_\nu) < 0$ by (23), (26) and (Y10), so (26) implies

(27) $$\overline{x}_\nu = x_{\text{left}}(E_0) \ .$$

Also, from (21) and the inequality $V(x_0) + c_2 S(x_0) < 0$ (contained in (Y3)), we get

(28) $$V(x_0) + \frac{1}{2} c_2 S(x_0) \le \min(V(x_\nu), 0) = E_0 \le 0 \ .$$

We prepare to pick δE for use in hypotheses (X0)...(X12) of the Airey density lemma.

By (28) and (23), we have $|x_{\text{left}}(E) - x_{\text{left}}(E_0)| \le C_\# B(x_\nu) \frac{|E-E_0|}{S(x_\nu)}$ for $|E - E_0| < c_\# S(x_\nu)$. Hence

(29)
$$|E - E_0| < \overline{c}_\# S(x_\nu) \quad \text{implies} \quad c_\# S(x_{\text{left}}(E_0)) < S(x_{\text{left}}(E)) < C_\# S(x_{\text{left}}(E_0))$$
$$\text{for a small constant } \overline{c}_\# \ .$$

We define $(\delta E) = \min\{\overline{c}_\# S(x_\nu), 10^{-9} c_2 S(x_0)\}$. Thus, (28) gives

$$\mathcal{J} = [E_0 - \delta E, E_0 + \delta E] \cap (-\infty, 0] \subset [V(x_0) + \frac{1}{4} c_2 S(x_0), 0] \ , \tag{30}$$

and (29) gives

$$c_\# S(x_{\text{left}}(E_0)) < S(x_{\text{left}}(E)) < C_\# S(x_{\text{left}}(E_0)) \quad \text{for all} \quad E \in \mathcal{J} \ . \tag{31}$$

Equation (30) and Lemma 1 show that for any $E \in \mathcal{J}$, the hypotheses of the WKB Theorems hold, with $E_\infty = 0$ and with E in place of E_0. This immediately implies hypotheses (X0)...(X4) and (X6), (X7) of the Airey density lemma. The constants called ε, K, N, c, C, c_1, C_α in (X0)...(X4), (X6), (X7) are determined by ε, K, N, c, C, c_1, c_2, C_α in (Y0)...(Y11), and we have $\Lambda_{\min} = \inf_{E \in \mathcal{J}} \Lambda(E) \geq \Lambda$. These conclusions are all contained in those of Lemma 1.

To check (X5), notice that

$$c_\# S(x_\nu) < \delta E < C_\# S(x_\nu) \ , \tag{32}$$

by (18) and the definition of δE. Hence (X5) follows at once from (31), and the inequality $|x_{\text{left}}(E_0) - x_\nu| = |\overline{x}_\nu - x_\nu| < 2\hat{c}B(x_\nu)$, which is contained in (26), (27).

Hypothesis (X8) follows trivially from (Y7), since $\Lambda_{\min} \geq \Lambda$.

Next we check (X9) with suitable quantities in place of $\hat{c}$, δx. Recall that $g_{\nu k_{\max}}(x, E)$ is supported in $\{|x - x_\nu| < \hat{c}B(x_\nu), \frac{|E - V(x)|}{S(x_\nu)} < C_* \Lambda^{-2/7}\}$, which is contained in $\{|E - V(x_\nu)| \leq C_\# \hat{c} S(x_\nu) + C_* \Lambda^{-2/7} S(x_\nu)\}$, which in turn is contained in $\{|E - V(x_\nu)| \leq C'_\# \hat{c}(\delta E)\}$ by (32) and (Y11).

¿From (26), (32) we get also $|V(x_\nu) - E_0| < C''_\# \hat{c} S(x_\nu) < C'''_\# \hat{c}(\delta E)$. Hence the support of $g_{\nu k_{\max}}(x, E)$ is contained in the set $\{|E - E_0| < \tilde{C}_\# \hat{c}(\delta E)\}$. So we take $\hat{c}_+ = \tilde{C}_\# \hat{c}$ in place of $\hat{c}$, and we have

$$\text{supp } g_{\nu k_{\max}}(x, E) \subset \{|E - E_0| < \hat{c}_+(\delta E)\} \ . \tag{33}$$

Again, supp $g_{\nu k_{\max}}(x,E)$ is contained in $\{|x-x_\nu| < \hat{c}B(x_\nu),\ |E-V(x)| < C_*\Lambda^{-2/7}S(x_\nu)\}$, and for $|x-x_\nu| < \hat{c}B(x_\nu)$ and $|E-E_0| < \hat{c}_+(\delta E)$ we have $c_\# \frac{|E-V(x)|}{S(x_\nu)} < \frac{|x-x_{\text{left}}(E)|}{B(x_\nu)} < C_\# \frac{|E-V(x)|}{S(x_\nu)}$, by (23). Hence supp $g_{\nu k_{\max}}(x,E)$ is contained in $\{|x-x_{\text{left}}(E)| < C_*\Lambda^{-2/7}B(x_\nu)\}$. So we take $(\delta x) = C_*\Lambda^{-2/7}B(x_\nu)$, and we have

$$\text{supp } g_{\nu k_{\max}}(x,E) \subset \{|x-x_{\text{left}}(E)| < (\delta x)\}\ . \tag{34}$$

In view of (33), (34), the hypothesis (X9) is reduced to checking that

$$\lambda^{2\varepsilon-2/3}(x_{\text{left}}(E_0))\cdot B(x_{\text{left}}(E_0)) < \delta x < \frac{1}{20}\lambda^{-2\varepsilon}(x_{\text{left}}(E_0))\cdot B(x_{\text{left}}(E_0))\ ,$$

i.e.

$$\lambda^{2\varepsilon-2/3}(\overline{x}_\nu) < \frac{\delta x}{B(\overline{x}_\nu)} < \frac{1}{20}\lambda^{-2\varepsilon}(\overline{x}_\nu) \qquad \text{(see (27))}\ .$$

By (26) we see that $c_\# B(x_\nu) < B(\overline{x}_\nu) < C_\# B(x_\nu)$ and $c_\#\lambda(\overline{x}_\nu) < \lambda(x_\nu) < C_\#\lambda(\overline{x}_\nu)$. Hence the hypothesis (X9) will follow from the definition of δx, if we can prove

$$\lambda^{3\varepsilon-2/3}(x_\nu) < \Lambda^{-2/7} < \lambda^{-3\varepsilon}(x_\nu)\ . \tag{35}$$

The first inequality in (35) is obvious, since $\lambda(x_\nu) \geq c_\#\Lambda$. To verify the second inequality in (35), we argue as follows. Hypothesis (Y7) shows that

$$\Lambda^{-1} = \int_{x_{\text{left}}(0)}^{x_{\text{rt}}(0)} \frac{dx}{\lambda(x)B(x)} \leq \Lambda^{10K}\lambda^{-1}(x_\nu)\cdot\int_I \frac{dx}{B(x)}$$
$$\leq \Lambda^{10K}\lambda^{-1}(x_\nu)\cdot\int_I \frac{dx}{\Lambda^{-K}|I|} = \Lambda^{11K}\lambda^{-1}(x_\nu)\ .$$

Hence $\Lambda^{-(11K+1)} \leq \lambda^{-1}(x_\nu)$, so $\Lambda^{-\frac{2}{7}} \leq \lambda^{-\frac{2}{7}\cdot\frac{1}{11K+1}}(x_\nu) \leq \lambda^{-\frac{1}{50K}}(x_\nu) \leq \lambda^{-3\varepsilon}(x_\nu)$ since $\varepsilon < \frac{1}{1000K}$. The second inequality in (35) is verified and the proof of (X9) is complete.

In view of our definition of δx and the fact that $B(x_{\text{left}}(E_0)) = B(\overline{x}_\nu) \sim B(x_\nu)$ and $S(x_{\text{left}}(E_0)) = S(\overline{x}_\nu) \sim S(x_\nu)$ as in the proof of (X9), the hypothesis (X10) amounts to the estimates $|\partial_x^\alpha \partial_E^\beta g_{\nu k_{\max}}(x,E)| \le$ $C_*^{\alpha\beta}[\Lambda^{-2/7}B(x_\nu)]^{-\alpha}[\Lambda^{-2/7}S(x_\nu)]^{-\beta}$. These estimates are equivalent to (19), since $2^{-k_{\max}} \sim \Lambda^{-2/7}$. Thus (X10) holds.

To verify (X11), we recall that in place of $\hat{c}$ we are using here $\hat{c}_+ = \tilde{C}_\# \hat{c}$. Since $\tilde{C}_\#$ and the quantities called ε, K, N, c, C, c_1, C_α in (X0)...(X10) are determined by ε, K, N, c, C, c_1, c_2, C_α in our present hypotheses (Y0)...(Y11), we see that (X11) follows at once from (Y10).

The last hypothesis (X12) follows from the WKB hypothesis (Y11), since $\Lambda_{\min} \ge \Lambda$ and the quantities called ε, K, N, c, c_1, C, C_α, $\hat{c}$, $\hat{C}_{\alpha\beta}$ in (X0)...(X12) are determined by ε, K, N, c, C, c_1, C_α, c_2, $\hat{c}$, $\hat{C}_\beta$ in (Y0)...(Y11).

Thus, we have verified all the hypotheses (X0)...(X12) of the Airey density lemma. We have already checked that $\Lambda_{\min} \ge \Lambda$, and equations (26), (27) yield $|x_\nu - x_{\text{left}}(E_0)| < 2\hat{c}B(x_\nu)$. Therefore, alternative (B) holds here. The proof of Lemma 3 is complete. ■

Now we can apply Lemmas 2 and 3, the Airey density lemma, and the Oscillatory density lemma to control $\rho(x, g_{\nu k})$. Let $I_\nu = \{x \mid |x - x_\nu| < \tilde{c}_\# B(x_\nu)\}$, with $\tilde{c}_\#$ small. If $0 \le k < k_{\max}$, then Lemma 2 and the Oscillatory density lemma allow us to compare $\rho(x, g_{\nu k})$ with $\rho_{sc}(x, g_{\nu k})$ on I_ν. If $k = k_{\max}$, then according to Lemma 3, we are in one of the cases (A), (B), (C). In case (A), both $\rho(x, g_{\nu k})$ and $\rho_{sc}(x, g_{\nu k})$ are identically zero. In case (B), we can apply the Airey density lemma to compare $\rho(x, g_{\nu k})$ with $\rho_{sc}(x, g_{\nu k})$ on I_ν. (Note that I_ν is contained in the interval called I_{left} in the Airey density lemma, since $|x_\nu - x_{\text{left}}(E_0)| \le 2\hat{c}B(x_\nu)$ in case (B).) Similarly, in case (C) we can apply the analogue of the Airey density lemma, with "left" and "rt" reversed, to compare $\rho(x, g_{\nu k})$ with $\rho_{sc}(x, g_{\nu k})$ on I_ν. Thus, in all cases we obtain a comparison of $\rho(x, g_{\nu k})$ with $\rho_{sc}(x, g_{\nu k})$. From this process, we obtain the following results.

$$\rho(x, g_{\nu k}) - \rho_{sc}(x, g_{\nu k}) = \frac{d}{dx} H_{\nu k}(x) \quad \text{on} \quad I_\nu \,. \tag{36}$$

CASE I: If $\min_{k\in\mathbb{Z}} |\phi(0) - \pi(k+1/2)| \geq \overline{C}_{\#}\Lambda^{-1}$, then:

$$(37)\quad (Av_{I_\nu}|H_{\nu k}|^2)^{1/2} \leq C_*(2^{-k})^{-3/4}\lambda^{-1/2}(x_\nu) + C_*\Lambda^{4\varepsilon-2}(2^{-k})^{-5/2}\lambda(x_\nu) + C_*\Lambda^{-\frac{N''}{2}}(\phi(0)+1) \quad \text{for} \quad 0 \leq k < k_{\max}$$

$$(38)\quad (Av_{I_\nu}|H_{\nu k}|^2)^{1/2} \leq C_*\lambda^{-1/3}(x_\nu) + C_*\lambda^{-1}(x_\nu)(\Lambda^{-2/7})^{-5/2} + C_*\Lambda^{4\varepsilon-2}\lambda(x_\nu)(\Lambda^{-2/7})^{1/2} + C_*\Lambda^{-1}\lambda(x_\nu)\Lambda^{-2/7} + C_*\Lambda^{-\frac{N''}{2}}(\phi(0)+1) \quad \text{for} \quad k = k_{\max}\ .$$

CASE II: If $\min_{k\in\mathbb{Z}} |\phi(0) - \pi(k+1/2)| \leq \overline{C}_{\#}\Lambda^{-1}$, then:

$$(39)\quad (Av_{I_\nu}|H_{\nu k}|^2)^{1/2} \leq C_*(2^{-k})^{1/2} + C_*(2^{-k})^{-3/4}\lambda^{-1/2}(x_\nu) + C_*\Lambda^{4\varepsilon-2}(2^{-k})^{-5/2}\lambda(x_\nu) + C_*\Lambda^{-\frac{N''}{2}}(\phi(0)+1) \quad \text{for} \quad 0 \leq k < k_{\max};$$

$$(40)\quad (Av_{I_\nu}|H_{\nu k}|^2)^{1/2} \leq C_*\lambda^{-1/3}(x_\nu) + C_*\lambda^{-1}(x_\nu)\cdot(\Lambda^{-2/7})^{-5/2} + C_*\Lambda^{-1}\lambda(x_\nu)(\Lambda^{-2/7})^{1/2} + C_*\Lambda^{-\frac{N''}{2}}(\phi(0)+1) \quad \text{for} \quad k = k_{\max}\ .$$

We apply (37)...(40) to bound $\sum\limits_{0\leq k\leq k_{\max}} (Av_{I_\nu}|H_{\nu k}|^2)^{1/2}$ for fixed ν. In *Case I*, we have to sum the right-hand side of (37) over $k = 0, 1, \ldots, k_{\max}-1$, and then add the result to the right-hand side of (38). Recalling that $2^{-k_{\max}} \sim \Lambda^{-2/7}$ and $\lambda(x_\nu) \geq c_{\#}\Lambda$, we obtain the following:

$$\sum_{0\leq k<k_{\max}} [1^{\text{rst}}\text{term on RHS of (37)}] \leq C_*(\Lambda^{-2/7})^{-3/4}(\Lambda^{-3/2}\lambda(x_\nu)) = C_*\Lambda^{-9/7}\lambda(x_\nu)\ .$$

$$\sum_{0\le k<k_{\max}} [2^{\text{nd}}\text{term on RHS of (37)}]$$
$$\le C_*\Lambda^{4\varepsilon-2}(\Lambda^{-2/7})^{-5/2}\lambda(x_\nu) = C_*\Lambda^{4\varepsilon-9/7}\lambda(x_\nu)\ .$$

$$\sum_{0\le k<k_{\max}} [3^{\text{rd}}\text{term on RHS of (37)}]$$
$$\le C_*\Lambda^{-\frac{N''}{2}}(\phi(0)+1)\log_2\Lambda \le C_*\Lambda^{-\frac{N''}{3}}(\phi(0)+1)\ .$$

$$[1^{\text{rst}}\text{ term on RHS of (38)}] \le C_*\Lambda^{-4/3}\lambda(x_\nu) < C_*\Lambda^{-9/7}\lambda(x_\nu)\ .$$

$$[2^{\text{nd}}\text{ term on RHS of (38)}] \le C_*(\Lambda^{-2}\lambda(x_\nu))(\Lambda^{-2/7})^{-5/2} = C_*\Lambda^{-9/7}\lambda(x_\nu)\ .$$

$$[3^{\text{rd}}\text{ term on RHS of (38)}] \le C_*\Lambda^{-9/7}\lambda(x_\nu)$$

$$[4^{\text{th}}\text{ term on RHS of (38)}] \le C_*\Lambda^{-9/7}\lambda(x_\nu)$$

$$[5^{\text{th}}\text{ term on RHS of (38)}] = C_*\Lambda^{-\frac{N''}{2}}(\phi(0)+1) \le C_*\Lambda^{-\frac{N''}{3}}(\phi(0)+1)\ .$$

In view of these observations, (37) and (38) imply

$$(41)\qquad \sum_{0\le k\le k_{\max}} (Av_{I_\nu}|H_{\nu k}|^2)^{1/2} \le C_*\Lambda^{4\varepsilon-9/7}\lambda(x_\nu) + C_*\Lambda^{-\frac{N''}{3}}(\phi(0)+1)$$
$$\text{in case}\quad \min_{k\in\mathbb{Z}}|\phi(0)-\pi(k+1/2)| \ge \overline{C}_\#\Lambda^{-1}\ .$$

On the right in (41), the exponent of $4\varepsilon - 9/7$ is not important to us. Any exponent strictly less than -1 would be enough for our purposes.

Similarly, in *CASE II* we have to sum the right-hand side of (39) over $k = 0, 1, \dots, k_{\max}-1$, and then add the result to the right-hand side of (40).
Again recalling that $2^{-k_{\max}} \sim \Lambda^{-2/7}$ and $\lambda(x_\nu) \ge c_\#\Lambda$, we obtain:

$$\sum_{0\le k<k_{\max}} [1^{\text{rst}}\text{ term on RHS in (39)}] \le C_* \le C_*\Lambda^{-1}\lambda(x_\nu)$$

$$\sum_{0\le k<k_{\max}} [2^{\text{nd}}\text{ term on RHS in (39)}] \le C_*(\Lambda^{-2/7})^{-3/4}(\Lambda^{-\frac{3}{2}}\lambda(x_\nu)) = C_*\Lambda^{-9/7}\lambda(x_\nu)$$

$$\sum_{0\le k<k_{\max}} [3^{\rm rd} \text{ term on RHS in (39)}]$$

$$\le C_*\Lambda^{4\varepsilon-2}(\Lambda^{-2/7})^{-5/2}\lambda(x_\nu) = C_*\Lambda^{-9/7+4\varepsilon}\lambda(x_\nu)$$

$$\sum_{0\le k<k_{\max}} [4^{\rm th} \text{ term on RHS in (39)}]$$

$$\le C_*\Lambda^{-\frac{N''}{2}}(\phi(0)+1)\log_2\Lambda < C_*\Lambda^{-\frac{N''}{3}}(\phi(0)+1)$$

$$[1^{\rm st} \text{ term on RHS in (40)}] \le C_*\Lambda^{-4/3}\lambda(x_\nu) < C_*\Lambda^{-9/7}\lambda(x_\nu)\ .$$

$$[2^{\rm nd} \text{ term on RHS in (40)}] \le C_*(\Lambda^{-2}\lambda(x_\nu))\cdot(\Lambda^{-2/7})^{-5/2} = C_*\Lambda^{-9/7}\lambda(x_\nu)\ .$$

$$[3^{\rm rd} \text{ term on RHS in 40)}] = C_*\Lambda^{-8/7}\lambda(x_\nu) < C_*\Lambda^{-1}\lambda(x_\nu)$$

$$[4^{\rm th} \text{ term on RHS in (40)}] = C_*\Lambda^{-\frac{N''}{2}}(\phi(0)+1) < C_*\Lambda^{-\frac{N''}{3}}(\phi(0)+1)\ .$$

In view of these observations, (39) and (40) imply

$$\sum_{0\le k\le k_{\max}} (Av_{I_\nu}|H_{\nu k}|^2)^{1/2} \le C_*\Lambda^{-1}\lambda(x_\nu) + C_*\Lambda^{-\frac{N''}{3}}(\phi(0)+1) \tag{42}$$

$$\text{in case} \quad \min_{k\in\mathbb{Z}} |\phi(0)-\pi(k+1/2)| \le \overline{C}_\#\Lambda^{-1}\ .$$

Let us examine how $H_{\nu k}(x)$ behaves on $I_\nu = \{|x-x_\nu| < \tilde{c}_\# B(x_\nu)\}$. We divide I_ν into three subintervals I_ν^-, I_ν^0, I_ν^+ by setting

$$I_\nu^0 = \{|x-x_\nu| \le \hat{c}B(x_\nu)\}$$

$$I_\nu^- = \text{ part of } I_\nu \text{ to the left of } I_\nu^0$$

$$I_\nu^+ = \text{ part of } I_\nu \text{ to the rt. of } I_\nu^0\ .$$

Recall that $g_{\nu k}(x,E)$ is supported in $\{(x,E)\mid x\in I_\nu^0\}$. Hence, both $\rho(x,g_{\nu k})$ and $\rho_{sc}(x,g_{\nu k})$ are supported in I_ν^0. Equation (36) therefore shows that $H_{\nu k}(x)$ is constant on I_ν^- and on I_ν^+. Say

$$H_{\nu k}(x) = c_{\nu k}^- \text{ for } x\in I_\nu^- \quad \text{and} \quad H_{\nu k}(x) = c_{\nu k}^+ \text{ for } x\in I_\nu^+\ . \tag{43}$$

Since $I_\nu^-, I_\nu^+ \subset I_\nu$ with $|I_\nu^-|, |I_\nu^+| > c_* |I_\nu|$, we have

$$|c_{\nu k}^-| = (Av_{I_\nu^-} |H_{\nu k}|^2)^{1/2} \le C_* (Av_{I_\nu} |H_{\nu k}|^2)^{1/2} \ , \tag{44}$$

and similarly

$$|c_{\nu k}^+| \le C_* (Av_{I_\nu} |H_{\nu k}|^2)^{1/2} \ . \tag{45}$$

Instead of $H_{\nu k}(x)$, we are really interested in

$$\tilde{H}_{\nu k}(x) = \int_{I_{\mathrm{BVP}} \cap (-\infty, x]} (\rho(x', g_{\nu k}) - \rho_{sc}(x', g_{\nu k}))\, dx' \ .$$

From (36) and (43), together with the fundamental theorem of calculus and the fact that $\rho(x, g_{\nu k})$, $\rho_{sc}(x, g_{\nu k})$ are supported in I_ν^0, we see that

$$\tilde{H}_{\nu k}(x) = 0 \quad \text{to the left of } I_\nu^0 \tag{46}$$

$$\tilde{H}_{\nu k}(x) = H_{\nu k}(x) - c_{\nu k}^- \quad \text{for } x \in I_\nu \text{ (and in particular for } x \in I_\nu^0) \tag{47}$$

$$\tilde{H}_{\nu k}(x) = c_{\nu k}^+ - c_{\nu k}^- \quad \text{to the right of } I_\nu^0 \ . \tag{48}$$

We use our estimates to bound the average size of

$$\tilde{H}(x) = \sum_{\nu k} \tilde{H}_{\nu k}(x) = \int_{I_{\mathrm{BVP}} \cap (-\infty, x]} (\rho(x', \sum_{\nu k} g_{\nu k}) - \rho_{sc}(x', \sum_{\nu k} g_{\nu k}))\, dx'$$

on an interval $J(\tilde{x}) = \{|x - \tilde{x}| \le \hat{c} B(\tilde{x})\} \subset I$. Note that $I_\nu^0 = J(x_\nu)$. For fixed $\tilde{x}$, we divide the set of all ν into three classes:

$$\begin{aligned} \nu \in \mathbb{N}_{\mathrm{nil}}(\tilde{x}) &\quad \text{if } I_\nu^0 \text{ lies entirely to the right of } J(\tilde{x}) \\ \nu \in \mathbb{N}_{\mathrm{const}}(\tilde{x}) &\quad \text{if } I_\nu^0 \text{ lies entirely to the left of } J(\tilde{x}) \\ \nu \in \mathbb{N}_{\mathrm{var}}(\tilde{x}) &\quad \text{if } I_\nu^0 \text{ intersects } J(\tilde{x}) \ . \end{aligned}$$

If $\nu \in \mathbb{N}_{\text{nil}}(\tilde{x})$,then $\tilde{H}_{\nu k} = 0$ throughout $J(\tilde{x})$, by (46) .

If $\nu \in \mathbb{N}_{\text{const}}(\tilde{x})$,then $\tilde{H}_{\nu k} = c^+_{\nu k} - c^-_{\nu k}$ throughout $J(\tilde{x})$, by (48) .

(49) If $\nu \in \mathbb{N}_{\text{var}}(\tilde{x})$,then $J(\tilde{x}) \subset I_\nu$ (by (Y10)), and $|J(\tilde{x})| > c_*|I_\nu|$.

Hence for $\nu \in \mathbb{N}_{\text{var}}(\tilde{x})$ we have $\tilde{H}_{\nu k} = H_{\nu k} - c^-_{\nu k}$ on $J(\tilde{x})$, and $Av_{J(\tilde{x})}|\tilde{H}_{\nu k}|^2 \leq C_* Av_{I_\nu}|H_{\nu k} - c^-_{\nu k}|^2 \leq C'_* Av_{I_\nu}|H_{\nu k}|^2$ by (44). Since also $|c^+_{\nu k} - c^-_{\nu k}|^2 \leq C_* Av_{I_\nu}|H_{\nu k}|^2$ by (44), (45), we have shown that

(50) $$\left(Av_{J(\tilde{x})}|\tilde{H}_{\nu k}|^2\right)^{1/2} \leq C_*\left(Av_{I_\nu}|H_{\nu k}|^2\right)^{1/2} \quad \text{if } \nu \in \mathbb{N}_{\text{const}}(\tilde{x}) \cup \mathbb{N}_{\text{var}}(\tilde{x})$$

Equations (49), (50) and the definition of $\tilde{H}(x)$ imply

$$\left(Av_{x\in J(\tilde{x})}|\tilde{H}(x)|^2\right)^{1/2} \leq C_* \sum_{\nu\in\mathbb{N}_{\text{const}}(\tilde{x})\cup\mathbb{N}_{\text{var}}(\tilde{x})} \left(\sum_{0\leq k\leq k_{\max}} \left(Av_{I_\nu}|H_{\nu k}|^2\right)^{1/2}\right) .$$

Applying (41) and (42) to estimate the right-hand side, we get

(51) $$\left(Av_{J(\tilde{x})}|\tilde{H}|^2\right)^{1/2} \leq C_*\Lambda^{4\varepsilon-9/7} \sum_{\nu\in\mathbb{N}_{\text{const}}(\tilde{x})\cup\mathbb{N}_{\text{var}}(\tilde{x})} \lambda(x_\nu) + \text{Junk}$$
$$\text{in case} \quad \min_{k\in\mathbb{Z}} |\phi(0) - \pi(k+1/2)| \geq \overline{C}_\#\Lambda^{-1} ;$$

(52) $$\left(Av_{J(\tilde{x})}|\tilde{H}|^2\right)^{1/2} \leq C_*\Lambda^{-1} \sum_{\nu\in\mathbb{N}_{\text{const}}(\tilde{x})\cup\mathbb{N}_{\text{var}}(\tilde{x})} \lambda(x_\nu) + \text{Junk}$$
$$\text{in case} \quad \min_{k\in\mathbb{Z}} |\phi(0) - \pi(k+1/2)| \leq \overline{C}_\#\Lambda^{-1} .$$

Here,

(53) $$\text{Junk} = C_*\Lambda^{-\frac{N''}{3}}(\phi(0)+1)\cdot(\text{Number of distinct } \nu) .$$

To interpret these estimates, we bound $\sum_{\nu\in\mathbb{N}_{\text{const}}(\tilde{x})\cup\mathbb{N}_{\text{var}}(\tilde{x})} \lambda(x_\nu)$, (Number of distinct ν), and $(\phi(0)+1)$. From the properties (1)...(5) of the θ_ν we can argue as follows: $\int_I \theta_\nu\, dx \geq c_*|I_\nu|$ for each ν, and $0 \leq \sum_\nu \theta_\nu \leq C_*$ everywhere so,

$$C_*|I| \geq \sum_\nu \int_I \theta_\nu\, dx \geq c_* \sum_\nu |I_\nu| .$$

Also, $|I_\nu| > c_* B(x_\nu) > c_* \Lambda^{-K}|I|$ by (Y7), so

$$C_*|I| \geq c_* \sum_\nu |I_\nu| \geq c_* \Lambda^{-K}|I| \cdot (\text{Number of distinct } \nu) \ .$$

Thus,

$$(54) \qquad (\text{Number of distinct } \nu) \leq C_* \Lambda^K \ .$$

Recall that (Y7) tells us that $\lambda(x) > \Lambda^{-\frac{3}{2}K}\lambda(\overline{x})$ for $x, \overline{x} \in I$, hence $\Lambda^{-1} \leq \int_I \frac{dx}{\lambda(x)B(x)} \leq \Lambda^{\frac{3}{2}K}\lambda^{-1}(\overline{x}) \int_I \frac{dx}{B(x)} < \Lambda^{\frac{3}{2}K}\lambda^{-1}(\overline{x}) \int_I \frac{dx}{\Lambda^{-K}|I|} = \Lambda^{\frac{5}{2}K}\lambda^{-1}(\overline{x})$, i.e. $\lambda(\overline{x}) \leq \Lambda^{\frac{5}{2}K+1}$. Hence

$$\begin{aligned}\phi(0) &\leq C_\# \int_I S^{1/2}(x)\,dx = C_\# \int_I \frac{\lambda(x)dx}{B(x)} \leq C_\# \Lambda^{\frac{5}{2}K+1} \int_I \frac{dx}{B(x)} \\ &\leq C_\# \Lambda^{\frac{5}{2}K+1} \int_I \frac{dx}{\Lambda^{-K}|I|} = C_\# \Lambda^{\frac{7}{2}K+1} \ , \text{ again by (Y7)} \ .\end{aligned}$$

Thus,

$$(55) \qquad (\phi(0)+1) < \Lambda^{5K} \ .$$

Combining (53), (54), (55) and recalling that $N'' > 1000K$, we get

$$(56) \qquad \text{Junk } < \Lambda^{-\frac{N''}{4}} \ .$$

To estimate $\sum_{\nu \in \mathbb{N}_{\text{const}}(\tilde{x}) \cup \mathbb{N}_{\text{var}}(\tilde{x})} \lambda(x_\nu)$, note that each x_ν satisfies

$$(57) \qquad \text{dist}(x_\nu, [x_{\text{left}}(0), x_{\text{rt}}(0)]) < 2\hat{c}B(x_\nu) \ .$$

In fact, (2) and (5a) show that $\text{dist}(x_\nu, \check{I}) \leq \hat{c}B(x_\nu)$, i.e.

$$(58) \qquad |x_\nu - \check{x}_\nu| \leq \hat{c}B(x_\nu) \quad \text{for an } \check{x}_\nu \in \check{I} \ .$$

Evidently, $B(\check{x}_\nu) \leq C_\# B(x_\nu)$. Moreover, by definition of $\check{I}$, any point $\check{x} \in \check{I}$ satisfies $\text{dist}(\check{x}, [x_{\text{left}}(0), x_{\text{rt}}(0)]) < C_\# \Lambda^{-\varepsilon} B(\check{x})$. Applying this to $\check{x}_\nu$ and using (58), we get

$$\text{dist}(x_\nu, [x_{\text{left}}(0), x_{\text{rt}}(0)]) \leq \hat{c}B(x_\nu) + C_\# \Lambda^{-\varepsilon} B(\check{x}_\nu) \leq \hat{c}B(x_\nu) + C'_\# \Lambda^{-\varepsilon} B(x_\nu)$$

which implies (57) by the WKB hypothesis (Y11).

Next, we use Lemma 1 with $E = 0$, to conclude that:

$$-V'(x) \geq c_{\#} S(x_{\text{left}}(0)) B^{-1}(x_{\text{left}}(0)) \qquad \text{for} \quad x \in [x_{\text{left}}(0), x_{\text{left}}(0) + \bar{c}_{\#} B(x_{\text{left}}(0))]$$

and hence

$$(59) \quad -V(x) > c_{\#} S(x_{\text{left}}(0)) B^{-1}(x_{\text{left}}(0)) \cdot (x - x_{\text{left}}(0)) \qquad \text{for} \quad x \in [x_{\text{left}}(0), x_{\text{left}}(0) + \bar{c}_{\#} B(x_{\text{left}}(0))]\ .$$

Similarly,

$$(60) \quad -V(x) > c_{\#} S(x_{\text{rt}}(0)) B^{-1}(x_{\text{rt}}(0)) \cdot (x_{\text{rt}}(0) - x) \qquad \text{for} \quad x \in [x_{\text{rt}}(0) - \bar{c}_{\#} B(x_{\text{rt}}(0)), x_{\text{rt}}(0)]\ .$$

Also from Lemma 1 with $E = 0$ we get

$$(61) \quad -V(x) > c_{\#} S(x) \quad \text{for} \quad x \in [x_{\text{left}}(0) + \bar{c}_{\#} B(x_{\text{left}}(0)), x_{\text{rt}}(0) - \bar{c}_{\#} B(x_{\text{rt}}(0))]\ .$$

From (57), (59), (60), (61) we will show that

$$(62) \qquad \int_{|x - x_\nu| < 3\hat{c}B(x_\nu)} \left(-V(x)\right)_+^{1/2} \geq c_* \lambda(x_\nu)\ .$$

To see (62), we shrink the region of integration on the left to $\{|x - x_\nu| < 3\hat{c}B(x_\nu)\}$ $\cap\, [x_{\text{left}}(0) + (\hat{c})^2 B(x_{\text{left}}(0)), x_{\text{rt}}(0) - (\hat{c})^2 B(x_{\text{rt}}(0))] \equiv \mathcal{E}$. Estimates (59), (60), (61) show that $-V(x) > c_* S(x_\nu)$ on $\mathcal{E}$, and (57) shows that $\mathcal{E}$ is an interval of length at least $\frac{1}{2}\hat{c}B(x_\nu)$. Hence, the left-hand side of (62) is at least $c_* S^{1/2}(x_\nu) B(x_\nu) = c_* \lambda(x_\nu)$, which proves (62).

Next we note that $\nu \in \mathbb{N}_{\text{const}}(\tilde{x}) \cup \mathbb{N}_{\text{var}}(\tilde{x})$ implies $x_\nu < \tilde{x} + C_{\#}\hat{c}B(\tilde{x})$ and therefore also $\{|x - x_\nu| < 3\hat{c}B(x_\nu)\} \subset (-\infty, \tilde{x} + C'_{\#}\hat{c}B(\tilde{x})]$. Hence (62) implies

$$\lambda(x_\nu) \leq C_* \int_{I_{\text{BVP}} \cap (-\infty, \tilde{x} + C'_{\#}\hat{c}B(\tilde{x})]} \chi_{|x - x_\nu| < 3\hat{c}B(x_\nu)} \left(-V(x)\right)_+^{1/2} dx$$

for $\nu \in \mathbb{N}_{\text{const}}(\tilde{x}) \cup \mathbb{N}_{\text{var}}(\tilde{x})$. Summing over ν and using (4), we get

$$(63) \qquad \sum_{\nu \in \mathbb{N}_{\text{const}}(\tilde{x}) \cup \mathbb{N}_{\text{var}}(\tilde{x})} \lambda(x_\nu) \leq C_* \int_{I_{\text{BVP}} \cap (-\infty, \tilde{x} + C'_\# \hat{c} B(\tilde{x})]} \big(-V(x)\big)_+^{1/2} \, dx \; .$$

Putting (56) and (63) into (51), (52) yields:

$$(64) \quad \left(Av_{|x-\tilde{x}|<\hat{c}B(\tilde{x})} |\tilde{H}(x)|^2\right)^{1/2} \leq C_* \Lambda^{4\varepsilon - 9/7} \int_{I_{\text{BVP}} \cap (-\infty, \tilde{x} + C_\# \hat{c} B(\tilde{x})]} \big(-V(x)\big)_+^{1/2} \, dx + \Lambda^{-\frac{N''}{4}}$$

$$\text{in case} \quad \min_{k \in \mathbb{Z}} |\phi(0) - \pi(k + 1/2)| \geq \overline{C}_\# \Lambda^{-1} \; ;$$

(65)

$$\left(Av_{|x-\tilde{x}|<\hat{c}B(\tilde{x})} |\tilde{H}(x)|^2\right)^{1/2} \leq C_* \Lambda^{-1} \int_{I_{\text{BVP}} \cap (-\infty, \tilde{x} + C_\# \hat{c} B(\tilde{x})]} \big(-V(x)\big)_+^{1/2} \, dx + \Lambda^{-\frac{N''}{4}}$$

$$\text{in case} \quad \min_{k \in \mathbb{Z}} |\phi(0) - \pi(k + 1/2)| \leq \overline{C}_\# \Lambda^{-1}.$$

Now set

$$g_{\text{err}}(x, E) = \varphi - \sum_{\nu k} g_{\nu k} = \varphi(E) \cdot \big(1 - \sum_{\nu k} h_{\nu k}(x, E)\big) \equiv \varphi(E) \cdot h_{\text{err}}(x, E) \; .$$

From (12) and (14) we get

$$(66) \quad |h_{\text{err}}(x, E)| \leq C_* \chi_{x \in I_{\text{BVP}} \setminus [x_{\text{left}}(E) - \Lambda^{-3/7} B(x_{\text{left}}(E)), x_{\text{rt}}(E) + \Lambda^{-3/7} B(x_{\text{rt}}(E))]} \; , \qquad \text{for} \quad V(x_0) < E \leq 0 \; .$$

This and hypothesis (Y9) show that $g_{\text{err}}(x, E) = 0$ whenever $V(x) \leq E \leq 0$. Hence $\rho_{sc}(x, g_{\text{err}}) \equiv 0$, by definition of the semiclassical density. On the other hand,

$$\rho(x, g_{\text{err}}) = \sum_k \varphi(E_k) h_{\text{err}}(x, E_k) |u_k(x)|^2 \; ,$$

so that

$$(67)\qquad \int\limits_{I_{\rm BVP}} |\rho(x, g_{\rm err})|\, dx \le C_* \sum_k |\varphi(E_k)| \cdot \int\limits_{x\in I_{\rm BVP}\setminus[x_{\rm left}(E_k)-\Lambda^{-3/7}B(x_{\rm left}(E_k)),x_{\rm rt}(E_k)+\Lambda^{-3/7}B(x_{\rm rt}(E_k))]} |u_k(x)|^2 dx$$

by virtue of (66). Here E_k are the eigenvalues ≤ 0, and u_k are the eigenfunctions of H.

Recall that $\lambda(x_{\rm left}(E_k)), \lambda(x_{\rm rt}(E_k)) \ge c_\# \Lambda(E_k) \ge c_\# \Lambda$, so $\Lambda^{-3/7} > \lambda^{\varepsilon-2/3}(x_{\rm left}(E)), \lambda^{\varepsilon-2/3}(x_{\rm rt}(E))$. Hence, Lemma 1 and the WKB Eigenfunction Theorem show that the integral on the right in (67) is at most $C_\# \Lambda^{-N''}$ for $V(x_0) + c_2 S(x_0) \le E_k \le 0$. For $E_k \le V(x_0) + c_2 S(x_0)$ we have $\varphi(E_k) = 0$ by hypothesis (Y9). Hence (67) implies

$$(68)\qquad \int\limits_{I_{\rm BVP}} |\rho(x, g_{\rm err})|\, dx \le C_\# \Lambda^{-N''} \cdot \text{(Number of eigenvalues } E_k \text{ in } [V(x_0)+c_2 S(x_0), 0]) .$$

We estimate the number of eigenvalues on the right in (68). If $E_k \in [V(x_0) + c_2 S(x_0), 0]$ is an eigenvalue, then by applying Lemma 1 with $E = E_k$ and the WKB Eigenvalue Theorem with $E_0 = E_k$, we learn the following:

$$(69)\qquad |\phi(E_k) - \pi(k' + 1/2)| \le C_\# \Lambda^{-1} \quad \text{for an integer } k' .$$

$$(70)\qquad E_k \text{ is the only eigenvalue } E \text{ satisfying } |E_k - E| \le c_\# S_{\min}(E_k) , \quad E \le 0, \text{ and } |\phi(E) - \pi(k' + 1/2)| \le C_\# \Lambda^{-1} .$$

Here, $S_{\min}(E_k) = \inf_{x\in[x_{\rm left}(E_k), x_{\rm rt}(E_k)]} S(x)$ as in the WKB Theorems.

For $|E - E_k| < c_\# S_{\min}(E_k)$ we have $|x_{\rm left}(E) - x_{\rm left}(E_k)| < c'_\# B(x_{\rm left}(E_k))$ and $|x_{\rm rt}(E) - x_{\rm rt}(E_k)| < c'_\# B(x_{\rm rt}(E_k))$, so $S_{\min}(E) \sim S_{\min}(E_k)$, provided we take $c_\#$

small. Recall that $S_{\min}(E) \sim S(\overline{x})$ for some $\overline{x} \in [x_{\text{left}}(E) + c_\# B(x_{\text{left}}(E)), x_{\text{rt}}(E) - c_\# B(x_{\text{rt}}(E))]$ and therefore

$$\begin{aligned}\frac{d\phi}{dE}(E) = \frac{1}{2}\int_I (E - V(x))_+^{-1/2}\,dx \geq c_\# S^{-1/2}(\overline{x})B(\overline{x}) \quad \text{(by Lemma 1)}\\ \geq c'_\# S^{-1}_{\min}(E)\cdot\lambda(\overline{x}) \geq c''_\# S^{-1}_{\min}(E)\Lambda(E) \geq c''_\# S^{-1}_{\min}(E)\Lambda\ .\end{aligned}$$

Here $\frac{d\phi}{dE}(E) \geq c'_\# S^{-1}_{\min}(E_k)\Lambda$ for $|E - E_k| \leq c_\# S_{\min}(E_k)$. (We have used this estimate repeatedly in other sections.) Also, $\frac{d\phi}{dE} \geq 0$ for all $E \in (V(x_0), 0]$. So if we are given E satisfying $E_k + c_\# S_{\min}(E_k) \leq E \leq 0$, then

$$\phi(E) \geq \phi(E_k) + \int_{E_k}^{E_k + c_\# S_{\min}(E_k)} \phi'(\tilde{E})d\tilde{E} \geq \phi(E_k) + c'_\#\Lambda > \pi(k'+1/2) + c''_\#\Lambda$$

by (69). In particular, we cannot have $|\phi(E) - \pi(k'+1/2)| \leq C_\#\Lambda^{-1}$. Similarly, if $V(x_0) < E \leq E_k - c_\# S_{\min}(E_k)$, then

$$\phi(E) \leq \phi(E_k) - \int_{E_k - c_\# S_{\min}(E_k)}^{E_k} \phi'(\tilde{E})d\tilde{E} \leq \phi(E_k) - c'_\#\Lambda < \pi(k'+1/2) - c''_\#\Lambda$$

by (69), so again we cannot have $|\phi(E) - \pi(k'+1/2)| \leq C_\#\Lambda^{-1}$. These remarks and (70) show that E_k is the only eigenvalue $E \in (V(x_0), 0]$ satisfying $|\pi(k'+1/2) - \phi(E)| \leq C_\#\Lambda^{-1}$. It follows that the number of eigenvalues $E_k \in [V(x_0) + c_2 S(x_0), 0]$ is less than or equal to the number of integers k' for which $|\phi(E) - \pi(k'+1/2)| \leq C_\#\Lambda^{-1}$ for some $E \in (V(x_0), 0]$. As E ranges over $(V(x_0), 0]$, $\phi(E)$ ranges over $(0, \phi(0)]$, so the number of k' as above is dominated by $C_\#(\phi(0)+1)$. Hence

$$\text{(70 bis)} \quad (\text{Number of } E_k \in [V(x_0)+c_2 S(x_0), 0]) \leq C_\#(\phi(0)+1) \leq C_\#\Lambda^{5K} \text{ by (55)}\ .$$

Substituting this in (68), we get

$$\text{(71)} \qquad \int_{I_{\text{BVP}}} |\rho(x, g_{\text{err}})|\,dx \leq \Lambda^{-\frac{N''}{4}}\ .$$

Recalling that $\rho_{sc}(x, g_{\text{err}}) \equiv 0$ and setting $H_{\text{err}}(x)$
$= \int_{I_{\text{BVP}} \cap (0,x]} (\rho(x', g_{\text{err}}) - \rho_{sc}(x', g_{\text{err}}))\, dx'$, we get from (71) the estimate $|H_{\text{err}}(x)| \leq \Lambda^{-\frac{N''}{4}}$ for all $x \in \mathbb{R}$. In particular

$$(72) \quad \left(Av_{|x-\tilde{x}|<\hat{c}B(\tilde{x})}|H_{\text{err}}(x)|^2\right)^{1/2} \leq \Lambda^{-\frac{N''}{4}} \text{ , for } \tilde{x} \text{ with } \{|x - \tilde{x}| < \hat{c}B(\tilde{x})\} \subset I \ .$$

We now know the following:

$$\varphi(E) = \sum_{\nu k} g_{\nu k}(x, E) + g_{\text{err}}(x, E) \quad (\text{definition of } g_{\text{err}}) \ .$$

$$\tilde{H}(x) = \int_{I_{\text{BVP}} \cap (-\infty, x]} (\rho(x', \sum_{\nu k} g_{\nu k}) - \rho_{sc}(x', \sum_{\nu k} g_{\nu k}))\, dx' \quad (\text{definition of } \tilde{H})$$

$$H_{\text{err}}(x) = \int_{I_{\text{BVP}} \cap (-\infty, x]} (\rho(x', g_{\text{err}}) - \rho_{sc}(x', g_{\text{err}}))\, dx' \quad (\text{definition of } H_{\text{err}}) \ .$$

Therefore,

$$(73) \qquad H(x) = \int_{I_{\text{BVP}} \cap (-\infty, x]} (\rho(x', \varphi) - \rho_{sc}(x', \varphi))\, dx' = \tilde{H}(x) + H_{\text{err}}(x) \ .$$

Combining estimates (64), (65), (72), we obtain;

$$(74) \quad \left(Av_{|x-\tilde{x}|<\hat{c}B(\tilde{x})}\, |H(x)|^2\right)^{1/2} \leq C_* \Lambda^{4\varepsilon - 9/7} \int_{I_{\text{BVP}} \cap (-\infty, \tilde{x} + C_\# \hat{c}B(\tilde{x})]} \left(-V(x)\right)_+^{1/2} dx + 2\Lambda^{-\frac{N''}{4}}$$

$$\text{in case} \quad \min_{k \in \mathbb{Z}} |\phi(0) - \pi(k + 1/2)| \geq \overline{C}_\# \Lambda^{-1} \ ;$$

$$(75) \quad \left(Av_{|x-\tilde{x}|<\hat{c}B(\tilde{x})}\, |H(x)|^2\right)^{1/2} \leq C_* \Lambda^{-1} \int_{I_{\text{BVP}} \cap (-\infty, \tilde{x} + C_\# \hat{c}B(\tilde{x})]} \left(-V(x)\right)_+^{1/2} dx + 2\Lambda^{\frac{-N''}{4}}$$

$$\text{in case} \quad \min_{k \in \mathbb{Z}} |\phi(0) - \pi(k + 1/2)| \leq \overline{C}_\# \Lambda^{-1} \ .$$

Estimates (74), (75) have been proven for arbitrary $\tilde{x}$ for which $\{|x - \tilde{x}| < \hat{c}B(\tilde{x})\} \subset I$. The set of such $\tilde{x}$ is slightly smaller than I. We also want estimates for $H(x)$, with x not necessarily belonging to I, or even to $I_{\rm BVP}$. To derive these estimates, we use Lemma 1 and the WKB Eigenfunction Theorem, to see that $E_k \in [V(x_0) + c_2 S(x_0), 0]$ implies $\int\limits_{I_{\rm BVP}\backslash[x_{\rm left}(0)-\hat{c}B(x_{\rm left}(0)),x_{\rm rt}(0)+\hat{c}B(x_{\rm rt}(0))]} |u_k(x)|^2\,dx \le \Lambda^{-N''}$. Therefore

$$
(76) \qquad \int\limits_{I_{\rm BVP}\backslash[x_{\rm left}(0)-\hat{c}B(x_{\rm left}(0)),x_{\rm rt}(0)+\hat{c}B(x_{\rm rt}(0))]} |\rho(x,\varphi)|\,dx
$$

$$
\le C_* \Lambda^{-N''} \cdot (\text{Number of } E_k \in [V(x_0) + c_2 S(x_0), 0]) \le \Lambda^{-\frac{N''}{4}} \text{ by (70 bis)} \ .
$$

Also, for $x \in I_{\rm BVP}\backslash[x_{\rm left}(0) - \hat{c}B(x_{\rm left}(0)), x_{\rm rt}(0) + \hat{c}B(x_{\rm rt}(0))]$ we have $V(x) > 0$, hence $\rho_{sc}(x,\varphi) = 0$.

For any x in $(-\infty, x_{\rm left}(0) - \hat{c}B(x_{\rm left}(0))]$ we have therefore

$$
(77) \quad |H(x)| = \Big| \int\limits_{I_{\rm BVP}\cap(-\infty,x]} (\rho(x',\varphi) - \rho_{sc}(x',\varphi))\,dx' \Big| = \Big| \int\limits_{I_{\rm BVP}\cap(-\infty,x]} \rho(x',\varphi)\,dx' \Big| \le \Lambda^{-\frac{N''}{4}} \ .
$$

Similarly, for any x, y in $[x_{\rm rt}(0) + \hat{c}B(x_{\rm rt}(0)), +\infty)$ we have

$$
(78) \quad |H(x) - H(y)| = \Big| \int\limits_{I_{\rm BVP}\cap[\min\{x,y\},\max\{x,y\}]} (\rho(x',\varphi) - \rho_{sc}(x',\varphi))\,dx' \Big| = \Big| \int\limits_{I_{\rm BVP}\cap[\min\{x,y\},\max\{x,y\}]} \rho(x',\varphi)\,dx' \Big| \le \Lambda^{-\frac{N''}{4}} \ .
$$

Take $\tilde{x} = x_{\rm rt}(0) + \sqrt{\hat{c}}\,B(x_{\rm rt}(0))$, so that $\{|x-\tilde{x}| < \hat{c}B(\tilde{x})\} \subset I$, and $|x - \tilde{x}| < \hat{c}B(\tilde{x})$ implies $x \ge x_{\rm rt}(0) + \hat{c}B(x_{\rm rt}(0))$. Using (74), (75), (78) and the obvious estimate $|H(y)| \le Av_{\{|x-\tilde{x}|<\hat{c}B(\tilde{x})\}}\,|H(x) - H(y)| + \big(Av_{\{|x-\tilde{x}|<\hat{c}B(\tilde{x})\}}|H(x)|^2\big)^{1/2}$, we find that

$$
(79) \quad |H(y)| \le C_* \Lambda^{4\varepsilon - 9/7} \int\limits_{I_{\rm BVP}} \big(-V(x)\big)_+^{1/2}\,dx + 10\Lambda^{-\frac{N''}{4}}
$$

$$
\text{when } \min_{k\in\mathbb{Z}} |\phi(0) - \pi(k+1/2)| \ge \overline{C}_\# \Lambda^{-1} \text{ and } y \ge x_{\rm rt}(0) + \hat{c}B(x_{\rm rt}(0)) \ ;
$$

and

$$(80)\quad |H(y)| \le C_* \Lambda^{-1} \int\limits_{I_{\rm BVP}} \big(-V(x)\big)_+^{1/2}\,dx + 10\Lambda^{-\frac{N''}{4}}$$

$$\text{when } \min_{k\in\mathbb{Z}} |\phi(0) - \pi(k+1/2)| \le \overline{C}_{\#}\Lambda^{-1} \text{ and } y \ge x_{\rm rt}(0) + \hat{c}B(x_{\rm rt}(0)) \ .$$

Our basic results on $\rho(x,\varphi)$ are (74), (75), (77), (79), (80). We collect these results in the following.

Lemma. *Assume hypotheses (Y0)…(Y11), and define* $H(x) = \int\limits_{I_{\rm BVP}\cap(-\infty,x]} (\rho(x',\varphi) - \rho_{sc}(x',\varphi))\,dx'$ *for arbitrary real* x*. Then we have the following estimates for* $H(x)$*.*

CASE I: Suppose $\min_{k\in\mathbb{Z}} |\phi(0) - \pi(k+1/2)| \ge \overline{C}_{\#}\Lambda^{-1}$*. Then for* $-\infty < x \le x_{\rm left}(0) - \hat{c}B(x_{\rm left}(0))$ *we have* $|H(x)| \le 10\Lambda^{-\frac{N''}{4}}$*; for* $x_{\rm left}(0) - \hat{c}B(x_{\rm left}(0)) \le \tilde{x} \le x_{\rm rt}(0) + \hat{c}B(x_{\rm rt}(0))$ *we have* $\big(Av_{|x-\tilde{x}|<\hat{c}B(\tilde{x})}|H(x)|^2\big)^{1/2} \le C_*\Lambda^{4\varepsilon-9/7} \int\limits_{I_{\rm BVP}\cap(-\infty,\tilde{x}+C_{\#}\hat{c}B(\tilde{x})]} \big(-V(x)\big)_+^{1/2}\,dx + 10\Lambda^{-\frac{N''}{4}}$*; and for* $x_{\rm rt}(0) + \hat{c}B(x_{\rm rt}(0)) \le x < +\infty$ *we have*

$$|H(x)| \le C_*\Lambda^{4\varepsilon-9/7} \int\limits_{I_{\rm BVP}} \big(-V(x')\big)_+^{1/2}\,dx' + 10\Lambda^{-\frac{N''}{4}} \ .$$

CASE II: Suppose $\min_{k\in\mathbb{Z}} |\phi(0) - \pi(k+1/2)| \le \overline{C}_{\#}\Lambda^{-1}$*. Then for* $-\infty < x \le x_{\rm left}(0) - \hat{c}B(x_{\rm left}(0))$ *we have* $|H(x)| \le 10\Lambda^{-\frac{N''}{4}}$*; for* $x_{\rm left}(0) - \hat{c}B(x_{\rm left}(0)) \le \tilde{x} \le x_{\rm rt}(0) + \hat{c}B(x_{\rm rt}(0))$ *we have* $\big(Av_{|x-\tilde{x}|<\hat{c}B(\tilde{x})}|H(x)|^2\big)^{1/2} \le C_*\Lambda^{-1} \int\limits_{I_{\rm BVP}\cap(-\infty,\tilde{x}+C_{\#}\hat{c}B(\tilde{x})]} \big(-V(x)\big)_+^{1/2}\,dx + 10\Lambda^{-\frac{N''}{4}}$*; and for* $x_{\rm rt}(0) + \hat{c}B(x_{\rm rt}(0)) \le x < +\infty$ *we have*

$$|H(x)| \le C_*\Lambda^{-1} \int\limits_{I_{\rm BVP}} \big(-V(x')\big)_+^{1/2}\,dx' + 10\Lambda^{-\frac{N''}{4}} \ .$$

The constants $C_{\#}$*,* $\overline{C}_{\#}$ *depend only on* ε*,* K*,* N*,* c*,* C*,* c_1*,* c_2*,* C_α *in (Y0)…(Y11) while* C_* *depends only* ε*,* K*,* N*,* c*,* C*,* c_1*,* c_2*,* C_α*,* $\hat{c}$*,* $\hat{C}_\beta$ *in (Y0)…(Y11).*

THE DENSITY FOR A ONE-DIMENSIONAL POTENTIAL

Our goal here is to extend the results of the previous section on $\rho(x,\varphi)-\rho_{sc}(x,\varphi)$ to cover the case $\varphi \equiv 1$. Thus, we at last gain control over the full density.

We suppose we are given a potential $V(x)$ defined on a (possibly unbounded) interval I_{BVP}, together with a subinterval I on which are defined positive functions $S(x)$, $B(x)$. We are given also positive numbers K, ε, N with $K > 100$, $\varepsilon < \frac{1}{1000K}$, $N > \frac{K}{\varepsilon^{50}}$; and we set $N' = [\varepsilon N/500]$, $N'' = \frac{3}{2}\varepsilon N' - 30000K - 33$. Define $H = -\frac{d^2}{dx^2} + V(x)$ on I, with Dirichlet or Neumann conditions. When we speak of eigenfunctions or eigenvalues, we mean those of H. We suppose we are given a positive constant $\hat{c}$. Our hypotheses are as follows.

We assume conditions (Y0)...(Y7) from the previous section. In addition, we assume (Y10) and the following variant of (Y11).

(Y11*) Λ, defined as in the previous section, is bounded below by a certain large, positive number determined by ε, K, N, c, C, c_1, c_2, C_α, $\hat{c}$.

Our present assumptions differ from those of the previous section in that we have dropped the function $\varphi(E)$ and the constants $\hat{C}_\beta$ used to control it.

We use $c_\#$, $C_\#$ etc. to denote constants that depend only on ε, K, N, c, C, c_1, c_2, C_α in (Y0)...(Y7) and (Y10), while constants denoted c_*, C_* etc. depend also on $\hat{c}$.

Our plan is to make a partition of unity $1 = \sum_{0\le k\le k_{\text{last}}} \varphi_k(E)$, and write

$$H(x) = \int_{I_{\text{BVP}}\cap(-\infty,x]} (\rho(x',1) - \rho_{sc}(x',1))dx'$$

as a sum

$$H(x) = \sum_{0\le k\le k_{\text{last}}} H_k(x) \ , \quad \text{with}$$

$$H_k(x) = \int_{I_{\text{BVP}}\cap(-\infty,x]} (\rho(x',\varphi_k) - \rho_{sc}(x',\varphi_k))dx' \ .$$

We will use the results of the previous section to control $H_k(x)$ for $0 \le k < k_{\text{last}}$, and we use our results on the microlocalized density near the minimum of the potential to control $H_{k_{\text{last}}}(x)$. The $\varphi_k(E)$ may be taken to satisfy the following conditions, for a small constant $c_{\#}^2 > 0$ to be picked later.

(1) $$\varphi_0(E) \quad \text{is supported in} \quad \{E \ge V(x_0) + c_{\#}^2 S(x_0)\}$$

(2) $$\left|\left(\frac{d}{dE}\right)^{\beta} \varphi_0(E)\right| \le C_{\#}^{\beta} (S(x_0))^{-\beta} \quad \text{for all} \quad E\ ,\beta\ .$$

For $1 \le k < k_{\text{last}}$,

(3) $$\varphi_k(E) \quad \text{is supported in} \quad \{2^{-2(k+3)} c_{\#}^2 S(x_0) < E - V(x_0) < 2^{-2(k-3)} c_{\#}^2 S(x_0)\}$$

(4) $$\left|\left(\frac{d}{dE}\right)^{\beta} \varphi_k(E)\right| \le C_{\#}^{\beta} (2^{-2k} S(x_0))^{-\beta} \quad \text{for all} \quad E\ ,\beta\ .$$

Regarding $k = k_{\text{last}}$,

(5) $$\varphi_{k_{\text{last}}}(E) \quad \text{is supported in} \quad \{|E - V(x_0)| < 2^{-2(k_{\text{last}}-3)} c_{\#}^2 S(x_0)\}\ , \quad \text{and}$$

(6) $$\left|\left(\frac{d}{dE}\right)^{\beta} \varphi_{k_{\text{last}}}(E)\right| \le C_{\#}^{\beta} (2^{-2k_{\text{last}}} S(x_0))^{-\beta} \quad \text{for all} \quad E\ ,\beta\ .$$

(7) $$\sum_{0 \le k \le k_{\text{last}}} \varphi_k(E) = 1 \quad \text{for} \quad V(x_0) \le E \le 0\ .$$

Since all the eigenvalues $E_k \le 0$ satisfy $V(x_0) \le E_k \le 0$, equation (7) is enough to show that

(8) $$H(x) = \sum_{0 \le k \le k_{\text{last}}} H_k(x) \quad \text{for all } x\ ,$$

with $H(x)$, $H_k(x)$ defined as above.

We take k_{last} to satisfy $2^{-k_{\text{last}}-1} < (\lambda(x_0))^{-\frac{18}{43}} \le 2^{-k_{\text{last}}}$. We pick $c_{\#}^2$ small enough that $V(x_0) + 10^3 c_{\#}^2 S(x_0) < 0$,

(9)

$$V(x_0)+c_{\#}S(x_0)B^{-2}(x_0)\cdot(x-x_0)^2 \le V(x) \le V(x_0)+C_{\#}S(x_0)B^{-2}(x_0)\cdot(x-x_0)^2 ,$$

and

$$(10) \quad c_{\#}S(x_0)B^{-1}(x_0) < \frac{V'(x)}{(x-x_0)} < C_{\#}S(x_0)B^{-1}(x_0)$$

$$\text{for} \quad V(x_0) < V(x) \le V(x_0) + 10^3 c_{\#}^2 S(x_0) \ .$$

For $V(x_0) < E < V(x_0) + 10^3 c_{\#}^2 S(x_0)$, we note that $\phi'(E) \sim \lambda(x_0)(S(x_0))^{-1}$. Hence, as E varies in $\left\{\frac{E-V(x_0)}{S(x_0)} \in \left[\frac{10^3}{3}c_{\#}^2 2^{-2k}, \frac{10^3}{2}c_{\#}^2 2^{-2k}\right]\right\}$, we find that $\phi(E)$ varies by $\sim \lambda(x_0)\cdot 2^{-2k} \ge \lambda(x_0)2^{-2k_{\max}} \sim \lambda(x_0)\cdot\lambda^{-\frac{36}{43}}(x_0) = \lambda^{7/43}(x_0) > c_{\#}\Lambda^{7/43}$ $>> 1$ when $1 \le k \le k_{\max}$.

Therefore, for $1 \le k \le k_{\max}$, we can find an energy $\tilde{E}_k \le -c_{\#}S(x_0)$ (<u>not</u> an eigenvalue – the eigenvalues are called E_k) with

$$(11) \qquad \tilde{E}_k - V(x_0) \ , \quad \tilde{E}_k - \max(\text{supp } \varphi_k) \sim c_{\#}^2 2^{-2k} S(x_0)$$

$$(12) \qquad \min_{k' \in \mathbb{Z}} \ |\phi(\tilde{E}_k) - \pi(k'+1/2)| \ge \frac{1}{20} \ .$$

We can take the $\tilde{E}_k$ to decrease with k, so that $x_{\text{left}}(\tilde{E}_k)$ increases with k, and $x_{\text{rt}}(\tilde{E}_k)$ decreases with k.

Now for $1 \le k \le k_{\text{last}}$ we set $S_k = 2^{-2k}S(x_0)$, $B_k = 2^{-k}B(x_0)$, $\lambda_k = S_k^{1/2}B_k = 2^{-2k}\lambda(x_0)$, $V_k(x) = V(x) - \tilde{E}_k$, and $I_k = \{|x-x_0| < C_{\#}^1 B_k\}$, with $C_{\#}^1$ and $c_{\#}^2$ picked as in the following lemma.

Lemma 1. *We can pick $c_{\#}^2$, $C_{\#}^1$ so that $V(x_0)+10^3 c_{\#}^2 S(x_0) < 0$ and (9) and (10) hold, and the following conditions are satisfied.*

(A) *Hypotheses (Y0)...(Y11) in the previous section are satisfied, with $\varphi_0(E)$ in place of $\varphi(E)$, with $c_{\#}^2$ in place of c_2, and with each $\hat{C}_\beta$ in (Y8) of the form $C_{\#}^{\beta}$.*

(B) *Suppose* $1 \leq k < k_{\text{last}}$. *Then hypotheses (Y0)... (Y11) are satisfied, with* $\varphi_k(E - \tilde{E}_k)$ *in place of* $\varphi(E)$, $V_k(x)$ *in place of* $V(x)$, S_k *in place of* $S(x)$, B_k *in place of* $B(x)$, I_k *in place of* I, $c_\#^2$ *in place of* c_2.

The constants called ε, K, N, c, C, c_1, c_2, C_α, $\hat{C}_\beta$ *in (Y0)... (Y11) are all of the form* $C_\#$. *In particular, they do not depend on* k.

(C) *The hypotheses of the WKB Theorem on Low Eigenvalues ((H0*)... (H6*)) are satisfied, with* $V_{k_{\text{last}}}(x)$ *in place of* $V(x)$, $S(x_0)$ *and* $B(x_0)$ *in place of* S *and* B, *and with* $E_\infty = 0$. *Also, the function* $\varphi_{k_{\text{last}}}(E)$ *satisfies the assumptions on* $g(E)$ *which we made in the section on the microlocalized density near the minimum of the potential, with* $C_\#(\lambda(x_0))^{-18/43}$ *in place of* τ, *and with the constants* $\hat{C}_\beta$ *of the form* $C_\#$.

Sketch of Proof.

Part (A) follows from (1) and (2), together with our assumptions (Y0)... (Y7) and (Y10), (Y11*). In fact, (1) and (2) amount to (Y8), (Y9) for $\varphi_0(E)$; and (Y11) follows from (Y11*), since each $\hat{C}_\beta$ has the form $C_\#$.

Part (B). We have to verify (Y0)... (Y11). (Y0)... (Y5) are trivial, and we leave them to the reader.

Here, the quantity called Λ in (Y0)... (Y11) for V_k, S_k, B_k, etc. is $\Lambda_{(V_k)} = 2^{-2k}\lambda(x_0) \geq 2^{-2k_{\max}}\lambda(x_0) \geq c_\#(\lambda(x_0))^{7/43} > c_\#\Lambda^{7/43}$. To check (Y6), note that $V(x) = \tilde{E}_k$ at the point $x_{k,\text{left}}$ that plays the rôle of $x_{\text{left}}(0)$ for V_k. We have $(x_0 - x_{k,\text{left}}) \sim B_k$ and $-V' \sim S_k B_k^{-1}$ whenever $(x_0 - x) \sim B_k$. Hence there is a point x_k^- with $(x_{k,\text{left}} - x_k^-) \sim B_k$ and $0 \geq V(x_k^-) \geq \tilde{E}_k + c_\# S_k$. Our hypotheses on $V(x)$ show that $V(x)$ is decreasing in $[x_{\text{left}}(0), x_k^-]$ and $V(x) > 0$ for $x < x_{\text{left}}(0)$. Hence for $x < x_k^-$ we have $V(x) > V(x_k^-) \geq \tilde{E}_k + c_\# S_k$, so $V_k(x) = V(x) - \tilde{E}_k \geq c_\# S_k = c_\# \lambda_k^2 B_k^{-2}$.

If $x < x_{k,\text{left}} - C_\# B_k$, then $B_k^{-2} \geq (x - x_{k,\text{left}})^{-2}$ and $x < x_k^-$, so $V_k(x) \geq c_\# \lambda_k^2 B_k^{-2} \geq$

$c_{\#}\lambda_k^2(x - x_{k,\text{left}})^{-2}$. Similarly, if $x_{k,\text{rt}}$ plays the rôle of $x_{\text{rt}}(0)$ for $V_k(x)$, and if $x > x_{k,\text{rt}} + C_{\#}B_k$, then $V_k(x) \geq c_{\#}\lambda_k^2 \cdot (x - x_{k,\text{rt}})^{-2}$. We have proven more than required for (Y6).

Property (Y7) is trivial, and (Y8), (Y9) for $\varphi_k(E)$ follow at once from (3), (4). Properties (Y10) and (Y11) for $V_k(x)$ follow at once from our present hypotheses (Y10), (Y11*), and the fact that $\hat{C}_\beta$ has here the form $C_{\#}^{\beta}$. Thus, part (B) of the Lemma is verified.

To prove part (C) we first check (H0*)...(H6*). Note that the quantity that plays the rôle of λ for $V_{k_{\text{last}}}$ is $\lambda(x_0) \geq c_{\#}\Lambda$. Properties (H0*)...(H6*) are trivial, except (perhaps) for (H5*), which follows from the same argument just used above for (Y6) in part B.

The properties of $\varphi_{k_{\text{last}}}(E)$, which we assumed for $g(E)$ in the section on the microlocalized density near the minimum of the potential, simply amount to (5) and (6).

The proof of Lemma 1 is complete. ■

Define $\tilde{\varphi}_k = \varphi_k(E - \tilde{E}_k)$ for $1 \leq k \leq k_{\text{last}}$. The functions $\rho(x, \varphi_k)$ and $\rho_{sc}(x, \varphi_k)$ arising from the potential $V(x)$ are equal to the functions $\rho(x, \tilde{\varphi}_k)$ and $\rho_{sc}(x, \tilde{\varphi}_k)$ respectively, arising from $V_k(x)$. (This fact is trivially verified, using the fact that $\varphi_k(E) = 0$ for $E \geq \tilde{E}_k$.)

Therefore, Lemma 1(B) and the lemma of the previous section allow us to control $H_k(x)$ for $1 \leq k \leq k_{\text{last}}$. Note that the phase $\phi(0)$ corresponding to $V_k(x)$ is equal to the phase $\phi(\tilde{E}_k)$ arising from $V(x)$, which is not too near any $\pi(k' + 1/2)$, in view of (12). So the lemma of the previous section gives us the following estimates on $H_k(x)$ $(1 \leq k \leq k_{\text{last}})$:

$$(13) \qquad |H_k(x)| \leq C_* \lambda_k^{-\frac{N''}{4}} \quad \text{for} \quad x < x_{\text{left}}(\tilde{E}_k) - \hat{c}B_k$$

$$(14)\quad \left(Av_{|x-\tilde{x}|<\hat{c}B_k}\ |H_k(x)|^2\right)^{1/2} \le C_* \lambda_k^{4\varepsilon-2/7} \quad \text{for}$$

$$x_{\text{left}}(\tilde{E}_k) - \hat{c}B_k \le \tilde{x} \le x_{\text{rt}}(\tilde{E}_k) + \hat{c}B_k$$

$$(15)\qquad |H_k(x)| \le C_* \lambda_k^{4\varepsilon-2/7} \quad \text{for} \quad x > x_{\text{rt}}(\tilde{E}_k) + \hat{c}B_k\ .$$

These estimates in turn trivially imply the following.

$$(16)\qquad |H_k(x)| \le C_* \Lambda^{-\frac{N''}{1000}} \quad \text{for} \quad x < x_{\text{left}}(\tilde{E}_1) - \hat{c}B(x_0) \quad (1 \le k < k_{\text{last}})$$

$$(17)\quad \left(Av_{|x-\tilde{x}|<\hat{c}B(x_0)}\ |H_k(x)|^2\right)^{1/2} \le C_* \lambda_k^{-\frac{2}{7}+4\varepsilon}$$

$$\text{for all} \quad \tilde{x} \in \mathbb{R} \quad (1 \le k < k_{\text{last}})$$

$$(17\text{bis})\qquad |H_k(x)| \le C_* \lambda_k^{-\frac{2}{7}+4\varepsilon} \quad \text{for} \quad x > x_{\text{rt}}(\tilde{E}_1) + \hat{c}B(x_0) \quad (1 \le k < k_{\text{last}})\ .$$

Next, applying Lemma 1 (A) and the Lemma of the previous section, we learn the following:

Case I: Suppose $\min_{k\in\mathbb{Z}}\ |\phi(0) - \pi(k+1/2)| \ge \overline{C}_\# \Lambda^{-1}$. Then

$$(18)\qquad |H_0(x)| \le C_* \Lambda^{-\frac{N''}{1000}} \quad \text{for} \quad x < x_{\text{left}}(0) - \hat{c}B(x_{\text{left}}(0))\ .$$

$$(19)\quad \left(Av_{|x-\tilde{x}|<\hat{c}B(\tilde{x})}\ |H_0(x)|^2\right)^{1/2}$$

$$\le C_* \Lambda^{4\varepsilon-\frac{9}{7}} \int_{I_{\text{BVP}} \cap (-\infty, \tilde{x}+C_\# \hat{c}B(\tilde{x})]} (-V(x))_+^{1/2} dx + C_* \Lambda^{-\frac{N''}{1000}}$$

$$\text{for} \quad x_{\text{left}}(0) - \hat{c}B(x_{\text{left}}(0)) \le \tilde{x} \le x_{\text{rt}}(0) + \hat{c}B(x_{\text{rt}}(0))\ .$$

$$(20)\quad |H_0(x)| \le C_* \Lambda^{4\varepsilon-9/7} \int_{I_{\text{BVP}}} (-V(x'))_+^{1/2} dx' + C_* \Lambda^{-\frac{N''}{1000}}$$

$$\text{for} \quad x > x_{\text{rt}}(0) + \hat{c}B(x_{\text{rt}}(0))\ .$$

Case II: Suppose $\min_{k\in\mathbb{Z}}\ |\phi(0) - \pi(k+1/2)| \le \overline{C}_\# \Lambda^{-1}$. Then

$$(21)\qquad |H_0(x)| \le C_* \Lambda^{-\frac{N''}{1000}} \quad \text{for} \quad x < x_{\text{left}}(0) - \hat{c}B(x_{\text{left}}(0))\ .$$

$$(22)\quad \left(Av_{|x-\tilde{x}|<\hat{c}B(\tilde{x})}\ |H_0(x)|^2\right)^{1/2} \le C_*\Lambda^{-1}\int_{I_{\mathrm{BVP}}\cap(-\infty,\tilde{x}+C_\#\hat{c}B(\tilde{x})]} (-V(x))_+^{1/2}dx + C_*\Lambda^{-\frac{N''}{1000}}$$

$$\text{for}\quad x_{\mathrm{left}}(0)-\hat{c}B(x_{\mathrm{left}}(0)) \le \tilde{x} \le x_{\mathrm{rt}}(0)+\hat{c}B(x_{\mathrm{rt}}(0))\ .$$

$$(23)\quad |H_0(x)| \le C_*\Lambda^{-1}\int_{I_{\mathrm{BVP}}} (-V(x'))_+^{1/2}dx' + C_*\Lambda^{-\frac{N''}{1000}}$$

$$\text{for}\quad x > x_{\mathrm{rt}}(0)+\hat{c}B(x_{\mathrm{rt}}(0))\ .$$

Applying Lemma 1(C) and the lemma from the section on the microlocalized density near the minimum of the potential, and recalling (12), we obtain

$$(24)\qquad |H_{k_{\mathrm{last}}}(x)| \le C_*(\lambda(x_0))^{1-N'} \quad \text{if} \quad x < x_0 - (\lambda(x_0))^{\varepsilon-\frac{18}{43}}B(x_0)$$

$$(25)\quad |H_{k_{\mathrm{last}}}(x)| \le C_*(\lambda(x_0))^{2\cdot(-\frac{18}{43})+1} = C_*\lambda^{\frac{7}{43}}(x_0)$$

$$\text{if} \quad |x-x_0| < \lambda^{\varepsilon-\frac{18}{43}}(x_0)B(x_0)$$

$$(26)\qquad |H_{k_{\mathrm{last}}}(x)| \le C_*\lambda^{-\frac{7}{43}}(x_0) \quad \text{if} \quad x > x_0 + (\lambda(x_0))^{\varepsilon-\frac{18}{43}}B(x_0)\ .$$

From (24) we get at once

$$(27)\qquad |H_{k_{\mathrm{last}}}(x)| \le C_*\Lambda^{-\frac{N''}{1000}} \quad \text{if} \quad x < x_0 - C_\#\hat{c}B(x_0)\ .$$

For any $\tilde{x}\in\mathbb{R}$, (24)...(26) imply

$$\int_{|x-\tilde{x}|<\hat{c}B(x_0)} |H_{k_{\mathrm{last}}}(x)|^2dx \le \int_{|x-x_0|<\lambda^{\varepsilon-18/43}(x_0)B(x_0)} |H_{k_{\mathrm{last}}}(x)|^2dx + \int_{\{|x-\tilde{x}|<\hat{c}B(x_0)\}\setminus\{|x-x_0|<\lambda^{\varepsilon-18/43}(x_0)B(x_0)\}} |H_{k_{\mathrm{last}}}(x)|^2dx$$

$$\le C_*\left(\lambda^{\frac{7}{43}}(x_0)\right)^2\cdot\lambda^{\varepsilon-\frac{18}{43}}(x_0)B(x_0)+C_*\left(\lambda^{-\frac{7}{43}}(x_0)\right)^2B(x_0) \le C_*\lambda^{\varepsilon-\frac{4}{43}}(x_0)B(x_0)\ ,$$

so

$$(28)\qquad \left(Av_{|x-\tilde{x}|<\hat{c}B(x_0)}\ |H_{k_{\mathrm{last}}}(x)|^2\right)^{1/2} \le C_*\lambda^{\varepsilon-\frac{2}{43}}(x_0)\ .$$

From (26) we get at once

$$|H_{k_{\text{last}}}(x)| \le C_* \lambda^{-\frac{7}{43}}(x_0) \quad \text{if} \quad x > x_0 + \hat{c}B(x_0) \; . \tag{29}$$

We use (16), (17), (17 bis) and (27), (28), (29) to control $\sum\limits_{1\le k\le k_{\text{last}}} H_k(x)$.

$$\text{If} \quad x \le x_{\text{left}}(\tilde{E}_1) - \hat{c}B(x_0) \;, \quad \text{then} \quad \Big| \sum_{1\le k\le k_{\text{last}}} H_k(x) \Big| \le \Lambda^{-10^{-4}N''} \tag{30}$$

$$\text{by (16), (27) and} \quad 2^{-k_{\text{last}}} \sim (\lambda(x_0))^{-\frac{18}{43}} \ge \Lambda^{-1000K} \; .$$

(The last estimate follows from (Y7)).

If $x \ge x_{\text{rt}}(\tilde{E}_1) + \hat{c}B(x_0)$, then $\Big| \sum\limits_{1\le k\le k_{\text{last}}} H_k(x) \Big| \le \sum\limits_{1\le k< k_{\text{last}}} C_* \lambda_k^{4\varepsilon - 2/7} + C_* \lambda^{-\frac{7}{43}}(x_0)$.
Recall that $\lambda_k \sim 2^{-2k}\lambda(x_0)$, which decreases geometrically to $\lambda_{k_{\max}} \sim (\lambda^{-\frac{18}{43}}(x_0))^2 \lambda(x_0) \sim \lambda^{+\frac{7}{43}}(x_0)$, so $\sum\limits_k \lambda_k^{4\varepsilon - \frac{2}{7}} \le C_*(\lambda(x_0))^{\varepsilon - \frac{2}{43}}$. Thus

$$\Big| \sum_{1\le k\le k_{\text{last}}} H_k(x) \Big| \le C_* \big(\lambda(x_0)\big)^{\varepsilon - \frac{2}{43}} \quad \text{for} \quad x > x_{\text{rt}}(\tilde{E}_1) + \hat{c}B(x_0) \; . \tag{31}$$

From (17) and (28), we get for any $\tilde{x} \in \mathbb{R}$ that

$$\Big(Av_{|x-\tilde{x}|<\hat{c}B(x_0)} \Big| \sum_{1\le k\le k_{\text{last}}} H_k(x) \Big|^2 \Big)^{1/2} \le \sum_{1\le k< k_{\text{last}}} C_* \lambda_k^{4\varepsilon - \frac{2}{7}} + C_* \lambda^{\varepsilon - \frac{2}{43}}(x_0) \; .$$

As in the proof of (31), the right-hand side is dominated by $(\lambda(x_0))^{\varepsilon - \frac{2}{43}}$, so

$$\Big(Av_{|x-\tilde{x}|<\hat{c}B(x_0)} \Big| \sum_{1\le k\le k_{\text{last}}} H_k(x) \Big|^2 \Big)^{1/2} \le C_* \big(\lambda(x_0)\big)^{\varepsilon - \frac{2}{43}} \; . \tag{32}$$

From (30), (31), (32) we conclude that:

$$\Big| \sum_{1\le k\le k_{\max}} H_k(x) \Big| \le C_* \Lambda^{-10^{-4}N''} \quad \text{if} \quad x < x_{\text{left}}(0) - \hat{c}B(x_{\text{left}}(0)) \; . \tag{33}$$

$$\Big(Av_{|x-\tilde{x}|<\hat{c}B(\tilde{x})} \Big| \sum_{1\le k\le k_{\max}} H_k(x) \Big|^2 \Big)^{1/2} \le C_* \Lambda^{-10^{-4}N''} \tag{34}$$

$$\text{if} \quad x_{\text{left}}(0) - \hat{c}B(x_{\text{left}}(0)) \le \tilde{x} < x_{\text{left}}(\tilde{E}_1) - C_\# \hat{c}B(x_0)$$

$$(35)\quad \Big(Av_{|x-\tilde{x}|<\hat{c}B(\tilde{x})}\Big|\sum_{1\le k\le k_{\max}} H_k(x)\Big|^2\Big)^{1/2} \le C_*\big(\lambda(x_0)\big)^{\varepsilon-\frac{2}{43}}$$
$$\text{if}\quad x_{\text{left}}(\tilde{E}_1)-C_\#\hat{c}B(x_0)\le\tilde{x}<x_{\text{rt}}(0)+\hat{c}B(x_{\text{rt}}(0))$$

$$(36)\quad \Big|\sum_{1\le k\le k_{\max}} H_k(x)\Big| \le C_*\big(\lambda(x_0)\big)^{\varepsilon-\frac{2}{43}} \quad \text{if}\quad x>x_{\text{rt}}(0)+\hat{c}B(x_{\text{rt}}(0))\ .$$

If $x_{\text{left}}(\tilde{E}_1)-C_\#\hat{c}B(x_0)\le\tilde{x}$, then $\lambda(x_0)\le C_*\int_{x_{\text{left}}(\tilde{E}_1)-2C_\#\hat{c}B(x_0)}^{x_{\text{left}}(\tilde{E}_1)-C_\#\hat{c}B(x_0)}\big(-V(x)\big)_+^{1/2}dx$, so

$$\lambda^{\varepsilon-\frac{2}{43}}(x_0)\le C_*\big(\lambda(x_0)\big)^{\varepsilon-\frac{45}{43}}\int_{I_{\text{BVP}}\cap(-\infty,\tilde{x}]}\big(-V(x)\big)_+^{1/2}dx$$
$$\le C_*\Lambda^{\varepsilon-\frac{45}{43}}\int_{I_{\text{BVP}}\cap(-\infty,\tilde{x}]}\big(-V(x)\big)_+^{1/2}dx\ .$$

Therefore (33)...(36) imply:

$$(37)\quad \Big|\sum_{1\le k\le k_{\text{last}}} H_k(x)\Big| \le C_*\Lambda^{-10^{-4}N''} \quad \text{if}\quad x\le x_{\text{left}}(0)-\hat{c}B(x_{\text{left}}(0))\ .$$

$$(38)\quad \Big(Av_{|x-\tilde{x}|<\hat{c}B(\tilde{x})}\Big|\sum_{1\le k\le k_{\text{last}}} H_k(x)\Big|^2\Big)^{1/2}$$
$$\le C_*\Lambda^{-10^{-4}N''}+C_*\Lambda^{\varepsilon-\frac{45}{43}}\int_{I_{\text{BVP}}\cap(-\infty,\tilde{x}+C_\#\hat{c}B(\tilde{x})]}\big(-V(x)\big)_+^{1/2}dx$$
$$\text{if}\quad x_{\text{left}}(0)-\hat{c}B(x_{\text{left}}(0))\le\tilde{x}\le x_{\text{rt}}(0)+\hat{c}B(x_{\text{rt}}(0))\ .$$

$$(39)\quad \Big|\sum_{1\le k\le k_{\text{last}}} H_k(x)\Big| \le C_*\Lambda^{-10^{-4}N''}+C_*\Lambda^{\varepsilon-\frac{45}{43}}\int_{I_{\text{BVP}}}\big(-V(x')\big)_+^{1/2}dx'$$
$$\text{if}\quad x>x_{\text{rt}}(0)+\hat{c}B(x_{\text{rt}}(0))\ .$$

Since $H(x)=H_0(x)+\big(\sum_{1\le k\le k_{\text{last}}} H_k(x)\big)$, estimates (18)...(23) and (37)...(39) yield the following results.

Case I: Suppose $\min_{k\in\mathbb{Z}}|\phi(0)-\pi(k+1/2)|\ge\overline{C}_\#\Lambda^{-1}$. Then

$$(40)\qquad |H(x)|\le C_*\Lambda^{-10^{-4}N''} \quad \text{for}\quad x\le x_{\text{left}}(0)-\hat{c}B(x_{\text{left}}(0))$$

$$(41)\quad \left(Av_{|x-\tilde{x}|<\hat{c}B(\tilde{x})}|H(x)|^2\right)^{1/2}$$
$$\leq C_*\Lambda^{-10^{-4}N''} + C_*\Lambda^{\varepsilon-\frac{45}{43}}\int_{I_{\mathrm{BVP}}\cap(-\infty,\tilde{x}+C_\#\hat{c}B(\tilde{x})]}\left(-V(x)\right)_+^{1/2}dx$$
$$\text{for}\quad x_{\mathrm{left}}(0)-\hat{c}B(x_{\mathrm{left}}(0))\leq\tilde{x}\leq x_{\mathrm{rt}}(0)+\hat{c}B(x_{\mathrm{rt}}(0))$$

$$(42)\quad |H(x)|\leq C_*\Lambda^{-10^{-4}N''}+C_*\Lambda^{\varepsilon-\frac{45}{43}}\int_{I_{\mathrm{BVP}}}\left(-V(x')\right)_+^{1/2}dx'$$
$$\text{for}\quad x\geq x_{\mathrm{rt}}(0)+\hat{c}B(x_{\mathrm{rt}}(0))\ .$$

Case II: Suppose $\min_{k\in\mathbb{Z}}|\phi(0)-\pi(k+1/2)|\leq\overline{C}_\#\Lambda^{-1}$. Then

$$(43)\qquad |H(x)|\leq C_*\Lambda^{-10^{-4}N''}\quad\text{for}\quad x\leq x_{\mathrm{left}}(0)-\hat{c}B(x_{\mathrm{left}}(0))$$

$$(44)\quad \left(Av_{|x-\tilde{x}|<\hat{c}B(\tilde{x})}|H(x)|^2\right)^{1/2}$$
$$\leq C_*\Lambda^{-10^{-4}N''}+C_*\Lambda^{-1}\int_{I_{\mathrm{BVP}}\cap(-\infty,\tilde{x}+C_\#\hat{c}B(\tilde{x})]}\left(-V(x)\right)_+^{1/2}dx$$
$$\text{for}\quad x_{\mathrm{left}}(0)-\hat{c}B(x_{\mathrm{left}}(0))\leq\tilde{x}\leq x_{\mathrm{rt}}(0)+\hat{c}B(x_{\mathrm{rt}}(0))$$

$$(45)\quad |H(x)|\leq C_*\Lambda^{-10^{-4}N''}+C_*\Lambda^{-1}\int_{I_{\mathrm{BVP}}}\left(-V(x')\right)_+^{1/2}dx'$$
$$\text{for}\quad x\geq x_{\mathrm{rt}}(0)+\hat{c}B(x_{\mathrm{rt}}(0))\ .$$

After a change of notation, we may drop the assumptions $K>100$, $\varepsilon<\frac{1}{1000K}$, $N>\frac{K}{\varepsilon^{50}}$; and we may replace $10^{-4}N''$ in (40)...(45) by N. To do this, we merely pick new K, ε, N by setting $K_*=K+100$, $\varepsilon_*=\min(\varepsilon,\frac{1}{2000K_*})$, $N_*=(N+1)\cdot\frac{K_*}{(\varepsilon_*)^{50}}$. Then K_*, ε_*, N_* satisfy $K_*>100$, $\varepsilon_*<\frac{1}{1000K_*}$, $N_*>\frac{K_*}{(\varepsilon_*)^{50}}$. The hypotheses of this section hold with K_*, ε_*, N_* in place of K, ε, N.

The quantity $10^{-4}N''$ arising from N_* is greater than N. Thus, we obtain the result of the next section, which is our main theorem on the density for a one-dimensional potential.

THE WKB DENSITY THEOREM IN ONE DIMENSION

Set-Up: We are given positive numbers ε, K, N, $\hat{c}$; two intervals $I \subset I_{\mathrm{BVP}}$ (possibly unbounded); a point $x_0 \in I$; a potential $V(x)$ defined on I_{BVP}; and two positive functions $S(x)$, $B(x)$ defined on I. Our assumptions are as follows.

Assumptions Concerning $V(x)$, $S(x)$, $B(x)$ on I.

(Z0) If $x, y \in I$ and $|x-y| < cB(x)$, then $c < \frac{B(y)}{B(x)} < C$ and $c < \frac{S(y)}{S(x)} < C$.

(Z1) If $x \in I$ and $\alpha \geq 0$, then $\left|\left(\frac{d}{dx}\right)^{\alpha} V(x)\right| \leq C_\alpha S(x) B^{-\alpha}(x)$.

(Z2) The set $\{x \in I \mid V(x) < 0\}$ is a non-empty interval $(x_{\mathrm{left}}, x_{\mathrm{rt}})$, with $\mathrm{dist}\,(x_{\mathrm{left}}, \partial I) > cB(x_{\mathrm{left}})$ and $\mathrm{dist}\,(x_{\mathrm{rt}}, \partial I) > cB(x_{\mathrm{rt}})$.

(Z3) We have $V(x_0) < -cS(x_0)$, $V'(x_0) = 0$; and for $|x - x_0| \leq c_1 B(x_0)$ we have $V''(x) \geq cS(x_0)B^{-2}(x_0)$.

(Z4) For $x_{\mathrm{left}} \leq x \leq x_0 - c_1 B(x_0)$ we have $-V'(x) > cS(x)B^{-1}(x)$; and for $x_0 + c_1 B(x_0) \leq x \leq x_{\mathrm{rt}}$ we have $+V'(x) > cS(x)B^{-1}(x)$.

Define $\lambda(x) = S^{1/2}(x)B(x)$ for $x \in I$, and set

$$\Lambda = \left(\int_{x_{\mathrm{left}}}^{x_{\mathrm{rt}}} \frac{dx}{\lambda(x)B(x)}\right)^{-1}.$$

Assumptions Concerning $V(x)$ on all of I_{BVP}.

(Z5) We have $V(x) > 0$ for all $x \in I_{\mathrm{BVP}} \setminus [x_{\mathrm{left}}, x_{\mathrm{rt}}]$.

(Z6) For all $x \in I_{\mathrm{BVP}}$ with $x < x_{\mathrm{left}} - \Lambda^K B(x_{\mathrm{left}})$, we have $V(x) \geq \frac{1000}{|x - x_{\mathrm{left}}|^2}$; and for all $x \in I_{\mathrm{BVP}}$ with $x > x_{\mathrm{rt}} + \Lambda^K B(x_{\mathrm{rt}})$, we have $V(x) \geq \frac{1000}{|x - x_{\mathrm{rt}}|^2}$.

Polynomial Growth Assumptions on $S(x)$, $B(x)$, I.

(Z7) We have $\max_{x \in I} B(x) < \Lambda^K \min_{x \in I} B(x)$; $\max_{x \in I} S(x) < \Lambda^K \min_{x \in I} S(x)$; and $|I| < \Lambda^K \cdot \min_{x \in I} B(x)$.

Smallness of the Constant $\hat{c}$.

(Z8) The constant $\hat{c}$ is bounded above by a certain small, positive number determined by ε, K, N, c, C, c_1, C_α.

The WKB Hypothesis.

(Z9) Λ is bounded below by a certain large, positive number determined by ε, K, N, c, C, c_1, $\hat{c}$, C_α.

Let E_k and $u_k(x)$ be the eigenvalues and (normalized) eigenfunctions of $-\frac{d^2}{dx^2}+V(x)$ on I_{BVP}, with Dirichlet or Neumann boundary conditions. Then define the density $\rho(x)$ and its semiclassical approximation $\rho_{sc}(x)$ on I_{BVP} by setting:

$$\rho(x)=\sum_{E_k\le 0}|u_k(x)|^2 \quad (x\in I_{\text{BVP}})\ ;$$

$$\rho_{sc}(x)=\frac{1}{\pi}\big(-V(x)\big)_+^{1/2}-\frac{\big(-V(x)\big)_+^{-1/2}}{\int_{I_{\text{BVP}}}\big(-V(y)\big)_+^{-1/2}dy}\chi_-\Big(\frac{1}{\pi}\int_{I_{\text{BVP}}}\big(-V(y)\big)_+^{1/2}dy-\frac{1}{2}\Big)$$
$$(x\in I_{\text{BVP}})\ .$$

Recall that $t_+^a=t^a$ for $t>0$, $t_+^a=0$ for $t\le 0$, and that $\chi_-(t)=x-k-\frac{1}{2}$ for $k=$ (largest integer $\le x$).

For any $x\in\mathbb{R}$, define

$$H(x)=\int_{I_{\text{BVP}}\cap(-\infty,x]}(\rho(\overline{x})-\rho_{sc}(\overline{x}))d\overline{x}\ .$$

WKB Density Theorem. *Assume (Z0)…(Z9).*

Case I: Suppose $\min_{k\in\mathbb{Z}}\left|\int_{I_{\text{BVP}}}\big(-V(y)\big)_+^{1/2}dy-\pi(k+1/2)\right|>\overline{C}\Lambda^{-1}$. *Then*

(A) $$|H(x)|\le\Lambda^{-N}\quad \textit{for}\quad x\le x_{\text{left}}-\hat{c}B(x_{\text{left}})\ .$$

(B) $$\Big(Av_{|x-\tilde{x}|<\hat{c}B(\tilde{x})}|H(x)|^2\Big)^{1/2}$$
$$\le\Lambda^{-N}+C_*\Lambda^{\varepsilon-\frac{45}{43}}\int_{I_{\text{BVP}}\cap(-\infty,\tilde{x}+\overline{C}\hat{c}B(\tilde{x})]}\big(-V(y)\big)_+^{1/2}dy$$
$$\textit{for}\quad x_{\text{left}}-\hat{c}B(x_{\text{left}})\le\tilde{x}\le x_{\text{rt}}+\hat{c}B(x_{\text{rt}})\ .$$

(C) $$|H(x)| \leq \Lambda^{-N} + C_* \Lambda^{\varepsilon - \frac{45}{43}} \int_{I_{\mathrm{BVP}}} \big(-V(y)\big)_+^{1/2} dy \quad \textit{for} \quad x \geq x_{\mathrm{rt}} + \hat{c} B(x_{\mathrm{rt}}) \ .$$

Case II: Suppose instead $\min_{k\in\mathbb{Z}} \left| \int_{I_{\mathrm{BVP}}} \big(-V(y)\big)_+^{1/2} dy - \pi(k+1/2) \right| \leq \overline{C}\Lambda^{-1}$. *Then*

(A) $$|H(x)| \leq \Lambda^{-N} \quad \textit{for} \quad x \leq x_{\mathrm{left}} - \hat{c} B(x_{\mathrm{left}}) \ .$$

(B) $$\Big(Av_{|x-\tilde{x}|<\hat{c}B(\tilde{x})} |H(x)|^2\Big)^{1/2} \leq \Lambda^{-N} + C_* \Lambda^{-1} \int_{I_{\mathrm{BVP}} \cap (-\infty, \tilde{x} + \overline{C}\hat{c}B(\tilde{x})]} \big(-V(y)\big)_+^{1/2} dy$$
$$\textit{for} \quad x_{\mathrm{left}} - \hat{c} B(x_{\mathrm{left}}) \leq \tilde{x} \leq x_{\mathrm{rt}} + \hat{c} B(x_{\mathrm{rt}}) \ .$$

(C) $$|H(x)| \leq \Lambda^{-N} + C_* \Lambda^{-1} \int_{I_{\mathrm{BVP}}} \big(-V(y)\big)_+^{1/2} dy \quad \textit{for} \quad x \geq x_{\mathrm{rt}} + \hat{c} B(x_{\mathrm{rt}}) \ .$$

Here $\overline{C}$ *depends only on* ε, K, N, c, C, c_1, C_α; *and* C_* *depends only on* ε, K, N, c, C, c_1, $\hat{c}$, C_α.

Remarks. The exponent $\varepsilon - \frac{45}{43}$ in *Case I* is not important to us, and surely not optimal. Any exponent strictly less than -1 would serve our purpose.

THE DENSITY FOR DEGENERATE ONE-DIMENSIONAL POTENTIALS I

In the next sections, we prove crude results on the density $\rho(x)$ in various degenerate cases in which the hypotheses in the preceding section break down. We begin by treating a potential $V(x)$ whose minimum $V(x_0)$ is negative but has relatively small absolute value.

Set-up. We are given positive numbers ε, K, N, S, B; a potential $V(x)$ defined on a (possibly unbounded) interval I_{BVP}; and a point $x_0 \in I_{\mathrm{BVP}}$. Our assumptions are as follows.

Hypotheses.

(Z0*) $I = \{x\colon |x - x_0| < cB\} \subset I_{\mathrm{BVP}}$.

(Z1*) $\left|\left(\frac{d}{dx}\right)^\alpha V(x)\right| \leq C_\alpha S B^{-\alpha}$ on I.

(Z2*) $V''(x) \geq c' S B^{-2}$ on I.

Set $\lambda = S^{1/2} B$, and make the following further assumptions.

(Z3*) $V'(x_0) = 0$ and $-\lambda^{-\frac{36}{43}} S \leq V(x_0) < 0$.

(Z4*) For $x \in I_{\mathrm{BVP}} \backslash I$ we have $V(x) > 0$.

(Z5*) For $x \in I_{\mathrm{BVP}}$ with $|x - x_0| > \frac{1}{2}\lambda^K B$, we have $V(x) \geq \frac{1000}{|x-x_0|^2}$.

(Z6*) λ is bounded below by a certain large, positive number determined by ε, K, N, c, c', C_α.

Let E_k, $u_k(x)$ be the eigenvalues and (normalized) eigenfunctions for $-\frac{d^2}{dx^2} + V(x)$ on I_{BVP}, with Dirichlet or Neumann boundary conditions.

As in the previous section, define the density $\rho(x)$ and its semiclassical approximation $\rho_{sc}(x)$ on I_{BVP}, by the formulas

$$\rho(x) = \sum_{E_k \le 0} |u_k(x)|^2$$

$$\rho_{sc}(x) = \frac{1}{\pi}\big(-V(x)\big)_+^{1/2} - \frac{\big(V(x)\big)_+^{-1/2}}{\int_{I_{\mathrm{BVP}}}\big(-V(y)\big)_+^{-1/2}dy}\ \chi_-\big(\frac{1}{\pi}\int_{I_{\mathrm{BVP}}}\big(-V(y)\big)_+^{1/2}dy - \frac{1}{2}\big)$$

Then set

$$H(x) = \int_{I_{\mathrm{BVP}}\cap(-\infty,x]} \big(\rho(\overline{x}) - \rho_{sc}(\overline{x})\big)\, d\overline{x}\ .$$

First Degenerate Density Lemma. *Assume (Z0*)...(Z6*).*

CASE I: Suppose $min_{k\in\mathbb{Z}}\Big|\int_{I_{\mathrm{BVP}}}\big(-V(y)\big)_+^{1/2}dy - \pi(k+1/2)\Big| \ge \overline{C}\lambda^{-1}$*. Then*

(A) $|H(x)| \le \lambda^{-N}$ *if* x *lies to the left of* I*.*

(B) $\big(Av_I|H|^2\big)^{1/2} \le C_*\lambda^{\varepsilon-2/43}$*.*

(C) $|H(x)| \le C_*\lambda^{\varepsilon-2/43}$ *if* x *lies to the right of* I*.*

CASE II: Suppose instead $\min_{k\in\mathbb{Z}}\Big|\int_{I_{\mathrm{BVP}}}\big(-V(y)\big)_+^{1/2}dy - \pi(k+1/2)\Big| \le \overline{C}\lambda^{-1}$*. Then*

(A) $|H(x)| \le \lambda^{-N}$ *if* x *lies to the left of* I*.*

(B) $(Av_I|H|^2)^{-1/2} \le C_*$*.*

(C) $|H(x)| \le C_*$ *if* x *lies to the right of* I*.*

The constants C_**,* $\overline{C}$ *depend only on* ε*,* K*,* N*,* c*,* c'*,* C_α*.*

Proof. Set $\tau \sim \big(\lambda(x_0)\big)^{-\frac{18}{43}}$, and pick $g(E)$ as in the section on the microlocalized density near the minimum of the potential, with $g(E) = 1$ for $V(x_0) \le E \le 0$. We can take $g(E)$ so that the constants $\hat{C}_\beta$ used to control it depend only on ε, K, N, c, c', C_α. Then $\rho(x,g) = \rho(x)$ because $g(E_k) = 1$ for all the eigenvalues $E_k \le 0$. Also, $\rho_{sc}(x,g) = \rho_{sc}(x)$ because $g(E) = 1$ for $V(x_0) \le E \le 0$. Hence our present function $H(x)$ is the same as the $H(x)$ in the Lemma in the section on the microlocalized density near the minimum of the potential. That lemma yields:

(a) $|H(x)| \le C_*\lambda^{1-N'}$ if $x \in I_{\mathrm{BVP}}$, $x < x_0 - \lambda^{\varepsilon-\frac{18}{43}}B$

(b) $|H(x)| \le C_* \lambda^{\frac{7}{43}}$ if $|x - x_0| < \lambda^{\varepsilon - \frac{18}{43}} B$.

(c) $|H(x)| \le C_* \lambda^{-\frac{7}{43}}$ if $x \in I_{\mathrm{BVP}}$, $x > x_0 + \lambda^{\varepsilon - \frac{18}{43}} B$ in *Case I*.

(d) $|H(x)| \le C_*$ $x \in I_{\mathrm{BVP}}$, $x > x_0 + \lambda^{\varepsilon - \frac{18}{43}} B$ in *Case II*.

The restriction to $x \in I_{\mathrm{BVP}}$ is irrelevant, since $H(x) = H(\inf I_{\mathrm{BVP}})$ for $x < \inf I_{\mathrm{BVP}}$ and $H(x) = H(\sup I_{\mathrm{BVP}})$ for $x >$ $\sup I_{\mathrm{BVP}}$.

¿From (a) we get Case I (A) and Case II (A), provided we enlarge the N in the Lemma on the microlocalized density near the minimum of the potential. ¿From (c) we get Case I (C), and from (d) we get Case II (C). To handle (B), we write

$$
(1) \qquad \int_I |H(x)|^2 dx \le \int_{|x-x_0|<\lambda^{\varepsilon - \frac{18}{43}} B} |H(x)|^2 dx + \int_{I \setminus \{|x-x_0|<\lambda^{\varepsilon - \frac{18}{43}} B\}} |H(x)|^2 dx \; .
$$

In Case I this yields

$$
\int_I |H(x)|^2 dx \le C_* \lambda^{\frac{14}{43}} \cdot \lambda^{\varepsilon - \frac{18}{43}} B + C_* \lambda^{-\frac{14}{43}} \cdot B \le C_* \lambda^{\varepsilon - \frac{4}{43}} B \; ,
$$

proving the assertion of Case I (B).

In Case II, (1) yields instead

$$
\int_I |H(x)|^2 dx \le C_* \lambda^{\frac{14}{43}} \cdot \lambda^{\varepsilon - \frac{18}{43}} B + C_* B \le C_*' B \; ,
$$

proving the assertion of Case II (B). ■

Remark. Again, the precise exponents in Case I (B), (C) are irrelevant to us, and are surely not optimal.

THE DENSITY FOR DEGENERATE ONE-DIMENSIONAL POTENTIALS II

In this section we derive (very) crude results for the density without making any polynomial growth assumptions on the weight functions $S(x)$, $B(x)$ or the interval I. These results will be used later for ODE arising from three-dimensional problems, with angular momentum ℓ in the range [Large Constant, Z^{ε}]. The precise formulation is as follows.

Set-Up. We are given a potential $V(x)$ defined on a (possibly unbounded) interval I_{BVP}; positive functions $S(x)$, $B(x)$, defined on a subinterval $I \subset I_{\text{BVP}}$; a point $x_{\text{crit}} \in I_{\text{BVP}}$; an energy $E_{\text{crit}} \leq 0$; and a number δ strictly between 0 and 1.

Assumptions.

($\overline{\text{Z0}}$) For $x, y \in I$ with $|x-y| < cB(x)$, we have $c < \frac{B(y)}{B(x)} < C$ and $c < \frac{S(y)}{S(x)} < C$, and $|I| > cB(x)$.

($\overline{\text{Z1}}$) For $x \in I$ and $\alpha \geq 0$ we have $\left|\left(\frac{d}{dx}\right)^{\alpha} V(x)\right| \leq C_{\alpha} S(x) B^{-\alpha}(x)$.

($\overline{\text{Z2}}$) For $E_{\text{crit}} \leq E \leq 0$, the set $\{x \in I_{\text{BVP}} \mid V(x) \leq E\}$ is a non-empty interval $(x_{\text{left}}(E), x_{\text{rt}}(E))$ contained in I, with $\operatorname{dist}(x_{\text{left}}(E), \partial I) > cB(x_{\text{left}}(E))$ and $\operatorname{dist}(x_{\text{rt}}(E), \partial I) > cB(x_{\text{rt}}(E))$.

($\overline{\text{Z3}}$) For $E_{\text{crit}} \leq E \leq 0$, we have $-V'(x) \geq cS(x)B^{-1}(x)$ for $x \in [x_{\text{left}}(E), x_{\text{left}}(E) + c_1 B(x_{\text{left}}(E))]$ and $+V'(x) \geq cS(x)B^{-1}(x)$ for $x \in [x_{\text{rt}}(E) - c_1 B(x_{\text{rt}}(E)), x_{\text{rt}}(E)]$.

($\overline{\text{Z4}}$) For $E_{\text{crit}} \leq E \leq 0$, we have $cS(x) < E - V(x) < CS(x)$ for $x \in [x_{\text{left}}(E) + c_1 B(x_{\text{left}}(E)), x_{\text{rt}}(E) - c_1 B(x_{\text{rt}}(E))]$

($\overline{\text{Z5}}$) $V(x)$ is decreasing and C^{∞} on $I_{\text{BVP}}^{\text{interior}} \cap (-\infty, x_{\text{left}}(0)]$.

($\overline{\text{Z6}}$) For $E_{\text{crit}} \leq E \leq 0$, we have $x_{\text{left}}(E) + cB(x_{\text{left}}(E)) \leq x_{\text{crit}}$.

($\overline{\text{Z7}}$) For $E_{\text{crit}} \leq E \leq 0$, we have

$$\int_{I_{\text{BVP}} \cap (-\infty, x_{\text{crit}}]} \left(E - V(t)\right)_{+}^{-1/2} dt \leq \delta \int_{I_{\text{BVP}}} \left(E - V(t)\right)_{+}^{-1/2} dt$$

(Z$\overline{8}$) $\Lambda \equiv \left(\int_{x_{\text{left}}(0)}^{x_{\text{rt}}(0)} \frac{dx}{\lambda(x)B(x)}\right)^{-1}$ is greater than a certain large, positive number determined by c, C, c_1, C_α above.

Here, $\lambda(x) = S^{1/2}(x)B(x)$ as usual.

If $\delta << 1$, then assumption (Z$\overline{7}$) says that in the semiclassical approximation, eigenfunctions $u_k(x)$ with eigenvalues $E_k \in [E_{\text{crit}}, 0]$ are almost entirely concentrated in $(x_{\text{crit}}, \infty) \cap I_{\text{BVP}}$. We want to show that the true density

$$\rho(x) = \sum_{E_k \leq 0} |u_k(x)|^2$$

is then highly concentrated in the same interval. Here, E_k and $u_k(x)$ are the eigenvalues and (normalized) eigenfunctions of $-\frac{d^2}{dx^2} + V(x)$ with Dirichlet boundary conditions on I_{BVP}. The precise result is as follows.

Second Degenerate Density Lemma. *Assume (Z$\overline{0}$)... (Z$\overline{8}$). Then*

$$\int_{I_{\text{BVP}} \cap (-\infty, x_{\text{crit}}]} \rho(x)dx \leq C_\# + C_\# \delta \int_{I_{\text{BVP}}} \left(-V(x)\right)_+^{1/2} dx + C_\# \int_{I_{\text{BVP}}} \left(E_{\text{crit}} - V(x)\right)_+^{1/2} dx$$

and $\int_{I_{\text{BVP}}} \rho(x)dx \leq C_\# + C_\# \int_{I_{\text{BVP}}} \left(-V(x)\right)_+^{1/2} dx$, *with* $C_\#$ *depending only on* c, C, c_1, C_α *in (Z$\overline{0}$)... (Z$\overline{8}$).*

Proof. Let E_k be an eigenvalue in $[E_{\text{critical}}, 0]$, and let $u_k(x)$ be the corresponding (normalized) eigenfunction. The hypotheses (H$\overline{0}$)...(H$\overline{6}$) of Theorem 2 in the discussion of WKB Theory with weak turning points are satisfied, with E_k in place of E_0, and with constants depending only on the constants in our present hypotheses (Z$\overline{0}$)...(Z$\overline{8}$). (See "Review of Earlier Results.") Theorem 2 shows that

$$(1) \quad |u_k(x)|^2 \leq C_\# \left[\int_{I_{\text{BVP}}} \left(E_k - V(t)\right)_+^{-1/2} dt\right]^{-1} \cdot \left(E_k - V(x)\right)^{-1/2} \quad \text{for} \quad x_{\text{left}}(E_k) < x < x_{\text{rt}}(E_k)$$

and

(2)

$$|u_k(x)|^2 \le C_\# \Big[\int_{I_{\rm BVP}} \big(E_k - V(t)\big)_+^{-1/2} dt\Big]^{-1} \cdot \Big[S(x_{\rm left}(E_k)) \cdot \lambda^{-2/3}(x_{\rm left}(E_k))\Big]^{-1/2}$$

$$\cdot \exp\Big(-c_\# \lambda^{2/3}(x_{\rm left}(E_k)) \cdot \frac{(x_{\rm left}(E_k) - x)}{B(x_{\rm left}(E_k))}\Big)$$

$$\text{for} \quad x \in I_{\rm BVP} \ , x \le x_{\rm left}(E_k) \ .$$

We write $C_\#$, $c_\#$ etc. for constants determined by c, C, c_1, C_α in $(Z\overline{0})\ldots(Z\overline{8})$. Integrating (2), we find that

$$(3) \quad \int_{I_{\rm BVP}\cap(-\infty, x_{\rm left}(E_k)]} |u_k(x)|^2 dx$$

$$\le C_\# \big[\int_{I_{\rm BVP}} \big(E_k - V(t)\big)_+^{-1/2} dt\big]^{-1} S^{-1/2}(x_{\rm left}(E_k)) \lambda^{+1/3}(x_{\rm left}(E_k))$$

$$\cdot \frac{B(x_{\rm left}(E_k))}{\lambda^{2/3}(x_{\rm left}(E_k))} \ .$$

With c in $(Z\overline{6})$ we have

$$S^{-1/2}(x_{\rm left}(E_k)) B(x_{\rm left}(E_k)) \le C_\# \int_{x_{\rm left}(E_k)}^{x_{\rm left}(E_k) + cB(x_{\rm left}(E_k))} \big(E_k - V(t)\big)^{-1/2} dt$$

$$\le C_\# \int_{I_{\rm BVP}\cap(-\infty, x_{\rm crit}]} \big(E_k - V(t)\big)_+^{-1/2} dt \quad (\text{by } (Z\overline{6}))$$

$$\le C_\# \delta \Big[\int_{I_{\rm BVP}} \big(E_k - V(t)\big)_+^{-1/2} dt\Big] \quad (\text{by } (Z\overline{7})) \ ,$$

so estimate (3) implies

$$(4) \qquad \int_{I_{\rm BVP}\cap(-\infty, x_{\rm left}(E_k)]} |u_k(x)|^2 dx \le C_\# \delta \lambda^{-1/3}(x_{\rm left}(E_k)) \le C_\# \delta \ ,$$

because $\lambda(x_{\rm left}(E_k)) \ge c_\# \Lambda \ge c_\#$ by $(Z\overline{8})$.

We know from $(Z\overline{6})$ that $x_{\rm left}(E_k) < x_{\rm crit}$, and we know from $(Z\overline{7})$ and $\delta < 1$ that $x_{\rm crit} < x_{\rm rt}(E_k)$. Hence we may integrate (1) to prove the estimate

$$\int_{x_{\rm left}(E_k)}^{x_{\rm crit}} |u_k(x)|^2 dx$$

$$\le C_\# \Big[\int_{I_{\rm BVP}} \big(E_k - V(t)\big)_+^{-1/2} dt\Big]^{-1} \int_{I_{\rm BVP}\cap(-\infty, x_{\rm crit}]} \big(E_k - V(t)\big)_+^{-1/2} dt$$

$$\le C_\# \delta \quad \text{by } (Z\overline{7}) \ .$$

Combining this with (4), we get

$$\int_{I_{\mathrm{BVP}}\cap(-\infty,x_{\mathrm{crit}}]} |u_k(x)|^2 dx \le C_\# \delta \quad \text{if} \quad E_{\mathrm{crit}} \le E_k \le 0 \ . \tag{5}$$

If $E_k < E_{\mathrm{crit}}$, then at least we know that

$$\int_{I_{\mathrm{BVP}}\cap(-\infty,x_{\mathrm{crit}}]} |u_k(x)|^2 dx \le 1 \ , \tag{6}$$

$$\text{since} \quad u_k \quad \text{is a normalized eigenfunction} \ .$$

From (5), (6) and the definition of the density $\rho(x)$, we get

$$\begin{aligned} \int_{I_{\mathrm{BVP}}\cap(-\infty,x_{\mathrm{crit}}]} \rho(x)dx &\le C_\#\delta \cdot (\text{Number of } E_k \in [E_{\mathrm{crit}}, 0]) \\ &\quad + C_\# \cdot (\text{Number of } E_k < E_{\mathrm{crit}}) \\ &\le C_\#\delta \cdot (\text{Number of } E_k \le 0) + C_\# \cdot (\text{Number of } E_k \le E_{\mathrm{crit}}) \end{aligned} \tag{7}$$

Again we invoke Theorem 2 on WKB Theory with weak turning points. This time, for E_0 we take either 0 or E_{crit}. Hypotheses $(\mathrm{H}\overline{0})\ldots(\mathrm{H}\overline{6})$ are easily verified for both values of E_0, and the constants in $(\mathrm{H}\overline{0})\ldots(\mathrm{H}\overline{6})$ depend only on the constants in our present hypotheses $(\mathrm{Z}\overline{0})\ldots(\mathrm{Z}\overline{8})$. Theorem 2 shows that

$$\begin{aligned} &\left|(\text{Number of } E_k \le 0) - \frac{1}{\pi}\int_{I_{\mathrm{BVP}}} \big(-V(x)\big)_+^{1/2} dx\right| \le C_\# \ , \quad \text{and} \\ &\left|(\text{Number of } E_k \le E_{\mathrm{crit}}) - \frac{1}{\pi}\int_{I_{\mathrm{BVP}}} \big(E_{\mathrm{crit}} - V(x)\big)_+^{1/2} dx\right| \le C_\# \ . \end{aligned} \tag{8}$$

These estimates and (7) imply

$$\begin{aligned} \int_{I_{\mathrm{BVP}}\cap(-\infty,x_{\mathrm{crit}}]} \rho(x)dx &\le C_\# + C_\#\delta \int_{I_{\mathrm{BVP}}} \big(-V(x)\big)_+^{1/2} dx \\ &\quad + C_\# \int_{I_{\mathrm{BVP}}} \big(E_{\mathrm{crit}} - V(x)\big)_+^{1/2} dx \ , \end{aligned}$$

which is the conclusion of the lemma regarding $\int_{I_{\mathrm{BVP}}\cap(-\infty,x_{\mathrm{crit}}]} \rho(x)dx$. The conclusion regarding $\int_{I_{\mathrm{BVP}}} \rho(x)dx$ is immediate from(8). ■

THE DENSITY FOR DEGENERATE ONE-DIMENSIONAL POTENTIALS III

The results in this section are for application to three-dimensional problems, with angular momentum ℓ in the range $[1,\ \text{Large Constant}\,]$. Our setting is as follows.

We are given a potential $V(x)$, smooth on $(0,\infty)$. We take $B(x)=x$, and let $S(x)$ be a positive function on $I=[x_0,x_1]\subset(0,\infty)$. As usual, we set $\lambda(x)=S^{1/2}(x)B(x)$ on I. In addition to x_0, x_1, we are given other points x_{small}, x_{big}, x_{crit}, $x_*\in(0,\infty)$, with

$$(1)\qquad 0<x_{\text{small}}<\frac{1}{2}x_0\ ,2x_0<x_{\text{crit}}<\frac{1}{2}x_*\ ,x_*<\frac{1}{16}x_1\ ,2x_1<x_{\text{big}}\ .$$

Set $H=-\frac{d^2}{dx^2}+V(x)$ on $(0,\infty)$ with Dirichlet boundary conditions. Let E_k, $u_k(x)$ be the eigenvalues and (normalized) eigenfunctions of H. Define the density $\rho(x)$ as usual by $\rho(x)=\sum\limits_{E_k\le 0}|u_k(x)|^2$.

Our goal is to make a (very crude) estimate of $\int_0^{x_{\text{crit}}}\rho(x)dx$. In addition to (1), we make the following assumptions.

Hypotheses.

(Z$\hat{0}$) If $x,y\in I$ and $|x-y|<\frac{1}{2}B(x)$, then $c<S(y)/S(x)<C$.

(Z$\hat{1}$) If $x\in I$, then $\left|\left(\frac{d}{dx}\right)^\alpha V(x)\right|\le C_\alpha S(x)B^{-\alpha}(x)$.

(Z$\hat{2}$) If $x\in I$, then $V(x)<-cS(x)$ and $V'(x)>cS(x)B^{-1}(x)$.

(Z$\hat{3}$) $\Lambda=\left(\int_I\frac{dx}{\lambda(x)B(x)}\right)^{-1}$ is greater than a certain large, positive number determined by c, C, C_α in (Z$\hat{0}$)...(Z$\hat{2}$).

(Z$\hat{4}$) For $x\in(0,x_{\text{small}}]$ we have $V(x)\ge\underline{c}x_0^{-2}$

(Z$\hat{5}$) For $x\in[x_{\text{small}},x_0]$ we have $|V(x)|\le\underline{C}x_0^{-2}$

(Z$\hat{6}$) We have $x_{\text{big}}<\underline{C}x_1$ and $V(x)$ is increasing in $[x_1,x_{\text{big}}]$.

(Z$\hat{7}$) For $x\in\left[\frac{x_1}{8},x_{\text{big}}\right]$, we have $|V(x)|\le\underline{C}x_1^{-2}$.

(Z$\hat{8}$) For $x\in[x_{\text{big}},\infty)$, we have $V(x)\ge 0$.

(Z$\hat{9}$) For $E \in [V(x_*), 0]$ we have

$$\int_{x_0}^{x_{\text{crit}}} \big(E - V(x)\big)^{-1/2} dx \le \delta \cdot \int_{x_0}^{\frac{1}{2}x_*} \big(E - V(x)\big)^{-1/2} dx.$$

When $\delta << 1$ hypothesis (Z$\hat{9}$) shows that in the semiclassical approximation, most of the L^2-norm of an eigenfunction $u_k(x)$ with $E_k \in [V(x_*), 0]$ is concentrated outside of $(0, x_{\text{crit}}]$.

We denote by $c_\#$, $C_\#$, etc. constants that depend only on c, C, C_α; while c_*, C_* etc. denote constants that depend also on $\underline{c}$, $\underline{C}$.

Lemma 1. *Set $E_0 = 0$, $I_{\text{center}} = [x_0, x_1]$, $I_{\text{left}} = [x_{\text{small}}, x_0]$, $I_{\text{rt}} = [x_1, x_{\text{big}}]$, $I_{\text{far left}} = (0, x_{\text{small}}]$, $I_{\text{far rt}} = [x_{\text{big}}, \infty)$. Then the hypotheses (H$\hat{0}$)... (H$\hat{7}$) of Theorem 1 in the section on WKB Theory with Weak Turning Points are satisfied. The constants called c, C, C_α in (H$\hat{0}$)... (H$\hat{7}$) may be taken to be of the form $C_\#$. The constants called $\underline{c}$, $\underline{C}$ in (H$\hat{0}$)... (H$\hat{7}$) may be taken to be of the form C_*.*

Proof. (H$\hat{0}$) follows from (Z$\hat{0}$), and from $B(x) \equiv x$, $2x_0 < x_1$. (H$\hat{1}$) follows from (Z$\hat{1}$) and (Z$\hat{2}$). (H$\hat{2}$) follows from (Z$\hat{3}$). (H$\hat{3}$) is verified as follows. By definition, I_{left} and I_{rt} are non-empty. Since $0 < x_{\text{small}} < x_0$, we have $|I_{\text{left}}| \le x_0 = B(x_0) = B(x_{\text{left}})$. Since $0 < x_{\text{big}} < \underline{C}x_1$ by (Z$\hat{6}$), we have $|I_{\text{rt}}| \le \underline{C}x_1 = \underline{C}B(x_{\text{rt}})$. By (Z$\hat{2}$) we have $S(x) \le C_\# |V(x)|$ for $x \in I$, hence for $x = x_0, x_1$. Therefore, $\lambda(x_{\text{left}}) = S^{1/2}(x_0)B(x_0) = S^{1/2}(x_0) \cdot x_0 \le C_\# |V(x_0)|^{1/2} \cdot x_0 \le C_*$ by (Z$\hat{5}$); and similarly, $\lambda(x_{\text{rt}}) = S^{1/2}(x_1)B(x_1) = S^{1/2}(x_1) \cdot x_1 \le C_\# |V(x_1)|^{1/2} \cdot x_1 \le C_*$ by (Z$\hat{7}$). This completes the verification of (H$\hat{3}$). (H$\hat{4}$) follows from (Z$\hat{5}$), since $|V(x)| \le \underline{C}x_0^{-2} \le C_* |I_{\text{left}}|^{-2}$ in I_{left}, and we can take $\hat{x}_{\text{far left}}$ so that $(x - \hat{x}_{\text{far left}}) \sim |I_{\text{left}}|$ in I_{left}. (H$\hat{5}$) follows from (Z$\hat{7}$) and the equation $|I_{\text{rt}}| \sim x_1$ contained in (Z$\hat{6}$). (H$\hat{6}$) follows from (Z$\hat{4}$) since $0 < x_{\text{small}} < \frac{1}{2}x_0$. (H$\hat{7}$) follows from (Z$\hat{8}$). ■

Lemma 2. *Suppose $2x_0 < \tilde{x} < \frac{1}{2}x_1$. Set $\tilde{E} = V(\tilde{x})$, and define: $\tilde{V}(x) = V(x) - \tilde{E}$, $\tilde{E}_0 = 0$, $\tilde{I}_{\text{center}} = [x_0, \tilde{x} - \hat{C}_\# \lambda^{-2/3}(\tilde{x}) \cdot \tilde{x}]$ with $\hat{C}_\#$ to be picked large enough,*

$\tilde{I}_{\rm left} = [x_{\rm small}, x_0]$, $\tilde{I}_{\text{far left}} = (0, x_{\rm small}]$, $\tilde{I}_{\rm rt} = [\tilde{x} - \hat{C}_\# \lambda^{-2/3}(\tilde{x}) \cdot \tilde{x}, \tilde{x} + \hat{C}_\# \lambda^{-2/3}(\tilde{x}) \cdot \tilde{x}]$, $\tilde{I}_{\text{far rt}} = [\tilde{x} + \hat{C}_\# \lambda^{-2/3}(\tilde{x}) \cdot \tilde{x}, \infty)$. On $\tilde{I}_{\rm center}$, define $\tilde{B}(x) = \min(x, \tilde{x} - x)$, $\tilde{S}(x) = S(x) \cdot \left(\frac{\tilde{x}-x}{\tilde{x}}\right)$. Then $\tilde{V}(x)$, $\tilde{E}_0$, $\tilde{I}_{\text{far left}} \ldots \tilde{I}_{\text{far rt}}$, $\tilde{S}(x)$, $\tilde{B}(x)$ satisfy hypotheses $(\hat{H0}) \ldots (\hat{H7})$ in the section on WKB Theory with weak turning points. The constants called c, C, C_α in $(\hat{H0}) \ldots (\hat{H7})$ may be taken to have the form $C_\#$. The constants called $\underline{c}$, $\underline{C}$ in $(\hat{H0}) \ldots (\hat{H7})$ may be taken to have the form C_*.

Remark. We have $\lambda(\tilde{x}) \geq c_\# \Lambda$, so $(\hat{Z3})$ shows that $\tilde{I}_{\rm rt} \subset I$ and $\tilde{I}_{\rm center} \supset [x_0, \frac{9}{10}\tilde{x}]$.

Proof. The above remark shows that $\tilde{I}_{\rm center}$ contains $[x_0, \frac{9}{10} \cdot 2x_0]$, so $|\tilde{I}_{\rm center}| \geq c_\# \tilde{B}(x)$ for $x = x_0$, hence for all $x \in \tilde{I}_{\rm center}$. If $x, y \in \tilde{I}_{\rm center}$ with $|x - y| < c_\# \tilde{B}(x)$, then one checks that $\tilde{B}(y) \sim \tilde{B}(x)$, $\left(\frac{\tilde{x}-x}{\tilde{x}}\right) \sim \left(\frac{\tilde{x}-y}{\tilde{x}}\right)$, and $S(x) \sim S(y)$ since $|x - y| < c_\# B(x)$. Hence $\tilde{B}(y) \sim \tilde{B}(x)$ and $\tilde{S}(y) \sim \tilde{S}(x)$. These remarks prove $(\hat{H0})$.

To prove $(\hat{H1})$ we argue as follows. For $x \in \tilde{I}_{\rm center}$, we know that $x \leq \tilde{x}$ and that $x, \tilde{x} \in I$. From $(\hat{Z2})$ we get $V(x) < V(\tilde{x}) < 0$, so $|V(x) - \tilde{E}| = |V(x) - V(\tilde{x})| \leq |V(x)| \leq C_\# S(x) \leq C'_\# \tilde{S}(x)$ provided $|x - \tilde{x}| > c_\# \tilde{x}$. Also, for $\alpha \geq 1$ we have $\left|\left(\frac{d}{dx}\right)^\alpha \tilde{V}(x)\right| = \left|\left(\frac{d}{dx}\right)^\alpha V(x)\right| \leq C^\alpha_\# S(x) B^{-\alpha}(x) \leq C^\alpha_\# \tilde{S}(x) \tilde{B}^{-\alpha}(x)$ since $\tilde{S}(x) = S(x) \cdot \left(\frac{\tilde{x}-x}{\tilde{x}}\right)$ and $\tilde{B}(x) \sim B(x) \cdot \left(\frac{\tilde{x}-x}{\tilde{x}}\right)$. Hence $|(\frac{d}{dx})^\alpha \tilde{V}(x)| \leq C^\alpha_\# \tilde{S}(x) \tilde{B}^{-\alpha}(x)$ is verified, except for $\alpha = 0$, $|x - \tilde{x}| < c_\# \tilde{x}$. However in this case, we have $|\tilde{V}(x)| = |V(x) - \tilde{E}| = |V(x) - V(\tilde{x})| \leq C_\# S(\tilde{x}) \cdot \left(\frac{\tilde{x}-x}{\tilde{x}}\right)$ by our estimates for $V'(x)$, and the right-hand side is $\leq C_\# \tilde{S}(x)$. Thus we know that $\left|\left(\frac{d}{dx}\right)^\alpha \tilde{V}(x)\right| \leq C^\alpha_* \tilde{S}(x) \tilde{B}^{-\alpha}(x)$, for all $\alpha \geq 0$, $x \in \tilde{I}_{\rm center}$.

To complete the proof of $(\hat{H1})$, we need to show that $-\tilde{V}(x) > c_\# \tilde{S}(x)$ for $x \in I_{\rm center}$. If $|\tilde{x} - x| \geq c_\# \tilde{x}$, then V is increasing on $[x, \tilde{x}]$, and $V' > c_\# S(x) x^{-1}$ on $[x, (1 + c_\#)x] \subset [x, \tilde{x}]$, as we see from $(\hat{Z2})$. Hence, $-\tilde{V}(x) = V(\tilde{x}) - V(x) \geq \int_x^{(1+c_\#)x} V' \geq c_\# S(x) \geq c_\# \tilde{S}(x)$, as required.

On the other hand, if $|\tilde{x} - x| \leq c_\# \tilde{x}$, then $(\hat{Z2})$ gives $V' > c_\# S(\tilde{x}) \tilde{x}^{-1}$ on $[x, \tilde{x}]$.

Hence

$$-\tilde{V}(x) = V(\tilde{x}) - V(x) = \int_x^{\tilde{x}} V' \geq c_\# S(\tilde{x})\big(\frac{\tilde{x}-x}{x}\big) \geq c_\# \tilde{S}(x) \ .$$

Thus, in all cases, we know that $-\tilde{V}(x) \geq c_\# \tilde{S}(x)$ for $x \in \tilde{I}_{\text{center}}$, completing the proof of $(\text{H}\hat{1})$.

Note that the constants called c, C, C_α in $(\text{H}\hat{0})$ and $(\text{H}\hat{1})$ do not depend on our choice of $\hat{C}_\#$.

To prove $(\text{H}\hat{2})$, we compute $\tilde{\lambda}(x) = \tilde{S}^{1/2}(x)\tilde{B}(x)$ and $\tilde{\Lambda}^{-1} = \int_{\tilde{I}_{\text{center}}} \frac{dx}{\tilde{\lambda}(x)\tilde{B}(x)}$. For $x \in \tilde{I}_{\text{center}}$ with $|x - \tilde{x}| > c_\# \tilde{x}$ we have $\tilde{S}(x) \sim S(x)$, $\tilde{B}(x) \sim B(x)$, hence $\tilde{\lambda}(x) \sim \lambda(x)$. For $x \in \tilde{I}_{\text{center}}$ with $|x - \tilde{x}| \leq c_\# \tilde{x}$, we have $\tilde{S}(x) = S(x) \cdot \big(\frac{\tilde{x}-x}{\tilde{x}}\big)$, $\tilde{B}(x) = (\tilde{x} - x) \sim B(x) \cdot \big(\frac{\tilde{x}-x}{\tilde{x}}\big)$, hence

$$\begin{aligned}\tilde{\lambda}(x) \sim \big[S(x) \cdot \big(\frac{\tilde{x}-x}{\tilde{x}}\big)\big]^{1/2}\big[B(x) \cdot \big(\frac{\tilde{x}-x}{\tilde{x}}\big)\big] = \lambda(x) \cdot \big(\frac{\tilde{x}-x}{\tilde{x}}\big)^{3/2} \\ \sim \lambda(\tilde{x}) \cdot \big(\frac{\tilde{x}-x}{\tilde{x}}\big)^{3/2} \ .\end{aligned}$$

So

$$\begin{aligned}\tilde{\Lambda}^{-1} &\sim \int_{\tilde{I}_{\text{center}} \setminus \{|x-\tilde{x}| \leq c_\# \tilde{x}\}} \frac{dx}{\lambda(x)B(x)} \\ &\quad + \int_{\tilde{I}_{\text{center}} \cap \{|x-\tilde{x}| \leq c_\# \tilde{x}\}} \frac{dx}{\lambda(\tilde{x}) \cdot (\frac{\tilde{x}-x}{\tilde{x}})^{3/2} \cdot (\tilde{x} - x)} \\ &\leq \int_I \frac{dx}{\lambda(x)B(x)} + \int_{|x-\tilde{x}| > \hat{C}_\# \lambda^{-2/3}(\tilde{x}) \cdot \tilde{x}} \frac{dx}{\lambda(\tilde{x}) \big|\frac{\tilde{x}-x}{\tilde{x}}\big|^{3/2} \cdot |\tilde{x} - x|} \\ &= \Lambda^{-1} + C_\# (\hat{C}_\#)^{-3/2} \ .\end{aligned}$$

Hence $(\text{H}\hat{2})$ follows from $(\text{Z}\hat{3})$ provided we take $\hat{C}_\#$ large enough. This completes the proof of $(\text{H}\hat{2})$.

To prove $(\text{H}\hat{3})$ we argue as follows. We have $\tilde{I}_{\text{left}}, \tilde{I}_{\text{rt}} \neq \emptyset$ by definition. We have $|\tilde{I}_{\text{left}}| = x_0 - x_{\text{small}} \leq x_0 \sim \tilde{B}(x_{\text{left}})$, since $\tilde{B}(x_{\text{left}}) = \min(x_0, \tilde{x} - x_0)$ and $\tilde{x} > 2x_0$. Next, note that $x_{\text{rt}} = \tilde{x} - \hat{C}_\# \lambda^{-2/3}(\tilde{x}) \cdot \tilde{x} \geq \frac{9}{10}\tilde{x}$ since $\tilde{I}_{\text{center}}$ has been seen to contain $[x, \frac{9}{10}\tilde{x}]$. Therefore, $\tilde{B}(x_{\text{rt}}) = \min(\tilde{x} - x_{\text{rt}}, x_{\text{rt}}) = \tilde{x} - x_{\text{rt}} = \hat{C}_\# \lambda^{-2/3}(\tilde{x}) \cdot \tilde{x}$. Hence,

$|\tilde{I}_{\rm rt}| = 2\hat{C}_\# \lambda^{-2/3}(\tilde{x}) \cdot \tilde{x} = 2\tilde{B}(x_{\rm rt})$, which proves the required bounds for $|\tilde{I}_{\rm left}|$, $|\tilde{I}_{\rm rt}|$. Regarding $\tilde{\lambda}(x_{\rm left})$, $\tilde{\lambda}(x_{\rm rt})$, we note that $\tilde{\lambda}(x_{\rm left}) = \tilde{\lambda}(x_0) \sim \lambda(x_0) \leq C_*$, as we saw in the proof of Lemma 1; and $\tilde{\lambda}(x_{\rm rt}) \sim \lambda(\tilde{x}) \cdot \left(\frac{\tilde{x}-x_{\rm rt}}{x}\right)^{3/2} = (\hat{C}_\#)^{3/2}$. The right-hand side has the form C_*, completing the proof of (H$\hat{3}$).

To prove (H$\hat{4}$), we need to show $|V(x) - V(\tilde{x})| \leq C_* x_0^{-2}$ for $x \in [x_{\rm small}, x_0]$. Hypothesis (Z$\hat{5}$) reduces this to $|V(\tilde{x})| \leq C_* x_0^{-2}$. From (Z$\hat{2}$) we get $V(x_0) < V(\tilde{x}) < 0$, so it's enough to check that $|V(x_0)| \leq C_* x_0^{-2}$. This follows from another application of (Z$\hat{5}$), completing the proof of (H$\hat{4}$).

To prove (H$\hat{5}$), we need to show that $|V(x) - V(\tilde{x})| \leq C_*(2\hat{C}_\#\lambda^{-2/3}(\tilde{x}) \cdot \tilde{x})^{-2}$ for $|x - \tilde{x}| \leq \hat{C}_\#\lambda^{-2/3}(\tilde{x}) \cdot \tilde{x}$. The range of x is contained in I, by the remark following the statement of Lemma 2. Hence $|V(x) - V(\tilde{x})| \leq C_\# S(\tilde{x}) B^{-1}(\tilde{x}) \cdot |x - \tilde{x}| \leq C_\# S(\tilde{x}) \cdot \hat{C}_\#\lambda^{-2/3}(\tilde{x}) = C_\#[\lambda^2(\tilde{x}) \cdot \tilde{x}^{-2}] \cdot \hat{C}_\#\lambda^{-2/3}(\tilde{x}) = C_\#\hat{C}_\#\lambda^{4/3}(\tilde{x}) \cdot \tilde{x}^{-2} = C_\#\hat{C}_\#^3(2\hat{C}_\#\lambda^{-2/3}(\tilde{x}) \cdot \tilde{x})^{-2}$, as required. The proof of (H$\hat{5}$) is complete.

To prove (H$\hat{6}$), we need to show that $V(x) - V(\tilde{x}) \geq c_* x_0^{-2}$ for $x \in (0, x_{\rm small}]$. From (Z$\hat{4}$) get $V(x) \geq c_* x_0^{-2}$, and (Z$\hat{2}$) gives $V(\tilde{x}) < 0$. Hence (H$\hat{6}$) is trivially correct. Finally, (H$\hat{7}$) asserts that $V(x) - V(\tilde{x}) \geq \frac{-10^{-9}}{(x - \max \tilde{I}_{\rm center})^2}$ for $x \geq \tilde{x} + \hat{C}_\#\lambda^{-2/3}(\tilde{x}) \cdot \tilde{x}$. We will prove the stronger inequality $V(x) - V(\tilde{x}) \geq 0$ for $x \geq \tilde{x}$.

In fact, in $[\tilde{x}, x_1]$ and in $[x_1, x_{\rm big}]$, the function V is increasing, by (Z$\hat{2}$), (Z$\hat{6}$). Hence, $\tilde{x} \leq x \leq x_{\rm big}$ implies $V(x) - V(\tilde{x}) \geq 0$. On the other hand, if $x \geq x_{\rm big}$, then $V(x) \geq 0$ by (Z$\hat{8}$), while $V(\tilde{x}) < 0$ by (Z$\hat{2}$). Hence again $V(x) - V(\tilde{x}) \geq 0$, completing the proof. ■

Now set $\tilde{E}_{\rm low} = V(x_*)$, $\tilde{E}_{\rm hi} = V(\frac{1}{4}x_1)$. For $E_k \in [\tilde{E}_{\rm low}, \tilde{E}_{\rm hi}]$ we can write

$$E_k = V(\tilde{x}) = \tilde{E} \quad \text{for some} \quad \tilde{x} \in [x_*, \frac{1}{4}x_1] \,, \quad \text{by (Z}\,\hat{2}) \,.$$

Lemma 2 and Theorem 1 on WKB Theory with weak turning points yield the following estimates for the normalized eigenfunction u_k. (See "Review of Earlier

Results.")

$$(2)\quad |u_k(x)|^2 \le C_*\Big(\int_{\tilde I_{\rm center}} \big(E_k - V(y)\big)^{-1/2} dy\Big)^{-1} \big(E_k - V(x)\big)^{-1/2}$$

$$\text{for}\quad x \in \tilde I_{\rm center}\ ;$$

$$|u_k(x)|^2 \le C_*\Big(\int_{\tilde I_{\rm center}} \big(E_k - V(y)\big)^{-1/2} dy\Big)^{-1} \big(E_k - V(x_0)\big)^{-1/2} \exp\Big(\frac{-c_*(x_0-x)}{x_0}\Big)$$

$$\text{for}\quad x \le x_0\ .$$

In particular,

$$\begin{aligned}\int_0^{x_0} |u_k(x)|^2 dx &\le C_*\Big(\int_{\tilde I_{\rm center}} \big(E_k - V(y)\big)^{-1/2} dy\Big)^{-1} \big(E_k - V(x_0)\big)^{-1/2} x_0 \\ &\le C_*\Big(\int_{\tilde I_{\rm center}} \big(E_k - V(y)\big)^{-1/2} dy\Big)^{-1} S^{-1/2}(x_0)\cdot x_0\end{aligned}$$

because $E_k = V(\tilde x) \ge V(2x_0) \ge V(x_0) + c_\# S(x_0) B^{-1}(x_0)\cdot(c_\# x_0)$ by $(\mathrm{Z}\hat{2})$.

Also, $E_k = V(\tilde x) \in [V(x_0), 0]$, so $|E_k| \le C_\# S(x_0)$; and $|V(x)| \le C_\# S(x_0)$ for $x \in [x_0, \frac{3}{2}x_0]$. For such x we have also $E_k - V(x) > 0$. Hence $\int_{x_0}^{\frac{3}{2}x_0}\big(E_k - V(x)\big)^{-1/2}dx \ge \int_{x_0}^{\frac{3}{2}x_0}\big[c_\# S^{-1/2}(x_0)\big]dx = c_\# S^{-1/2}(x_0)\cdot x_0$, so $\int_0^{x_0}|u_k(x)|^2dx \le C_*\Big(\int_{\tilde I_{\rm center}}\big(E_k - V(y)\big)^{-1/2}dy\Big)^{-1}\int_{x_0}^{\frac{3}{2}x_0}\big(E_k - V(x)\big)^{-1/2}dx$.

Recall from the Remark following the statement of Lemma 2 that $\tilde I_{\rm center}$ contains $[x_0, \frac{9}{10}\tilde x]$, which in turn contains $[x_0, \frac{1}{2}x_*]$. Hence

$$(3)\qquad \Big(\int_{\tilde I_{\rm center}} \big(E_k - V(y)\big)^{-1/2} dy\Big)^{-1} \le \Big(\int_{x_0}^{\frac{1}{2}x_*} \big(E_k - V(y)\big)^{-1/2} dy\Big)^{-1}\ .$$

Also $\frac{3}{2}x_0 \le x_{\rm crit}$, so we get

$$\int_0^{x_0} |u_k(x)|^2 dx \le C_*\Big(\int_{x_0}^{x_{\rm crit}} \big(E_k - V(y)\big)^{-1/2} dy\Big)\Big(\int_{x_0}^{\frac{1}{2}x_*} \big(E_k - V(y)\big)^{-1/2} dy\Big)^{-1}\ .$$

Applying $(\mathrm{Z}\hat{9})$, we conclude that

$$(4)\qquad \int_0^{x_0} |u_k(x)|^2 dx \le C_*\delta \quad \text{for}\quad E_k \in [\tilde E_{\rm low}, \tilde E_{\rm hi}]\ .$$

Similarly, (2), (3) imply

$$\int_{x_0}^{x_{\rm crit}} |u_k(x)|^2 dx \le C_* \Big(\int_{x_0}^{x_{\rm crit}} \big(E_k - V(x)\big)^{-1/2} dx \Big) \Big(\int_{x_0}^{\frac{1}{2}x_*} \big(E_k - V(y)\big)^{-1/2} dy \Big)^{-1} ,$$

so another application of (Z$\hat{9}$) gives

$$\int_{x_0}^{x_{\rm crit}} |u_k(x)|^2 dx \le C_* \delta \quad \text{for} \quad E_k \in [\tilde{E}_{\rm low}, \tilde{E}_{\rm hi}] \ .$$

This and (4) imply

$$\int_{0}^{x_{\rm crit}} |u_k(x)|^2 dx \le C_* \delta \quad \text{for} \quad E_k \in [\tilde{E}_{\rm low}, \tilde{E}_{\rm hi}] \ .$$

Of course, for arbitrary E_k we have $\int_0^{x_{\rm crit}} |u_k(x)|^2 dx \le 1$, since $u_k(x)$ is normalized. Hence,

$$\begin{aligned} \int_0^{x_{\rm crit}} \rho(x) dx \le\ & C_* \delta \cdot (\text{ Number of } E_k \in [\tilde{E}_{\rm low}, \tilde{E}_{\rm hi}]) + C_* \cdot \big(\text{Number of } E_k < \tilde{E}_{\rm low}\big) \\ & + C_* \cdot (\text{ Number of } E_k \in (\tilde{E}_{hi}, 0]) \\ \le\ & C_* \delta \cdot (\text{ Number of } E_k \le 0) + C_* (\text{ Number of } E_k \le \tilde{E}_{\rm low}) \\ & + C_* \cdot (\text{Number of } E_k \in (\tilde{E}_{hi}, 0]) \ . \end{aligned} \tag{5}$$

Again applying Theorem 1 (on WKB with weak turning points) and Lemma 2, with $\tilde{E} = \tilde{E}_{\rm hi}$, we obtain

$$\Big| \big(\text{Number of } E_k \le \tilde{E}_{\rm hi}\big) - \frac{1}{\pi} \int_{\tilde{I}_{\rm center}} \big(\tilde{E}_{\rm hi} - V(x)\big)^{1/2} dx \Big| \le C_* \ ,$$

with $\tilde{I}_{\rm center}$ containing $[x_0, \frac{9}{10} \cdot \frac{1}{4} x_1]$ by the remark following the statement of Lemma 2. Hence

$$(\text{ Number of } E_k \le \tilde{E}_{\rm hi}) \ge \frac{1}{\pi} \int_{x_0}^{\frac{1}{8}x_1} \big(\tilde{E}_{\rm hi} - V(x)\big)^{1/2} dx - C_* \ . \tag{6}$$

Similarly, applying Theorem 1 and Lemma 2, with $\tilde{E} = \tilde{E}_{\rm low}$, and noting that $\tilde{I}_{\rm center} \subset I$, we get

$$\Big| \big(\text{Number of } E_k \le \tilde{E}_{\rm low}\big) - \frac{1}{\pi} \int_{\tilde{I}_{\rm center}} \big(\tilde{E}_{\rm low} - V(x)\big)^{1/2} dx \Big| \le C_* \ ,$$

so

$$\text{(7)} \qquad (\text{Number of } E_k \le \tilde{E}_{\text{low}}) \le \frac{1}{\pi}\int_{x_0}^{x_1} \big(\tilde{E}_{\text{low}} - V(x)\big)_+^{1/2} dx + C_* \ .$$

Lemma 1 and Theorem 1 (on WKB with weak turning points) imply

$$\text{(8)} \qquad (\text{Number of } E_k \le 0) \le \frac{1}{\pi}\int_{x_0}^{x_1} \big(-V(x)\big)^{1/2} dx + C_* \ .$$

From (6) and (8) we get

$$\text{(9)} \quad (\text{Number of } E_k \in (\tilde{E}_{\text{hi}}, 0]) \le C_* + \frac{1}{\pi}\int_{x_0}^{\frac{x_1}{8}} \big\{\big(-V(x)\big)^{1/2} - \big(\tilde{E}_{\text{hi}} - V(x)\big)^{1/2}\big\} dx + \frac{1}{\pi}\int_{\frac{x_1}{8}}^{x_1} \big(-V(x)\big)^{1/2} dx \ .$$

We estimate the right-hand side of (9). From $(\hat{Z7})$, we see that the last term on the right is dominated by C_*. Moreover, in $[x_0, \frac{x_1}{8}]$, we have $V(x) < V(\frac{1}{4}x_1) = \tilde{E}_{\text{hi}} < 0$, so

$$\big|\big(-V(x)\big)^{1/2} - \big(\tilde{E}_{\text{hi}} - V(x)\big)^{1/2}\big| \le C_\# |\tilde{E}_{\text{hi}}| \big(-V(x)\big)^{-1/2} \le C_\# |\tilde{E}_{\text{hi}}| S^{-1/2}(x) \ .$$

Hence

$$\int_{x_0}^{x_1/8} \big\{\big(-V(x)\big)^{1/2} - \big(\tilde{E}_{\text{hi}} - V(x)\big)^{1/2}\big\} dx \le C_\# |\tilde{E}_{\text{hi}}| \int_{x_0}^{x_1} \frac{dx}{S^{1/2}(x)}$$
$$\le C_\# |\tilde{E}_{\text{hi}}| x_1^2 \int_{x_0}^{x_1} \frac{dx}{S^{1/2}(x) x^2} = \frac{C_\#}{\Lambda} |\tilde{E}_{\text{hi}}| x_1^2 \quad (\text{by definition of } \Lambda)$$
$$\le |\tilde{E}_{\text{hi}}| x_1^2 \quad (\text{by } (Z\,\hat{3})) = |V(\frac{x_1}{4})| x_1^2 \le C_* \quad \text{by } (Z\,\hat{7}).$$

Thus, all terms on the right in (9) are dominated by C_*, so that (9) becomes

$$\text{(10)} \qquad \big(\text{Number of } E_k \in (\tilde{E}_{\text{hi}}, 0]\big) \le C_*$$

Substituting (7), (8), (10) into (5), we obtain

$$\text{(11)} \quad \int_0^{x_{\text{crit}}} \rho(x) dx \le C_* \delta \int_{x_0}^{x_1} \big(-V(x)\big)^{1/2} dx + C_* \int_{x_0}^{x_1} \big(\tilde{E}_{\text{low}} - V(x)\big)_+^{1/2} dx + C_* \ .$$

Immediately from (8), (11) and the definition $\tilde{E}_{\text{low}} = V(x_*)$, we get the following.

Third Degenerate Density Lemma. *Assume (1) and* $(\hat{Z0})\ldots(\hat{Z9})$*. Set* $E_{\text{crit}} = V(x_*)$*. Then*

$$\int_0^{x_{\text{crit}}} \rho(x)dx \le C_* + C_*\delta \int_0^\infty \big(-V(x)\big)_+^{1/2} dx + C_* \int_0^\infty \big(E_{\text{crit}} - V(x)\big)_+^{1/2} dx$$

and

$$\int_0^\infty \rho(x)dx \le C_* + C_* \int_0^\infty \big(-V(x)\big)_+^{1/2} dx$$

with C_* *depending only on* c, C, C_α, $\underline{c}$, $\underline{C}$, *in* $(\hat{Z0})\ldots(\hat{Z9})$.

THE DENSITY FOR DEGENERATE ONE-DIMENSIONAL POTENTIALS IV

The results in this section will be used to handle angular momentum $\ell = 0$ in three-dimensional problems. Our setting is as follows.

We are given a smooth potential $V(x)$ on $(0,\infty)$. We take $B(x) = x$, and let $S(x)$ be a positive function on $I = [x_0, x_1] \subset (0,\infty)$. Let $\lambda(x) = S^{1/2}(x)B(x)$ as usual. We are given x_{crit}, x_*, x_{big}, satisfying

$$16x_0 < x_{\text{crit}}\ , 16x_{\text{crit}} < x_*\ , 16x_* < x_1\ , 16x_1 < x_{\text{big}}\ . \tag{1}$$

Set $H = -\frac{d^2}{dx^2} + V(x)$ on $(0,\infty)$, with Dirichlet boundary conditions. Let E_k, $u_k(x)$ be the eigenvalues and (normalized) eigenfunctions of H. Define the density $\rho(x)$ as usual by

$$\rho(x) = \sum_{E_k \le 0} |u_k(x)|^2\ .$$

Our goal is to make a crude estimate of $\int_0^{x_{\text{crit}}} \rho(x)dx$. In addition to (1), we make the following assumptions.

Hypotheses.

(Z0†) If $x, y \in I$ and $|x-y| < \frac{1}{2}B(x)$, then $c < S(y)/S(x) < C$.

(Z1†) If $x \in I$ and $\alpha \ge 0$, then $\left|\left(\frac{d}{dx}\right)^\alpha V(x)\right| \le C_\alpha S(x) B^{-\alpha}(x)$.

(Z2†) If $x \in I$, then $V(x) < -cS(x)$ and $V'(x) > cS(x)B^{-1}(x)$.

(Z3†) $\Lambda = \left(\int_I \frac{dx}{\lambda(x)B(x)}\right)^{-1}$ is greater than a certain large, positive number determined by c, C, C_α in (Z0†)...(Z2†).

(Z4†) $|V(x)| \le \underline{C}/(x_0 x)$ for $x \in (0, x_0]$.

(Z5†) $V(x)$ is increasing and negative in $[\frac{x_1}{8}, x_{\text{big}}]$, and satisfies there $|V(x)| < \underline{C} x_1^{-2}$. Also, $x_{\text{big}} < \underline{C} x_1$.

(Z6†) $V(x) \ge -10^{-9} x^{-2}$ for $x \in [x_{\text{big}}, \infty)$.

(Z7†) For $E \in [V(x_*), 0]$, we have

$$\int_{x_0}^{x_{\text{crit}}} \big(E - V(x)\big)^{-1/2} dx \leq \delta \cdot \int_{x_0}^{\frac{1}{2}x_*} \big(E - V(x)\big)^{-1/2} dx.$$

As usual, (Z7†) says that certain eigenfunctions live mostly outside of $[x_0, x_{\text{crit}}]$ in the semiclassical approximation.

We denote by $c_\#$, $C_\#$, etc. constants that depend only on c, C, C_α in (Z0†)...(Z7†); while c_*, C_*, etc. denote constants that depend also on $\underline{C}$.

Lemma 1. *Set $I_{\text{far left}} = \emptyset$, $I_{\text{left}} = (0, x_0]$, $I_{\text{center}} = [x_0, x_1]$, $I_{\text{rt}} = [x_1, x_{\text{big}}]$, $I_{\text{far rt}} = [x_{\text{big}}, \infty)$, $E_0 = 0$. Then the hypotheses (H$\hat{0}$)...(H$\hat{7}$), from the section on WKB with weak turning points, are satisfied. The constants called c, C, C_α in (H$\hat{0}$)...(H$\hat{7}$) may be taken of the form $C_\#$. The constants called $\underline{c}$, $\underline{C}$ in (H$\hat{0}$)...(H$\hat{7}$) may be taken of the form C_*.*

Proof. (H$\hat{0}$) follows from (Z0†) and from $B(x) \equiv x$, $x_1 > 2x_0$. (H$\hat{1}$) follows from (Z1†) and (Z2†). (H$\hat{2}$) follows from (Z3†). (H$\hat{3}$) is proven as follows. $I_{\text{left}}, I_{\text{rt}} \neq \emptyset$ by definition. $|I_{\text{left}}| = x_0 = B(x_{\text{left}})$, $|I_{\text{rt}}| = x_{\text{big}} - x_1 \leq \underline{C}x_1$ (by (Z5†)) $= \underline{C}B(x_{\text{rt}})$. Also $\lambda(x_{\text{left}}) = S^{1/2}(x_0) \cdot x_0 \leq C_*$, since $-\underline{C}x_0^{-2} \leq V(x_0) \leq -cS(x_0) < 0$ by (Z2†) and (Z4†). Similarly, $\lambda(x_{\text{rt}}) = S^{1/2}(x_1) \cdot x_1 \leq C_*$, since $-\underline{C}x_1^{-2} \leq V(x_1) \leq -cS(x_1) < 0$ by (Z2†) and (Z5†). We have proven all the assertions in (H$\hat{3}$). (H$\hat{4}$) is (Z4†). (H$\hat{5}$) follows from (Z5†). (H$\hat{6}$) holds vacuously, since $I_{\text{far left}} = \emptyset$. (H$\hat{7}$) follows from (Z6†). ■

Lemma 2. *Suppose $\tilde{E} = V(\tilde{x})$ with $\tilde{x} \in [x_*, \frac{1}{4}x_1]$. Take $\tilde{V}(x) = V(x) - \tilde{E}$, $\tilde{S}(x) = S(x) \cdot \left(\frac{\tilde{x}-x}{\tilde{x}}\right)$, $\tilde{B}(x) = \min(x, \tilde{x} - x)$, $\tilde{E}_0 = 0$, $\tilde{I}_{\text{far left}} = \emptyset$, $\tilde{I}_{\text{left}} = (0, x_0]$, $\tilde{I}_{\text{center}} = [x_0, \tilde{x} - \hat{C}_\# \lambda^{-2/3}(\tilde{x}) \cdot \tilde{x}]$ with $\hat{C}_\#$ to be picked large enough, $\tilde{I}_{\text{rt}} = [\tilde{x} - \hat{C}_\# \lambda^{-2/3}(\tilde{x}) \cdot \tilde{x}, \tilde{x} + \hat{C}_\# \lambda^{-2/3}(\tilde{x}) \cdot \tilde{x}]$, $\tilde{I}_{\text{far rt}} = [\tilde{x} + \hat{C}_\# \lambda^{-2/3}(\tilde{x}) \cdot \tilde{x}, \infty)$. Then the hypotheses (H$\hat{0}$)...(H$\hat{7}$), in the section on WKB with weak turning points, are satisfied. The constants called c, C, C_α in (H$\hat{0}$)...(H$\hat{7}$) may be taken of the form $C_\#$.*

The constants called $\underline{c}$, $\underline{C}$ *in* $(\mathrm{H}\hat{0})\ldots(\mathrm{H}\hat{7})$ *may be taken of the form* C_*.

Sketch of Proof. $(\mathrm{H}\hat{0})$, $(\mathrm{H}\hat{1})$, $(\mathrm{H}\hat{2})$ are proven exactly as in the proof of Lemma 2 in the preceding section. Regarding $(\mathrm{H}\hat{3})$, the assertions $|\tilde{I}_{\mathrm{rt}}| \leq C_* \tilde{B}(x_{\mathrm{rt}})$ and $\tilde{\lambda}(x_{\mathrm{rt}}) \leq C_*$ are proven just as in the discussion of $(\mathrm{H}\hat{3})$ in the proof of Lemma 2 in the preceding section. We have also $|\tilde{I}_{\mathrm{left}}| = x_0 = \tilde{B}(x_{\mathrm{left}})$, since $x_{\mathrm{left}} = x_0 < \frac{1}{2}\tilde{x}$. In addition, $\tilde{\lambda}(x_{\mathrm{left}}) \sim \lambda(x_0)$ (since $x_{\mathrm{left}} = x_0 < \frac{1}{2}\tilde{x}$) $= S^{1/2}(x_0) \cdot x_0 \leq C_*$ (as we saw in the proof of Lemma 1 above.) The intervals $\tilde{I}_{\mathrm{left}}$, $\tilde{I}_{\mathrm{rt}}$ are non-empty by definition. We have thus proven all the assertions in $(\mathrm{H}\hat{3})$. To check $(\mathrm{H}\hat{4})$, let $x \in \tilde{I}_{\mathrm{left}}$.$(Z4^\dagger)$ gives $|V(x)| \leq \frac{C_*}{x_0 x}$, $|V(x_0)| \leq \frac{C_*}{x_0^2}$; and $(\mathrm{Z}2^\dagger)$ gives $V(x_0) < V(\tilde{x}) < 0$. Hence $|\tilde{V}(x)| = |V(x) - V(\tilde{x})| \leq |V(x)| + |V(\tilde{x})| \leq \frac{2C_*}{x_0 x}$, proving $(\mathrm{H}\hat{4})$. $(\mathrm{H}\hat{5})$ is proven here just as it was in the proof of Lemma 2 in the previous section. $(\mathrm{H}\hat{6})$ holds vacuously, since $\tilde{I}_{\text{far left}} = \emptyset$. $(\mathrm{H}\hat{7})$ asserts that $V(x) - V(\tilde{x}) \geq -10^{-9}(x - [\tilde{x} - \hat{C}_{\#}\lambda^{-2/3}(\tilde{x}) \cdot \tilde{x}])^{-2}$ for $x \geq \tilde{x} + \hat{C}_{\#}\lambda^{-2/3}(\tilde{x}) \cdot \tilde{x}$. We shall verify the following inequalities, which together are stronger than $(\mathrm{H}\hat{7})$.

$$V(x) - V(\tilde{x}) \geq 0 \quad \text{for} \quad \tilde{x} \leq x \leq x_{\mathrm{big}} \tag{2}$$

$$V(x) - V(\tilde{x}) \geq -10^{-9} x^{-2} \quad \text{for} \quad x \geq x_{\mathrm{big}}\ . \tag{3}$$

From $(\mathrm{Z}2^\dagger)$, $(\mathrm{Z}5^\dagger)$, we have V increasing on $[\tilde{x}, x_1]$ and on $[x_1, x_{\mathrm{big}}]$, so (2) holds. From $(\mathrm{Z}2^\dagger)$, $(\mathrm{Z}6^\dagger)$ we have $V(\tilde{x}) \leq 0$ and $V(x) \geq -10^{-9}x^{-2}$ for $x \geq x_{\mathrm{big}}$, so (3) holds. This completes the proof of $(\mathrm{H}\hat{7})$. ■

Using Lemmas 1 and 2 above, and Theorem 1 in the section on WKB with weak turning points, we derive the following result, exactly as in the previous section.

Fourth Degenerate Density Lemma. *Assume (1) and* $(Z0^\dagger)\ldots(Z7^\dagger)$. *Set* $E_{\mathrm{crit}} = V(x_*)$. *Then*

$$\int_0^{x_{\mathrm{crit}}} \rho(x)dx \leq C_* + C_*\delta \int_0^{\infty} \left(-V(x)\right)_+^{1/2} dx + C_* \int_0^{\infty} \left(E_{\mathrm{crit}} - V(x)\right)_+^{1/2} dx$$

and $\int_0^\infty \rho(x)dx \le C_* + C_* \int_0^\infty \big(-V(x)\big)_+^{1/2} dx$ *with* C_* *depending only on* c, C, C_α, $\underline{C}$ *in (Z0*†*)...(Z7*†*).*

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