THE EIGENVALUE SUM FOR A ONE-DIMENSIONAL POTENTIAL

by

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Introduction

This paper is part of a series [FS2...7] proving the asymptotic formula announced in [FS1] for the ground-state energy of an atom. Our goal here is to give precise estimates for the sum of the negative eigenvalues of an ordinary differential operator

$$H = -\frac{d^2}{dx^2} + V(x) \quad \text{on an interval } I_{\text{BVP}} \ . \tag{1}$$

We denote this sum by $\text{sneg}(H)$.

The potentials of interest to us are large and slowly varying. A basic example is

$$V(x) = \lambda^2 V_1(x) \quad \text{on} \quad I_{\text{BVP}} = [-1,1] \ , \tag{2}$$

with V_1 a fixed smooth function and λ a large parameter. We suppose

$$V_1(0) < 0 \ , \ V_1'(0) = 0 \ , \ V_1'' > c > 0 \text{ on } [-1,1] \ , \ \{V_1 < 0\} \subset\subset (-1,1) \ . \tag{3}$$

For such potentials, a standard approximate formula for the eigenvalue sum is

$$\text{sneg}(H) \approx -\frac{2}{3\pi} \int_{I_{\text{BVP}}} \big(-V(x)\big)_+^{3/2} \, dx \ , \tag{4}$$

where $t_+^s = t^s$ if $t > 0$, $t_+^s = 0$ if $t \le 0$.

Unfortunately, (4) is too crude for our purposes. To give a sharper approximation, we set

$$\phi(0) = \int_{I_{\text{BVP}}} \big(-V(x)\big)_+^{1/2} \, dx \ , \quad \phi'(0) = \frac{1}{2} \int_{I_{\text{BVP}}} \big(-V(x)\big)_+^{-1/2} \, dx \ , \quad \text{and} \tag{5}$$

$$\tilde{\chi}(t) = \min_{k \in \mathbb{Z}} \Big\{ |t - (k + \frac{1}{2})|^2 - \frac{1}{12} \Big\} \quad \text{for } t \in \mathbb{R} \ . \tag{6}$$

Our basic result for potentials of the form (2), (3) is as follows.

Theorem 1. *The sum of the negative eigenvalues of H is given by*

(7)

$$\text{sneg}(H) = -\frac{2}{3\pi} \int_{I_{\text{BVP}}} \big(-V(x)\big)_+^{3/2} dx + \frac{1}{24\pi} \int_{I_{\text{BVP}}} V''(x) \big(-V(x)\big)_+^{-1/2} \, dx$$
$$+ \frac{\pi}{2} \big(\phi'(0)\big)^{-1} \tilde{\chi}\Big(\frac{1}{\pi}\phi(0) - \frac{1}{2}\Big) + \text{ Error} \ ,$$

with $|\text{Error}| < C_\varepsilon \lambda^\varepsilon$. *Here,* C_ε *may depend on* $\varepsilon > 0$ *and on* V_1*, but not on* λ.

On the right in (7), the first term has size $\sim \lambda^3$, while the next two terms are $\sim \lambda$.

To motivate Theorem 1, we recall the textbook derivation of (4). At the heart of the argument is the WKB approximation (see Erdélyi [E]). According to WKB the eigenvalues E_k of H satisfy

$$\phi(E_k) \approx \pi(k + 1/2) \quad \text{for integers } k\ , \text{ with} \tag{8}$$

$$\phi(E) = \int\limits_{I_{\mathrm{BVP}}} \big(E - V(x)\big)_+^{1/2}\, dx\ . \tag{9}$$

Defining $E(t)$ as the solution of $\phi(E) = \pi(t + 1/2)$, we rewrite (8) as $E_k \approx E(k)$, which gives

$$\text{sneg}(H) \approx \sum_{k \in [a,b] \cap \mathbb{Z}} E(k)\ , \text{ with } a = \frac{1}{\pi}\phi(\min V) - \frac{1}{2},\ b = \frac{1}{\pi}\phi(0) - \frac{1}{2}\ . \tag{10}$$

It is reasonable to guess that

$$\sum_{k \in [a,b] \cap \mathbb{Z}} E(k) \approx \int_a^b E(t)dt\ . \tag{11}$$

Elementary manipulations show that $\int_a^b E(t)dt = -\frac{2}{3\pi} \int_{I_{\mathrm{BVP}}} \big(-V(x)\big)_+^{-3/2}\, dx$. Hence, the usual approximation (4) for $\text{sneg}(H)$ follows from (8) and (11).

Our previous papers [FS2,3] prove refinements of the basic approximations (8) and (11). Specifically, the eigenvalues E_k satisfy

$$\phi(E_k) + \frac{1}{48}\psi(E_k) \approx \pi(k + 1/2)\ , \text{ where} \tag{12}$$

$$\psi(E) = \lim_{\delta \to 0+} \Big\{ \int\limits_{E - V(x) > \delta} V''(x)\big(E - V(x)\big)_+^{-3/2}\, dx - q(E)\delta^{-1/2} \Big\} \tag{13}$$

and $q(E)$ is defined to make the limit finite in (13).

Equation (8) holds modulo errors $O(\lambda^{-1})$, while (12) holds modulo errors $O(\lambda^{\varepsilon-2})$. (See [FS2].) Regarding (11), we studied $\sum_{k\in\mathbb{Z}\cap[a,b]} f(k)$ for general, slowly varying functions f. We showed in [FS3] that

$$(14)\quad \sum_{k\in\mathbb{Z}\cap[a,b]} f(k) \approx \int_a^b f(t)dt - f(b)\chi_-(b) - f(a)\chi_+(a) + \frac{1}{2}f'(b)\tilde{\chi}(b) - \frac{1}{2}f'(a)\tilde{\chi}(a)$$

with

$$(15)\qquad \chi_-(t) = (t-k-1/2) \text{ for } k = \text{ (greatest integer } \leq t\text{);}$$

$$(16)\quad \chi_+(t) = (k-t-1/2) \text{ for } k = \text{ (least integer } \geq t\text{); and } \tilde{\chi}(t) \text{ given by (6) .}$$

Here we take $f(t) = E(t)$. Equation (14) then holds modulo errors $O(1)$, while (11) merely holds modulo $O(\lambda)$. Using (12) and (14) in place of the crude approximations (8), (11) in the derivation of (4), we arrive at the sharp formula (7) for $\text{sneg}(H)$. This concludes our introductory remarks on Theorem 1.

To understand atoms, we need to deal with potentials having a Coulomb singularity

$$(17)\qquad V(x) \approx \frac{\ell(\ell+1)}{x^2} - \frac{Z}{x} + E^0 \text{ near } x = 0\ .$$

In our application [FS4,5], we have $Z >> 1$, $E^0 \sim Z^{4/3}$ and $\ell = O(Z^{1/3})$. When $\ell << Z^{1/3}$, the potential (17) is too singular to allow us to apply Theorem 1. Hence we need a more general result. In [FS2,3] we studied potentials $V(x)$ satisfying estimates

$$(18)\qquad \left|\left(\frac{d}{dx}\right)^\alpha V(x)\right| \leq C_\alpha S(x)\big(B(x)\big)^{-\alpha} \text{ when } x \in I\ ,$$

for suitable weight functions $S(x)$, $B(x)$. Here I is an interval containing $\{x \in I_{\text{BVP}}\colon V(x) < 0\}$. Hypothesis (18) lets us treat all the potentials we need, simply by picking the proper weight functions $S(x)$, $B(x)$ and interval I. For instance, when V is given by (2), (3), then (18) holds with $S(x) = \lambda^2$, $B(x) = 1$, $I = I_{\mathrm{BVP}}$. If V is given instead by (17), then we take $I \subset \left[\frac{\ell(\ell+1)}{2Z}, \infty\right)$ and pick $S(x) = \frac{Z}{x}$, $B(x) = x$ in the region where (17) holds. Here again, V satisfies (18).

The first main result of this paper is a version of Theorem 1 for general potentials satisfying (18). In the setting of (18), the natural analogue of hypothesis (3) is as follows.

(19)
$$V(x) \text{ takes its minimum at an interior point } x_0 \in I \text{ , and } V(x_0) < -cS(x_0)\ .$$
$$\text{For } |x - x_0| \leq c_1 B(x_0) \text{ we have } V''(x) > cS(x)\big(B(x)\big)^{-2}; \text{ while for}$$
$$|x - x_0| \geq c_1 B(x_0) \text{ we have } |V'(x)| > cS(x)\big(B(x)\big)^{-1}\ .$$

The number that plays the rôle of λ in Theorem 1 is

$$\Lambda = \left(\int_{V(x)<0} \frac{dx}{\big(S(x)\big)^{1/2}\big(B(x)\big)^2}\right)^{-1}, \tag{20}$$

as explained in [FS2]. Now we can state our first main result, modulo technicalities.

First WKB Eigenvalue Sum Theorem. *Let V satisfy (18), (19) and various technical conditions; and suppose Λ is large. Then* $\mathrm{sneg}(H)$ *is given by (7), with* $|\mathrm{Error}| < C_\varepsilon \Lambda^{\varepsilon-2}|V(x_0)|$.

For a precise statement of the result, we refer the reader to the relevant section of this paper.

When V satisfies (17) with relatively small ℓ, we can make a crucial improvement over the preceding theorem, by comparing the eigenvalue sum for V with that of the Coulomb potential

$$V_c(x) = \frac{\ell(\ell+1)}{x^2} - \frac{Z}{x} + E^0\ . \tag{21}$$

We set $H_c = -\frac{d^2}{dx^2} + V_c(x)$ on $(0,\infty)$. Our second main result is as follows.

Second WKB Eigenvalue Sum Theorem. *Let V satisfy (18), (19) and various technical conditions; and suppose Λ is large. Assume also that*

$$V(x) = V_c(x) \text{ for all } x \in (0, x_*] \ , \tag{22}$$

with $x_0 << x_* << Z^{-1/3}$. Then

$$\begin{aligned}
\operatorname{sneg}(H) - \operatorname{sneg}(H_c) &= -\frac{2}{3\pi}\int_0^\infty \big(-V(x)\big)_+^{3/2}\, dx \\
&+ \frac{1}{24\pi}\int_0^\infty V''(x)\big(-V(x)\big)_+^{-1/2}\, dx + \frac{\pi}{2}\big(\phi'(0)\big)^{-1}\tilde{\chi}\big(\frac{1}{\pi}\phi(0) - \frac{1}{2}\big) \\
&+ \frac{2}{3\pi}\int_0^\infty \big(-V_c(x)\big)_+^{-3/2}\, dx - \frac{1}{24\pi}\int_0^\infty V_c''(x)\big(-V_c(x)\big)_+^{-1/2}\, dx \\
&- \frac{2(E^0)^{3/2}}{Z}\tilde{\chi}\big(\frac{Z}{2(E^0)^{1/2}} - \sqrt{\ell(\ell+1)} - \frac{1}{2}\big) + \text{ Error } ,
\end{aligned} \tag{23}$$

with

$$|\text{Error}| < C_\varepsilon \Lambda^{\varepsilon - 2}\big(\frac{Z}{x_*}\big) \ .$$

Again we refer the reader to the relevant section of this paper for a precise statement of the result.

This paper contains additional results on $\operatorname{sneg}(H)$ under various degenerate hypotheses, but we omit them from the introduction. In [FS5] we will apply the results of this paper to radially symmetric three-dimensional Schrödinger operators.

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Review of Earlier Results

Here we gather results from [FS2,3] which we will need in this paper.

A. The WKB Eigenvalue Theorem.

Set-up.

We are given the following: A potential $V(x)$ defined on a (possibly unbounded) interval I_{BVP}; two positive functions $S(x)$, $B(x)$ defined on a subinterval $I \subset I_{\mathrm{BVP}}$; two real numbers $E_0 \le E_\infty$; positive numbers $\varepsilon < \frac{1}{100}$, $K > 1$ and $N > K\varepsilon^{-10}$. We define $N' = [\varepsilon N/500]$ and $N'' = \frac{3}{2}\varepsilon N' - K - 33$.

Our goal is to understand the eigenvalues of the self–adjoint operator $H = \frac{-d^2}{dx^2} + V(x)$ on $L^2(I_{\mathrm{BVP}})$, with Dirichlet or Neumann conditions at the endpoints.

Hypotheses.

Assumptions on $V(x)$, $S(x)$, $B(x)$ in I

(Hyp0) If $x, y \in I$ and $|x-y| < cB(x)$, then $c < \frac{B(y)}{B(x)} < C$ and $c < \frac{S(y)}{S(x)} < C$.

(Hyp1) For $x \in I$ and $\alpha \ge 0$ we have $|(\frac{d}{dx})^\alpha V(x)| \le C_\alpha S(x) B^{-\alpha}(x)$.

(Hyp2) The equation $V(x) = E_0$ has two solutions $x_{\mathrm{left}} < x_{\mathrm{rt}}$ in I, and they satisfy $\mathrm{dist}(x_{\mathrm{left}}, \partial I) > cB(x_{\mathrm{left}}), \mathrm{dist}(x_{\mathrm{rt}}, \partial I) > cB(x_{\mathrm{rt}})$.

(Hyp3) For $x_{\mathrm{left}} \le x \le x_{\mathrm{left}} + c_1 B(x_{\mathrm{left}})$ we have $-V'(x) > cS(x_{\mathrm{left}})B^{-1}(x_{\mathrm{left}})$, and for $x_{\mathrm{rt}} - c_1 B(x_{\mathrm{rt}}) \le x \le x_{\mathrm{rt}}$ we have $+V'(x) > cS(x_{\mathrm{rt}})B^{-1}(x_{\mathrm{rt}})$.

(Hyp4) For $x_{\mathrm{left}} + c_1 B(x_{\mathrm{left}}) \le x \le x_{\mathrm{rt}} - c_1 B(x_{\mathrm{rt}})$ we have $cS(x) < E_0 - V(x) < CS(x)$.

To state the remaining hypotheses, we establish some notation. Set $\lambda(x) = S^{1/2}(x)B(x)$ for $x \in I$. Then set

$$B_{\mathrm{left}} = B(x_{\mathrm{left}}), \quad S_{\mathrm{left}} = S(x_{\mathrm{left}}), \quad \lambda_{\mathrm{left}} = \lambda(x_{\mathrm{left}}) \ .$$

$$B_{\mathrm{rt}} = B(x_{\mathrm{rt}}), \quad S_{\mathrm{rt}} = S(x_{\mathrm{rt}}), \quad \lambda_{\mathrm{rt}} = \lambda(x_{\mathrm{rt}}) \ .$$

For $|E - E_0| < cS_{\text{left}}$, let $x_{\text{left}}(E)$ be the solution of $V(x) = E$ nearest to x_{left}, and for $|E - E_0| < cS_{\text{rt}}$, let $x_{\text{rt}}(E)$ be the solution of $V(x) = E$ nearest to x_{rt}. Define $S_{\min} = \inf_{x_{\text{left}} < x < x_{\text{rt}}} S(x)$ and $\Lambda = \left(\int_{x_{\text{left}}}^{x_{\text{rt}}} \frac{dx}{S^{1/2}(x) B^2(x)} \right)^{-1}$.

Our remaining hypotheses are as follows.

Assumptions on $V(x)$ in all of I_{BVP}.

(Hyp5) If $|E - E_0| < c_2 S_{\min}$ and $E \le E_\infty$, then $V(x) > E$ for all $x \in I_{\text{BVP}} \backslash [x_{\text{left}}(E), x_{\text{rt}}(E)]$.

(Hyp6) If $x \in I_{\text{BVP}}$ satisfies $x < x_{\text{left}} - \frac{1}{2} \lambda_{\text{left}}^K B_{\text{left}}$ then $V(x) \ge E_\infty + \frac{100}{|x - x_{\text{left}}|^2}$, and if $x \in I_{\text{BVP}}$ satisfies $x > x_{\text{rt}} + \frac{1}{2} \lambda_{\text{rt}}^K B_{\text{rt}}$, then $V(x) \ge E_\infty + \frac{100}{|x - x_{\text{rt}}|^2}$.

Technical Assumptions.

(Hyp7) $\max_{x \in I} S(x) \le \lambda_{\text{left}}^K S_{\text{left}}$ and $\max_{x \in I} S(x) \le \lambda_{\text{rt}}^K S_{\text{rt}}$

(Hyp8) $\int_{x_{\text{left}}}^{x_{\text{rt}}} \frac{dx}{S^{1/2}(x)} \le \Lambda^K \cdot \min(S_{\text{left}}^{-1/2} B_{\text{left}}, S_{\text{rt}}^{-1/2} B_{\text{rt}})$

(Hyp9) $[\int_{x_{\text{left}}}^{x_{\text{rt}}} \frac{dx}{S^{1/2}(x) B^4(x)}] \cdot [\int_{x_{\text{left}}}^{x_{\text{rt}}} \frac{dx}{S^{1/2}(x)}] \le \Lambda^K$

WKB Condition.

(Hyp10) Λ is bounded below by a positive constant depending only on ε, K, N, and on c, C, c_1, c_2, C_α in (Hyp0)...(Hyp4).

Definitions and Basic Properties of Phases.

Assume hypotheses (Hyp0)...(Hyp10). For $|E - E_0| < c_\# S_{\min}$, define

$$\phi(E) = \int_{x_{\text{left}}(E)}^{x_{\text{rt}}(E)} (E - V(x))^{1/2} \, dx \quad \text{and}$$

$$\psi(E) = \lim_{\delta \to 0+} \Big[\int_{x_{\text{left}}(E)+\delta}^{x_{\text{rt}}(E)-\delta} V''(x)(E - V(x))^{-\frac{3}{2}} \, dx - q(E)\delta^{-1/2} \Big]$$

with $q(E)$ uniquely specified by demanding the finiteness of the limit.

Lemma A1. *For* $|E-E_0|<c_\# S_{\min}$ *we have*

$$\left|\left(\frac{d}{dE}\right)^\beta \phi(E)\right| \le C_\#^\beta \int_{x_{\rm left}}^{x_{\rm rt}} S^{\frac12-\beta}(x)\,dx \quad \textit{and}$$

$$\left|\left(\frac{d}{dE}\right)^\beta \psi(E)\right| \le C_\#^\beta \int_{x_{\rm left}}^{x_{\rm rt}} \frac{dx}{S^{\frac12+\beta}(x)B^2(x)}\ .$$

Also $c_\# \int_{x_{\rm left}}^{x_{\rm rt}} \frac{dx}{S^{1/2}(x)} < \frac{d\phi(E)}{dE} < C_\# \int_{x_{\rm left}}^{x_{\rm rt}} \frac{dx}{S^{1/2}(x)}$.

The constants $c_\#$, $C_\#$, $C_\#^\beta$ *depend only on* ε, K, N, c, C, c_1, c_2, C_α *in the hypotheses (Hyp 0)... (Hyp 4).*

WKB Eigenvalue Theorem. *If (Hyp0)... (Hyp10) hold, then there is a function* $\Phi(E)$ *on* $[E_0-c_\# S_{\min}, E_0+c_\# S_{\min}]$ *and there are numbers* $E_{k_{\min}}, E_{k_{\min}+1}, \dots, E_{k_{\max}} \le E_\infty$ *with the following properties.*

(A) $\Phi(E)=\pm\frac{\pi}{2}+\phi(E)+\frac{1}{48}\psi(E)+\phi_{\rm error}(E)$, *with*

$$\left|\left(\frac{d}{dE}\right)^\beta \phi_{\rm error}\right| \le C_\#^\beta \Lambda^{-1} \int_{x_{\rm left}}^{x_{\rm rt}} \frac{dx}{S^{\frac12+\beta}(x)B^2(x)}, \quad \textit{all } \beta\ge 0\ .$$

(B) $\{k_{\min}, k_{\min}+1, \dots, k_{\max}\}$ *is exactly the set of integers* k *for which* $|\Phi(E)-\pi k| < C_\# \Lambda^{-N''}$ *for some* $E\in[E_0-\frac14 c_\# S_{\min}, E_0+\frac14 c_\# S_{\min}]\cap(-\infty, E_\infty]$.

(C) *If* $k_{\min}\le k<k_{\max}$, *then* E_k *is an eigenvalue of* H.

(D) *If* $k=k_{\max}$, *then* E_k *is either an eigenvalue of* H *or equal to* E_∞.

(E) *For* $k_{\min}\le k\le k_{\max}$ *we have* $|E_k-E_0|<c_\# S_{\min}$ *and* $|\Phi(E_k)-\pi k| < C_\# \Lambda^{-N''}$.

(F) *Every eigenvalue of* H *in the interval* $[E_0-\frac14 c_\# S_{\min}, E_0+\frac14 c_\# S_{\min}]$ $\cap(-\infty, E_\infty]$ *is one of the* E_k $(k_{\min}\le k\le k_{\max})$.

The constants $c_\#$, $C_\#^\beta$ depend only on ε, K, N, c, C, c_1 c_2, C_α in (Hyp0)...(Hyp5).

Remark.

Perhaps $E_{k_{\min}}$ or $E_{k_{\max}}$ or both lie slightly outside

$$[E_0-\frac14 c_\# S_{\min}, E_0+\frac14 c_\# S_{\min}]\ .$$

Note that the choice of sign in (A) affects only the indexing of the eigenvalues, not the content of the theorem.

B. The WKB Theorem on Low Eigenvalues. Let $\varepsilon, K, N > 0$ be given, with $\varepsilon N \geq 100$. Let $V(x)$ be a potential defined on a (possibly unbounded) interval I_{BVP}. Let S, B be positive numbers, and let $x_0 \in I_{\text{BVP}}$ be given. Define $\lambda = S^{1/2}B$. Let E_∞ be a given energy, with $E_\infty > V(x_0)$. We make the following assumptions.

(H0*) $I = \{|x - x_0| < cB\} \subset I_{\text{BVP}}$

(H1*) $|(\frac{d}{dx})^\alpha V(x)| \leq C_\alpha S B^{-\alpha}$ in I

(H2*) $\frac{d^2}{dx^2} V \geq c' S B^{-2}$ in I

(H3*) $V'(x_0) = 0$

(H4*) For $x \in I_{\text{BVP}} \backslash I$ we have $V(x) \geq \min\{E_\infty, V(x_0) + c'' \lambda^{-2\varepsilon} S\}$.

(H5*) For $x \in I_{\text{BVP}}$ with $|x - x_0| > \frac{1}{2}\lambda^K B$, we have $V(x) \geq E_\infty + \frac{1000}{|x-x_0|^2}$.

(H6*) λ is bounded below by a positive constant depending only on c, c', c'', C_α in (H0*)...(H4*), and on ε, K, N.

Let $H = -\frac{d^2}{dx^2} + V(x)$ on $L^2(I_{\text{BVP}})$ with Dirichlet or Neumann conditions at the endpoints.

For $V(x_0) < E < V(x_0) + \lambda^{-2\varepsilon} S$, define $x_{\text{left}}(E) < x_{\text{rt}}(E)$ to be the two values of $x \in I$ at which $V(x) = E$. Then define

$$\phi(E) = \int_{x_{\text{left}}(E)}^{x_{\text{rt}}(E)} (E - V(x))^{1/2}\, dx$$

$$\begin{aligned}\psi(E) &= \lim_{\delta \to 0+} \Big[\int\limits_{\substack{x \in I \\ E - V(x) > \delta}} V''(x)(E - V(x))^{-3/2}\, dx - q(E)\delta^{-1/2} \Big] \\ &= \lim_{\delta_{\text{left}}, \delta_{\text{rt}} \to 0+} \Big[\int_{x_{\text{left}}(E)+\delta_{\text{left}}}^{x_{\text{rt}}(E)-\delta_{\text{rt}}} V''(x)(E - V(x))^{-3/2}\, dx - q_{\text{left}}(E)\delta_{\text{left}}^{-1/2} \\ &\qquad - q_{\text{rt}}(E)\delta_{\text{rt}}^{-1/2} \Big]\end{aligned}$$

with $q(E)$, $q_{\text{left}}(E)$, $q_{\text{rt}}(E)$ uniquely specified by demanding the finiteness of the limits.

Lemma B1. *The phases $\phi(E)$, $\psi(E)$ satisfy the estimates*

$$\left|\left(\frac{d}{dE}\right)^{\beta}\phi(E)\right| \le C_{\#}^{\beta}\lambda S^{-\beta}$$

$$\left|\left(\frac{d}{dE}\right)^{\beta}\psi(E)\right| \le C_{\#}^{\beta}\lambda^{-1} S^{-\beta}$$

$$\frac{d}{dE}\phi(E) \ge c_{\#}\lambda S^{-1}$$

for $V(x_0) < E < V(x_0) + \lambda^{-2\varepsilon}S$.

The constants $c_{\#}$, $C_{\#}^{\beta}$ depend only on c, c',c'', C_α, ε, K, N in hypotheses (H0)... (H6*).*

WKB Theorem on Low Eigenvalues. *Assume (H0*)... (H6*). Then there is a finite sequence $E_0, E_1, \dots, E_{k_{\max}}$ with the following properties.*

(a) *Let $w = \phi(E_*) + \frac{1}{48}\psi(E_*)$ with $E_* = min\{E_\infty, V(x_0) + c_{\#}\lambda^{-2\varepsilon}S\}$, and let $\overline{n}$ be the largest integer with $\pi(\overline{n}+1/2) \le w$. If $\min_{k\in\mathbb{Z}}|w - \pi(k+1/2)| > C_{\#}\lambda^{-2+4\varepsilon}$, then $k_{\max} = \overline{n}$. In any case, $|k_{\max} - \overline{n}| \le 1$.*

(b) *If $0 \le k < k_{\max}$, then E_k is an eigenvalue of H.*

(c) *Either $E_{k_{\max}} = E_\infty$ or else $E_{k_{\max}}$ is an eigenvalue of H.*

(d) *Every eigenvalue E of H satisfying $E \le E_\infty$, $|E - V(x_0)| < c_{\#}\lambda^{-2\varepsilon}S$ is one of the E_k.*

(e) *For $0 \le k \le k_{\max}$ we have $V(x_0) < E_k < V(x_0) + 2c_{\#}\lambda^{-2\varepsilon}S$ and $|\phi(E_k) + \frac{1}{48}\psi(E_k) - \pi(k+1/2)| \le C_{\#}\lambda^{-2+4\varepsilon}$.*

The constants $c_{\#}$, $C_{\#}$ depend only on ε, K, N, c, c', c'', C_α in hypotheses (H0)... (H6*).*

C. The Basic Hypotheses for Potentials.

Set-Up: We are given positive numbers ε, K, N, $\hat{c}$; two intervals $I \subset I_{\text{BVP}}$ (possibly unbounded); a point $x_0 \in I$; a potential $V(x)$ defined on I_{BVP}; and two positive functions $S(x)$, $B(x)$ defined on I. Our assumptions are as follows.

Assumptions Concerning $V(x)$, $S(x)$, $B(x)$ on I.

(Z0) If $x, y \in I$ and $|x - y| < cB(x)$, then $c < \frac{B(y)}{B(x)} < C$ and $c < \frac{S(y)}{S(x)} < C$.

(Z1) If $x \in I$ and $\alpha \geq 0$, then $\left|\left(\frac{d}{dx}\right)^{\alpha} V(x)\right| \leq C_{\alpha} S(x) B^{-\alpha}(x)$.

(Z2) The set $\{x \in I \mid V(x) < 0\}$ is a non-empty interval $(x_{\text{left}}, x_{\text{rt}})$, with $\text{dist}\,(x_{\text{left}}, \partial I) > cB(x_{\text{left}})$ and $\text{dist}\,(x_{\text{rt}}, \partial I) > cB(x_{\text{rt}})$.

(Z3) We have $V(x_0) < -cS(x_0)$, $V'(x_0) = 0$; and for $|x - x_0| \leq c_1 B(x_0)$ we have $V''(x) \geq cS(x_0) B^{-2}(x_0)$.

(Z4) For $x_{\text{left}} \leq x \leq x_0 - c_1 B(x_0)$ we have $-V'(x) > cS(x)B^{-1}(x)$; and for $x_0 + c_1 B(x_0) \leq x \leq x_{\text{rt}}$ we have $+V'(x) > cS(x)B^{-1}(x)$.

Define $\lambda(x) = S^{1/2}(x)B(x)$ for $x \in I$, and set

$$\Lambda = \left(\int_{x_{\text{left}}}^{x_{\text{rt}}} \frac{dx}{\lambda(x)B(x)}\right)^{-1}.$$

Assumptions Concerning $V(x)$ on all of I_{BVP}.

(Z5) We have $V(x) > 0$ for all $x \in I_{\text{BVP}} \backslash [x_{\text{left}}, x_{\text{rt}}]$.

(Z6) For all $x \in I_{\text{BVP}}$ with $x < x_{\text{left}} - \Lambda^K B(x_{\text{left}})$, we have $V(x) \geq \frac{1000}{|x - x_{\text{left}}|^2}$; and for all $x \in I_{\text{BVP}}$ with $x > x_{\text{rt}} + \Lambda^K B(x_{\text{rt}})$, we have $V(x) \geq \frac{1000}{|x - x_{\text{rt}}|^2}$.

Polynomial Growth Assumptions on $S(x)$, $B(x)$, I.

(Z7) We have $\max_{x \in I} B(x) < \Lambda^K \min_{x \in I} B(x)$; $\max_{x \in I} S(x) < \Lambda^K \min_{x \in I} S(x)$; and $|I| < \Lambda^K \cdot \min_{x \in I} B(x)$.

Smallness of the Constant $\hat{c}$.

(Z8) The constant $\hat{c}$ is bounded above by a certain small, positive number determined by ε, K, N, c, C, c_1, C_α.

The WKB Hypothesis.

(Z9) Λ is bounded below by a certain large, positive number determined by ε, K, N, c, C, c_1, $\hat{c}$, C_α.

Let E_k and $u_k(x)$ be the eigenvalues and (normalized) eigenfunctions of $-\frac{d^2}{dx^2} + V(x)$ on I_{BVP}, with Dirichlet or Neumann boundary conditions. In [FS3] we studied the density $\rho(x) = \sum_{E_k \le 0} |u_k(x)|^2$ under the assumptions (Z0)...(Z9). Our result was called the "WKB Density Theorem." This paper will study $\mathrm{sneg}(H) = \sum_{E_k \le 0} E_k$ under the same assumptions.

Remark.

We have kept hypothesis (Z8) from [FS3], even though the constant $\hat{c}$ plays no role in this paper, simply to keep the same hypotheses as before.

D. The WKB Eigenvalue Theorems for Potentials with Weak Turning Points.

Set–up.

We are given an energy E_0 and a potential $V(x)$ defined on a (possibly unbounded) interval I_{BVP}. The interval I_{BVP} is partitioned into subintervals $I_{\text{far left}}$, I_{left}, I_{center}, I_{rt}, $I_{\text{far rt}}$ with $I_{\text{far left}}$ to the left of I_{left}, I_{left} to the left of I_{center}, etc. Here, $I_{\text{far left}}$ and $I_{\text{far right}}$ may be empty. On I_{center} we are given positive weight functions $S(x)$, $B(x)$. Set $\lambda(x) = S^{1/2}(x)B(x)$ and $\Lambda = (\int_{I_{\text{center}}} \frac{dx}{\lambda(x)B(x)})^{-1}$. We make the following assumptions.

Hypotheses.

(H$\hat{0}$) I_{center} is non–empty, and for $x, y \in I_{\text{center}}$ with $|x - y| < cB(x)$ we have $c < B(y)/B(x) < C$ and $c < S(y)/S(x) < C$, and $|I_{\text{center}}| > cB(x)$.

(H$\hat{1}$) For $x \in I_{\text{center}}$ we have $|(\frac{d}{dx})^\alpha V(x)| \le C_\alpha S(x) B^{-\alpha}(x)$ and $cS(x) < E_0 - V(x) < CS(x)$.

($\hat{\text{H2}}$) Λ is bounded below by a large number depending only on c, C, C_α in ($\hat{\text{H0}}$), ($\hat{\text{H1}}$).

($\hat{\text{H3}}$) I_{left}, I_{rt} are non–empty. If $I_{\text{center}} = [x_{\text{left}}, x_{\text{rt}}]$, then we have $|I_{\text{left}}| \leq \underline{C}B(x_{\text{left}})$, $|I_{\text{rt}}| \leq \underline{C}B(x_{\text{rt}})$, $\lambda(x_{\text{left}}) \leq \underline{C}$, $\lambda(x_{\text{rt}}) \leq \underline{C}$.

($\hat{\text{H4}}$) If $I_{\text{left}} = [x_{\text{far left}}, x_{\text{left}}]$, then we have $|V(x) - E_0| \leq \underline{C}|I_{\text{left}}|^{-1}(x - \hat{x}_{\text{far left}})^{-1}$ in I_{left}. Here $\hat{x}_{\text{far left}} \leq x_{\text{far left}}$ with strict inequality unless $I_{\text{far left}} = \emptyset$.

($\hat{\text{H5}}$) For $x \in I_{\text{rt}}$ we have $|V(x) - E_0| \leq \underline{C}|I_{\text{rt}}|^{-2}$.

($\hat{\text{H6}}$) For $x \in I_{\text{far left}}$ we have $V(x) - E_0 \geq \underline{c}|I_{\text{left}}|^{-2}$. $V(x)$ is C^∞ in the interior of $I_{\text{far left}}$.

($\hat{\text{H7}}$) For $x \in I_{\text{far rt}}$ we have $V(x) - E_0 \geq \frac{-10^{-9}}{(x - x_{\text{rt}})^2}$.

Our goal here is to understand the eigenvalues of $H = -\frac{d^2}{dx^2} + V(x)$ on $L^2(I_{\text{BVP}})$ with Dirichlet boundary conditions.

Denote by $C_\#$ a constant depending on c, C, C_α in ($\hat{\text{H0}}$), ($\hat{\text{H1}}$). Denote by C_*, c_* etc. constants depending only on c, C, C_α, $\underline{c}$, $\underline{C}$ in ($\hat{\text{H0}}$)…($\hat{\text{H7}}$).

Note that I_{left} and I_{rt} don't play completely analogous rôles in our hypotheses.

Theorem D1. *Under the assumptions ($\hat{H0}$)…($\hat{H7}$) we have*

$$|(\textit{Number of eigenvalues of } H < E_0) - \frac{1}{\pi}\int_{I_{\text{center}}} (E_0 - V(t))^{1/2}\, dt| \leq C_* \; .$$

As an application of this theorem, we study the following situation.

Set–up.

$V(x)$ is a potential defined on a (possibly unbounded) interval I_{BVP}. We are given a subinterval $I \subset I_{\text{BVP}}$ and weight functions $S(x)$, $B(x) > 0$ defined on I. Set $\lambda(x) = S^{1/2}(x)B(x)$. We are given an energy E_0.

We make the following assumptions.

Hypotheses.

(H$\bar{0}$) For $x, y \in I$ with $|x-y| < cB(x)$ we have $c < \frac{B(y)}{B(x)} < C$ and $c < \frac{S(y)}{S(x)} < C$, and $|I| > cB(x)$.

(H$\bar{1}$) For $x \in I$ we have $|(\frac{d}{dx})^\alpha V(x)| \leq C_\alpha S(x) B^{-\alpha}(x)$.

(H$\bar{2}$) $\{x \in I_{\text{BVP}} \mid V(x) < E_0\} = (x_{\text{left}}, x_{\text{right}}) \subset I$ with $\text{dist}(x_{\text{left}}, \partial I) > cB(x_{\text{left}})$, $\text{dist}(x_{\text{right}}, \partial I) > cB(x_{\text{right}})$.

(H$\bar{3}$) In $[x_{\text{left}}, x_{\text{left}} + c_1 B(x_{\text{left}})]$ we have $-V'(x) \geq cS(x_{\text{left}})/B(x_{\text{left}})$, and in $[x_{\text{right}} - c_1 B(x_{\text{right}}), x_{\text{right}}]$ we have $+V'(x) \geq cS(x_{\text{right}})/B(x_{\text{right}})$.

(H$\bar{4}$) In $[x_{\text{left}} + c_1 B(x_{\text{left}}), x_{\text{right}} - c_1 B(x_{\text{right}})]$ we have $cS(x) < E_0 - V(x) < CS(x)$.

(H$\bar{5}$) In $I_{\text{BVP}}^{\text{interior}} \cap (-\infty, x_{\text{left}})$, $V(x)$ is decreasing and C^∞.

(H$\bar{6}$) $\Lambda = (\int_{x_{\text{left}}}^{x_{\text{right}}} \frac{dx}{\lambda(x) B(x)})^{-1}$ is bigger than a large positive number depending only on c, C, c_1, C_α in (H$\bar{0}$)... (H$\bar{4}$).

Let $H = -\frac{d^2}{dx^2} + V(x)$ on $L^2(I_{\text{BVP}})$ with Dirichlet boundary conditions.

Theorem D2. *Under assumptions (H$\bar{0}$)... (H$\bar{6}$) above, we have*

$$\left|(\textit{Number of eigenvalues of } H < E_0) - \frac{1}{\pi} \int_{I_{\text{BVP}}} (E_0 - V(t))_+^{1/2}\, dt\right| \leq C_*.$$

The constant C_* depends only on c, C, c_1, C_α in (H$\bar{0}$)...(H$\bar{4}$).

Properties of Degenerate Potentials.

E. From the section "The Density for Degenerate One-Dimensional Potentials III" in [FS3], we recall the following results.

We are given a potential $V(x)$, smooth on $(0, \infty)$. We take $B(x) = x$, and let $S(x)$ be a positive function on $I = [x_0, x_1] \subset (0, \infty)$. As usual, we set $\lambda(x) = S^{1/2}(x) B(x)$ on I. In addition to x_0, x_1, we are given other points x_{small}, x_{big}, x_{crit}, $x_* \in (0, \infty)$, with

$$0 < x_{\text{small}} < \frac{1}{2} x_0 \;, 2x_0 < x_{\text{crit}} < \frac{1}{2} x_* \;, x_* < \frac{1}{16} x_1 \;, 2x_1 < x_{\text{big}} \;. \tag{1}$$

Set $H = -\frac{d^2}{dx^2} + V(x)$ on $(0,\infty)$ with Dirichlet boundary conditions. Let E_k, $u_k(x)$ be the eigenvalues and (normalized) eigenfunctions of H.

In addition to (1), we make the following assumptions.

Hypotheses.

(Z$\hat{0}$) If $x, y \in I$ and $|x-y| < \frac{1}{2}B(x)$, then $c < S(y)/S(x) < C$.

(Z$\hat{1}$) If $x \in I$, then $\left|\left(\frac{d}{dx}\right)^\alpha V(x)\right| \le C_\alpha S(x) B^{-\alpha}(x)$.

(Z$\hat{2}$) If $x \in I$, then $V(x) < -cS(x)$ and $V'(x) > cS(x)B^{-1}(x)$.

(Z$\hat{3}$) $\Lambda = \left(\int_I \frac{dx}{\lambda(x)B(x)}\right)^{-1}$ is greater than a certain large, positive number determined by c, C, C_α in (Z$\hat{0}$)...(Z$\hat{2}$).

(Z$\hat{4}$) For $x \in (0, x_{\text{small}}]$ we have $V(x) \ge \underline{c}x_0^{-2}$

(Z$\hat{5}$) For $x \in [x_{\text{small}}, x_0]$ we have $|V(x)| \le \underline{C}x_0^{-2}$

(Z$\hat{6}$) We have $x_{\text{big}} < \underline{C}x_1$ and $V(x)$ is increasing in $[x_1, x_{\text{big}}]$.

(Z$\hat{7}$) For $x \in \left[\frac{x_1}{8}, x_{\text{big}}\right]$, we have $|V(x)| \le \underline{C}x_1^{-2}$.

(Z$\hat{8}$) For $x \in [x_{\text{big}}, \infty)$, we have $V(x) \ge 0$.

(Z$\hat{9}$) For $E \in [V(x_*), 0]$ we have

$$\int_{x_0}^{x_{\text{crit}}} \big(E - V(x)\big)^{-1/2} dx \le \delta \cdot \int_{x_0}^{\frac{1}{2}x_*} \big(E - V(x)\big)^{-1/2} dx.$$

We denote by $c_\#$, $C_\#$, etc. constants that depend only on c, C, C_α; while c_*, C_* etc. denote constants that depend also on $\underline{c}$, $\underline{C}$.

Lemma E1. *Set $E_0 = 0$, $I_{\text{center}} = [x_0, x_1]$, $I_{\text{left}} = [x_{\text{small}}, x_0]$, $I_{\text{rt}} = [x_1, x_{\text{big}}]$, $I_{\text{far left}} = (0, x_{\text{small}}]$, $I_{\text{far rt}} = [x_{\text{big}}, \infty)$. Then the hypotheses (H$\hat{0}$)... (H$\hat{7}$) of Theorem 1 in the section on WKB Theory with Weak Turning Points are satisfied. The constants called c, C, C_α in (H$\hat{0}$)... (H$\hat{7}$) may be taken to be of the form $C_\#$. The constants called $\underline{c}$, $\underline{C}$ in (H$\hat{0}$)... (H$\hat{7}$) may be taken to be of the form C_*.*

Lemma E2. *Suppose $2x_0 < \tilde{x} < \frac{1}{2}x_1$. Set $\tilde{E} = V(\tilde{x})$, and define: $\tilde{V}(x) = V(x) - \tilde{E}$, $\tilde{E}_0 = 0$, $\tilde{I}_{\text{center}} = [x_0, \tilde{x} - \hat{C}_\# \lambda^{-2/3}(\tilde{x}) \cdot \tilde{x}]$ with $\hat{C}_\#$ picked large enough,*

$\tilde{I}_{\text{left}} = [x_{\text{small}}, x_0]$, $\tilde{I}_{\text{far left}} = (0, x_{\text{small}}]$, $\tilde{I}_{\text{rt}} = [\tilde{x} - \hat{C}_\# \lambda^{-2/3}(\tilde{x}) \cdot \tilde{x}, \tilde{x} + \hat{C}_\# \lambda^{-2/3}(\tilde{x}) \cdot \tilde{x}]$, $\tilde{I}_{\text{far rt}} = [\tilde{x} + \hat{C}_\# \lambda^{-2/3}(\tilde{x}) \cdot \tilde{x}, \infty)$. *On* $\tilde{I}_{\text{center}}$, *define* $\tilde{B}(x) = \min(x, \tilde{x} - x)$, $\tilde{S}(x) = S(x) \cdot \left(\frac{\tilde{x}-x}{\tilde{x}}\right)$. *Then* $\tilde{V}(x)$, $\tilde{E}_0$, $\tilde{I}_{\text{far left}} \dots \tilde{I}_{\text{far rt}}$, $\tilde{S}(x)$, $\tilde{B}(x)$ *satisfy hypotheses* ($\hat{H0}$)... ($\hat{H7}$) *in the section on WKB Theory with weak turning points. The constants called* c, C, C_α *in* ($\hat{H0}$)... ($\hat{H7}$) *may be taken to have the form* $C_\#$. *The constants called* $\underline{c}$, $\underline{C}$ *in* ($\hat{H0}$)... ($\hat{H7}$) *may be taken to have the form* C_*.

F. From the section "The Density for Degenerate One-Dimensional Potentials IV" in [FS3], we recall the following results.

We are given a smooth potential $V(x)$ on $(0, \infty)$. We take $B(x) = x$, and let $S(x)$ be a positive function on $I = [x_0, x_1] \subset (0, \infty)$. Let $\lambda(x) = S^{1/2}(x) B(x)$ as usual. We are given x_{crit}, x_*, x_{big}, satisfying

$$(1) \qquad 16x_0 < x_{\text{crit}} \; , 16x_{\text{crit}} < \frac{1}{10} x_* \; , \frac{16}{10} x_* < x_1 \; , 16x_1 < x_{\text{big}} \; .$$

Set $H = -\frac{d^2}{dx^2} + V(x)$ on $(0, \infty)$, with Dirichlet boundary conditions. (We have changed notation slightly from [FS3]; our present x_* differs from that in [FS3] by a factor of ten.) In addition to (1), we make the following assumptions.

Hypotheses.

($Z0^\dagger$) If $x, y \in I$ and $|x - y| < \frac{1}{2} B(x)$, then $c < S(y)/S(x) < C$.

($Z1^\dagger$) If $x \in I$ and $\alpha \geq 0$, then $\left|\left(\frac{d}{dx}\right)^\alpha V(x)\right| \leq C_\alpha S(x) B^{-\alpha}(x)$.

($Z2^\dagger$) If $x \in I$, then $V(x) < -cS(x)$ and $V'(x) > cS(x) B^{-1}(x)$.

($Z3^\dagger$) $\Lambda = \left(\int_I \frac{dx}{\lambda(x) B(x)}\right)^{-1}$ is greater than a certain large, positive number determined by c, C, C_α in ($Z0^\dagger$)...($Z2^\dagger$).

($Z4^\dagger$) $|V(x)| \leq \underline{C}/(x_0 x)$ for $x \in (0, x_0]$.

($Z5^\dagger$) $V(x)$ is increasing and negative in $[\frac{x_1}{8}, x_{\text{big}}]$, and satisfies there $|V(x)| < \underline{C} x_1^{-2}$. Also, $x_{\text{big}} < \underline{C} x_1$.

($Z6^\dagger$) $V(x) \geq -10^{-9} x^{-2}$ for $x \in [x_{\text{big}}, \infty)$.

(Z7†) For $E \in [V(\frac{x_*}{10}), 0]$, we have

$$\int_{x_0}^{x_{\text{crit}}} \big(E - V(x)\big)^{-1/2} dx \le \delta \cdot \int_{x_0}^{\frac{1}{20}x_*} \big(E - V(x)\big)^{-1/2} dx.$$

We denote by $c_\#$, $C_\#$, etc. constants that depend only on c, C, C_α in (Z0†)...(Z7†); while c_*, C_*, etc. denote constants that depend also on $\underline{C}$.

Lemma F1. *Set* $I_{\text{far left}} = \emptyset$, $I_{\text{left}} = (0, x_0]$, $I_{\text{center}} = [x_0, x_1]$, $I_{\text{rt}} = [x_1, x_{\text{big}}]$, $I_{\text{far rt}} = [x_{\text{big}}, \infty)$, $E_0 = 0$. *Then the hypotheses* ($\hat{H0}$)...($\hat{H7}$), *from the section on WKB with weak turning points, are satisfied. The constants called* c, C, C_α *in* ($\hat{H0}$)...($\hat{H7}$) *may be taken of the form* $C_\#$. *The constants called* $\underline{c}$, $\underline{C}$ *in* ($\hat{H0}$)...($\hat{H7}$) *may be taken of the form* C_*.

Lemma F2. *Suppose* $\tilde{E} = V(\tilde{x})$ *with* $\tilde{x} \in [\frac{1}{10}x_*, \frac{1}{4}x_1]$. *Take* $\tilde{V}(x) = V(x) - \tilde{E}$, $\tilde{S}(x) = S(x) \cdot \left(\frac{\tilde{x} - x}{\tilde{x}}\right)$, $\tilde{B}(x) = \min(x, \tilde{x} - x)$, $\tilde{E}_0 = 0$, $\tilde{I}_{\text{far left}} = \emptyset$, $\tilde{I}_{\text{left}} = (0, x_0]$, $\tilde{I}_{\text{center}} = [x_0, \tilde{x} - \hat{C}_\# \lambda^{-2/3}(\tilde{x}) \cdot \tilde{x}]$ *with* $\hat{C}_\#$ *picked large enough,* $\tilde{I}_{\text{rt}} = [\tilde{x} - \hat{C}_\# \lambda^{-2/3}(\tilde{x}) \cdot \tilde{x}, \tilde{x} + \hat{C}_\# \lambda^{-2/3}(\tilde{x}) \cdot \tilde{x}]$, $\tilde{I}_{\text{far rt}} = [\tilde{x} + \hat{C}_\# \lambda^{-2/3}(\tilde{x}) \cdot \tilde{x}, \infty)$. *Then the hypotheses* ($\hat{H0}$)...($\hat{H7}$), *in the section on WKB with weak turning points, are satisfied. The constants called* c, C, C_α *in* ($\hat{H0}$)...($\hat{H7}$) *may be taken of the form* $C_\#$. *The constants called* $\underline{c}$, $\underline{C}$ *in* ($\hat{H0}$)...($\hat{H7}$) *may be taken of the form* C_*.

G. In the section "The Density for Degenerate One-Dimensional Potentials II" in [FS3], we studied the following class of degenerate potentials.

Set-Up. We are given a potential $V(x)$ defined on a (possibly unbounded) interval I_{BVP}; positive functions $S(x)$, $B(x)$, defined on a subinterval $I \subset I_{\text{BVP}}$; a point $x_{\text{crit}} \in I_{\text{BVP}}$; an energy $E_{\text{crit}} \le 0$; and a number δ strictly between 0 and 1.

Assumptions.

($\overline{Z0}$) For $x, y \in I$ with $|x - y| < cB(x)$, we have $c < \frac{B(y)}{B(x)} < C$ and $c < \frac{S(y)}{S(x)} < C$, and $|I| > cB(x)$.

($\overline{Z1}$) For $x \in I$ and $\alpha \ge 0$ we have $\left|\left(\frac{d}{dx}\right)^\alpha V(x)\right| \le C_\alpha S(x) B^{-\alpha}(x)$.

(Z$\overline{2}$) For $E_{\text{crit}} \le E \le 0$, the set $\{x \in I_{\text{BVP}} \mid V(x) \le E\}$ is a non-empty interval $(x_{\text{left}}(E), x_{\text{rt}}(E))$ contained in I, with $\text{dist}(x_{\text{left}}(E), \partial I) > cB(x_{\text{left}}(E))$ and $\text{dist}\,(x_{\text{rt}}(E), \partial I) > cB(x_{\text{rt}}(E))$.

(Z$\overline{3}$) For $E_{\text{crit}} \le E \le 0$, we have $-V'(x) \ge cS(x)B^{-1}(x)$ for $x \in [x_{\text{left}}(E), x_{\text{left}}(E) + c_1 B(x_{\text{left}}(E))]$ and $+V'(x) \ge cS(x)B^{-1}(x)$ for $x \in [x_{\text{rt}}(E) - c_1 B(x_{\text{rt}}(E)), x_{\text{rt}}(E)]$.

(Z$\overline{4}$) For $E_{\text{crit}} \le E \le 0$, we have $cS(x) < E - V(x) < CS(x)$ for $x \in [x_{\text{left}}(E) + c_1 B(x_{\text{left}}(E)), x_{\text{rt}}(E) - c_1 B(x_{\text{rt}}(E))]$

(Z$\overline{5}$) $V(x)$ is decreasing and C^∞ on $I_{\text{BVP}}^{\text{interior}} \cap (-\infty, x_{\text{left}}(0)]$.

(Z$\overline{6}$) For $E_{\text{crit}} \le E \le 0$, we have $x_{\text{left}}(E) + cB(x_{\text{left}}(E)) \le x_{\text{crit}}$.

(Z$\overline{7}$) For $E_{\text{crit}} \le E \le 0$, we have

$$\int_{I_{\text{BVP}} \cap (-\infty, x_{\text{crit}}]} \big(E - V(t)\big)_+^{-1/2} dt \le \delta \int_{I_{\text{BVP}}} \big(E - V(t)\big)_+^{-1/2} dt$$

(Z$\overline{8}$) $\Lambda \equiv \left(\int_{x_{\text{left}}(0)}^{x_{\text{rt}}(0)} \frac{dx}{\lambda(x)B(x)} \right)^{-1}$ is greater than a certain large, positive number determined by c, C, c_1, C_α above.

Here, $\lambda(x) = S^{1/2}(x)B(x)$ as usual.

Note that (Z$\overline{0}$)...(Z$\overline{8}$) imply the hypotheses (H$\overline{0}$)...(H$\overline{6}$) of Theorem 2 in the section on WKB Theory with weak turning points, for any $E_0 \in [E_{\text{crit}}, 0]$.

H. Approximating Sums by Integrals.

¿From [FS3] we recall the following result.

Lemma on Riemann Sums. *Let $f(t)$, $\sigma(t)$, $\tau(t)$ be defined on a non–empty interval $[a, b]$. Suppose $\sigma(t) > 0$, $\tau(t) \ge 1$ in $[a, b]$; and assume that whenever $t_1, t_2 \in [a, b]$ with $|t_1 - t_2| < c\tau(t_1)$, we have $c < \frac{\tau(t_2)}{\tau(t_1)} < C$ and $c < \frac{\sigma(t_2)}{\sigma(t_1)} < C$. Finally assume $|(\frac{d}{dt})^m f(t)| \le C_m \sigma(t) \tau^{-m}(t)$ for $t \in [a, b]$. Then $\sum\limits_{k \in \mathbb{Z} \cap [a,b]} f(k) = \int\limits_a^b f(t)dt - f(b)\chi_-(b) - f(a)\chi_+(a) + \frac{1}{2} f'(b)\tilde{\chi}(b) - \frac{1}{2} f'(a)\tilde{\chi}(a) +$ Error with $|\text{Error}| \le$*

$C'\sigma(a)\tau^{-2}(a)+C'\sigma(b)\tau^{-2}(b)+C'_N\int_a^b\sigma(t)\tau^{-N}(t)dt$. *Here,* C' *depends only on* c, C, C_m*; and* C'_N *depends only on* c, C, C_m, N. *If* $f(t)=0$ *to infinite order at* $t=a$, *then we have the sharper estimate* $|\mathrm{Error}|\leq C'\sigma(b)\tau^{-2}(b)+C'_N\int_a^b\sigma(t)\tau^{-N}(t)dt$, *with* C', C'_N *as before. Similarly, if* $f(t)=0$ *to infinite order at* $t=b$, *then* $|\mathrm{Error}|\leq C'\sigma(a)\tau^{-2}(a)+C'_N\int_a^b\sigma(t)\tau^{-N}(t)dt$. *If* $f(t)=0$ *to infinite order at both* $t=a$ *and* $t=b$, *then* $|\mathrm{Error}|\leq C'_N\int\limits_a^b\sigma(t)\tau^{-N}(t)dt$.

We adopt the notation and hypotheses of (Z0)...(Z9) of the WKB Density Theorem, and let E_* be a given energy strictly between zero and the minimum of the potential. Our goal is to compute the sum of the eigenvalues in $[E_*, 0]$ modulo a small error. We assume that the phase $\phi(E_*)$ differs from $\pi\cdot$(integer) by at most Λ^{-2}. Thus, $\phi(E_*) + \frac{1}{48}\psi(E_*)$ is not close to $\pi(k + 1/2)$ for any integer k. We assume also $\phi(E_*) < \phi(0) - 1$.

We use $c_\#$, $C_\#$, etc. to denote constants that depend only on ε, K, N, c, C, c_1, C_α, $\hat{c}$ in the hypotheses (Z0)...(Z9) of the WKB Density Theorem.

Our first step is to use the WKB Eigenvalue Theorem and the WKB Theorem on low eigenvalues to identify the eigenvalues of $-\frac{d^2}{dx^2} + V(x)$. Later, we will use the lemma on Riemann sums to compute the sum of the eigenvalues. We start by checking the hypotheses of the WKB Eigenvalue Theorem.

Lemma 1. *Suppose* $V(x_0) + c_\# S(x_0) \le E_0 \le 0$. *Then the hypotheses (Hyp0) ...(Hyp10) of the WKB Theorems are satisfied, with* $E_\infty = 0$ *and with* $300K$ *in place of* K. *The constants called* c, C, c_1, c_2, C_α *in (Hyp0)...(Hyp10) all have the form* $C_\#$. *The number called* Λ *in (Hyp0)...(Hyp10) is greater than or equal to our present* Λ.

Proof.

This is just Lemma 1 from the section "Combining Microlocalized Results" in [FS3] with $\varphi(E) \equiv 0$ and $\hat{C}_\beta \equiv 0$. ■

Lemma 2. *Suppose* $2^{-2(m+1)} c_\# S(x_0) < E_0 - V(x_0) \le 2^{-2m} c_\# S(x_0)$, *with* $\lambda^{-3\varepsilon}(x_0) \le 2^{-m} \le 1$. *Set* $S_m = 2^{-2m} S(x_0)$, $B_m = 2^{-m} B(x_0)$, $I_m = \{|x - x_0| < C_\# B_m\}$. *Then we can find an energy* $\tilde{E}_m \le 0$ *with* $\tilde{E}_m - E_0 \sim 2^{-2m} S(x_0)$,

$\min_{k\in\mathbb{Z}} |\phi(\tilde{E}_m) - \pi(k+1/2)| \geq \frac{1}{20}$, *and satisfying the following. The hypotheses (Hyp0)...(Hyp10) of the WKB Theorems are satisfied, with* $V(x) - \tilde{E}_m$ *in place of* $V(x)$*;* S_m *and* B_m *in place of* S *and* B*;* I_m *in place of* I*; with* $E_\infty = 0$*; with* $E_0 - \tilde{E}_m$ *in place of* E_0*; and with* $300K$ *in place of* K*. The constants called* c*,* C*,* c_1*,* c_2*,* C_α *in (Hyp0)...(Hyp10) all have the form* $C_\#$*. The number called* Λ *in (Hyp0)...(Hyp10) is* $\sim 2^{-2m}\lambda(x_0)$*.*

Proof.

For $V(x_0) < E < V(x_0)+c_\# S(x_0)$ we have $\phi'(E) \sim \lambda(x_0)S^{-1}(x_0)$. Hence we can easily find an $\tilde{E}_m \leq 0$ with $\tilde{E}_m - E_0 \sim 2^{-2m}S(x_0)$ and $\min_{k\in\mathbb{Z}} |\phi(\tilde{E}_m) - \pi(k+1/2)| > \frac{1}{20}$. Lemma 1(B) from the section "The Density for a One-Dimensional Potential" in [FS3] shows that $V(x) - \tilde{E}_m$, S_m, B_m, I_m satisfy hypotheses (Y0)... (Y11) for a suitable $\varphi_m(E)$. (For hypotheses (Y0)...(Y11), see [FS3]). Therefore, again applying Lemma 1 from the section "Combining the Microlocalized Results" in [FS3], we get the conclusion of Lemma 2. ■

Lemma 3. *For a suitable* $\tilde{E}$ *with* $\tilde{E} - V(x_0) \sim \lambda^{-2\varepsilon}(x_0)S(x_0)$*, and with* $\min_{k'\in\mathbb{Z}} |\phi(\tilde{E}) - \pi(k'+1/2)| \geq \frac{1}{20}$*, the hypotheses (H0*)...(H6*) of the WKB Theorem on Low Eigenvalues are satisfied, with* $V(x) - \tilde{E}$ *in place of* $V(x)$*; with* $S(x_0)$ *in place of* S*; with* $B(x_0)$ *in place of* B*; and with* $100K$ *in place of* K*. The constants called* c*,* c'*,* c''*,* C_α *in (H0*)...(H6*) all have the form* $C_\#$*. The number called* λ *in (H0*)...(H6*) is equal to* $\lambda(x_0)$*.*

Proof.

For $E - V(x_0) \sim \lambda^{-2\varepsilon}(x_0)S(x_0)$ we have $\frac{d\phi(E)}{dE} \sim \lambda(x_0)S^{-1}(x_0)$. Hence we can pick an $\tilde{E}$ with $\tilde{E} - V(x_0) \sim \lambda^{-2\varepsilon}(x_0)S(x_0)$ and $\min_{k'\in\mathbb{Z}} |\phi(\tilde{E}) - \pi(k'+1/2)| \geq \frac{1}{20}$. With $S = S(x_0)$, $B = B(x_0)$ and with $V(x) - \tilde{E}$ in place of $V(x)$, we check

(H0*)…(H6*). (H0*) is trivial from $x_0 \in [x_{\text{left}}(0), x_{\text{rt}}(0)] \subset I$, $\text{dist}(x_{\text{left}}(0), \partial I) > cB(x_{\text{left}}(0))$, $\text{dist}(x_{\text{rt}}(0), \partial I) > cB(x_{\text{rt}}(0))$. (H1*) is trivial from (Z1); (H2*), (H3*) are trivial from (Z3).

To prove (H4*), we argue as follows. Since $\tilde{E} - V(x_0) \sim \lambda^{-2\varepsilon}(x_0)S(x_0)$, we have $V(x) = \tilde{E}$ at $x = x_{\text{left}}(\tilde{E})$ and at $x = x_{\text{rt}}(\tilde{E})$, with $x_{\text{left}}(\tilde{E}) < x_0 < x_{\text{rt}}(\tilde{E})$, $x_{\text{left}}(\tilde{E})$ and $x_{\text{rt}}(\tilde{E})$ in $\{|x - x_0| < C_{\#}\lambda^{-\varepsilon}(x_0)B(x_0)\}$. Now $V(x)$ is decreasing in $[x_{\text{left}}(0), x_{\text{left}}(\tilde{E})]$ and positive in $I_{\text{BVP}} \cap (-\infty, x_{\text{left}}(0)]$, so $V(x) > \tilde{E}$ for $x \in I_{\text{BVP}}$ with $x < x_{\text{left}}(\tilde{E})$. Similarly, $V(x) > \tilde{E}$ for $x \in I_{\text{BVP}}$ with $x > x_{\text{rt}}(\tilde{E})$. Hence, $V(x) - \tilde{E} > 0$ for $x \in I_{\text{BVP}} \backslash \{|x - x_0| < C_{\#}\lambda^{-\varepsilon}(x_0)B(x_0)\}$, which is stronger than (H4*) since we take $E_\infty = 0$.

To prove (H5*), we pick E_+ with $E_+ - \tilde{E} \sim \tilde{E} - V(x_0)$. The proof of (H4*) above shows that $V(x) > E_+$ outside $[x_{\text{left}}(E_+), x_{\text{rt}}(E_+)]$. For $|x - x_0| > \frac{1}{2}\lambda^K(x_0)B(x_0)$ we have $x \notin [x_{\text{left}}(E_+), x_{\text{rt}}(E_+)]$, so $V(x) - \tilde{E} \geq E_+ - \tilde{E} \geq \lambda^{-2\varepsilon}(x_0)S(x_0) \geq \frac{1000}{|x-x_0|^2}$ as needed.

To see the last inequality, we write $\frac{1000}{|x-x_0|^2} \leq \frac{4000}{\lambda^{2K}(x_0)B^2(x_0)} = \frac{4000S(x_0)}{\lambda^{2K+2}(x_0)} \leq \lambda^{-2\varepsilon}(x_0)S(x_0)$. The proof of (H5*) is complete.

Finally, (H6*) is immediate from $\lambda(x_0) \geq c_{\#}\Lambda$ and the WKB hypothesis (Z9). The proof of Lemma 3 is complete. ■

We want to use the above Lemmas, together with the WKB Eigenvalue Theorem and the WKB Theorem on Low Eigenvalues, to give a precise description of the negative eigenvalues of $-\frac{d^2}{dx^2} + V(x)$. To carry this out, and also to be able to apply the lemma on Riemann sums, we need to understand how the phases $\phi(E)$, $\psi(E)$ vary with E. For $V(x_0) < E \leq 0$, define

$$S_{\min}(E) = \inf\{S(x) \mid x \in I, V(x) < E\} \quad \text{and} \tag{1}$$

$$\Gamma(E) = S_{\min}(E) \cdot \int_I (E - V(x))_+^{-1/2} \, dx \, . \tag{2}$$

For $E > 0$, define $S_{\min}(E) = S_{\min}(0)$ and $\Gamma(E) = \Gamma(0)$. Note that

(3) $\quad S_{\min}(E) \sim S(x_0)$ and $\Gamma(E) \sim \lambda(x_0)$

when $V(x_0) < E < V(x_0) + c_\# S(x_0)$.

Also, note that

$$\Gamma(E) \geq c_\# \Lambda \quad \text{for all } E \in \big(V(x_0), \infty\big) \ . \tag{4}$$

To see this, we need only consider $E \in \big(V(x_0) + c_\# S(x_0), 0\big]$, in view of (3) and the definition of $\Gamma(E)$ for $E > 0$. For such E, we have $S_{\min}(E) \sim S(x_*)$ for some x_* with $[x_* - c_\# B(x_*), x_* + c_\# B(x_*)] \subset \{x \in I \mid V(x) < E\}$. Hence (2) gives $\Gamma(E) \geq \big[c_\# S(x_*)\big] \int_{|x-x_*|<c_\# B(x_*)} \big[C_\# S(x_*)\big]^{-1/2} \, dx_* = c'_\# S^{1/2}(x_*) B(x_*) = c'_\# \lambda(x_*) \geq c''_\# \Lambda$, proving (4). We have used (4) several times already in [FS3]. For $E \in \big(V(x_0), +c_\# S_{\min}(0)\big]$ we have

$$\Big|\big(\frac{d}{dE}\big)^\beta \phi(E)\Big| \leq C_\#^\beta \Gamma(E) S_{\min}^{-\beta}(E) \quad (\beta \geq 1) \tag{5}$$

$$\Big|\big(\frac{d}{dE}\big)^\beta \psi(E)\Big| \leq C_\#^\beta \Lambda^{-2} \Gamma(E) S_{\min}^{-\beta}(E) \quad (\beta \geq 1) \tag{6}$$

$$\Big[\big(\frac{d}{dE}\big) \phi(E)\Big] \geq c_\# \Gamma(E) S_{\min}^{-1}(E) \ . \tag{7}$$

In fact, for $E < V(x_0) + c_\# S(x_0)$, estimates (5) and (6) are contained in Lemma 6 in the section of [FS2] on Eigenvalues Near the Minimum of the Potential, by virtue of (3) and Lemma 3 above. We leave the proof of (7) for $E < V(x_0) + c_\# S(x_0)$ to the reader. For $E_0 \in \big(V(x_0) + c_\# S(x_0), 0\big]$, we apply Lemma 1 above, and Lemma 1 in the section on the WKB Theorems, to get

$$\Big|\big(\frac{d}{dE}\big)^\beta \phi(E)\Big| \leq C_\#^\beta \Gamma(E_0) S_{\min}^{-\beta}(E_0) \quad (\beta \geq 1) \tag{8}$$

$$\Big|\big(\frac{d}{dE}\big)^\beta \psi(E)\Big| \leq C_\#^\beta \Gamma(E_0) S_{\min}^{-\beta}(E_0) \Lambda^{-2} \quad (\beta \geq 1) \tag{9}$$

$$\frac{d\phi}{dE} \geq c_{\#}\Gamma(E_0)S_{\min}^{-1}(E_0) \tag{10}$$

for $|E - E_0| \leq c_{\#}S_{\min}(E_0)$.

Taking $E_0 = 0$ in (8), (9), (10) proves (5), (6), (7) for $E \in [0, c_{\#}S_{\min}(0)]$. Taking $E_0 = E$ in (8), (9), (10) gives (5), (6), (7) for $E \in \big[V(x_0) + c_{\#}S(x_0), 0\big]$, which completes the proof of (5), (6), (7).

Note that for $|E_1 - E_0| \leq c_{\#}S_{\min}(E_0)$, E_1 and $E_0 \in \big(V(x_0), c_{\#}S_{\min}(0)\big]$, we have

$$S_{\min}(E_1) \sim S_{\min}(E_0) \quad \text{and} \quad \Gamma(E_1) \sim \Gamma(E_0)\ . \tag{11}$$

This is easily verified by looking separately at the cases $E_0 \in \big(V(x_0), V(x_0) + c_{\#}S(x_0)\big]$ and $E_0 \in \big(V(x_0) + c_{\#}S(x_0), c_{\#}S_{\min}(0)\big]$. From (5), (6), (7) we see that

$$E \mapsto \Phi(E) \equiv \frac{1}{\pi}\phi(E) + \frac{1}{48\pi}\psi(E) - \frac{1}{2} \tag{12}$$

is strictly increasing on $\big(V(x_0), c_{\#}S_{\min}(0)\big]$.

¿From (5)...(11) follow these quantitative results:

(13) Suppose $E_1 \geq E_0 + c_{\#}S_{\min}(E_0)$, with $E_0, E_1 \in \big(V(x_0), c_{\#}S_{\min}(0)\big]$.

Then $\Phi(E_1) \geq \Phi(E_0) + c'_{\#}\Gamma(E_0)$.

(14) Suppose $E_1 \leq E_0 - c_{\#}S_{\min}(E_0)$, with $E_0, E_1 \in \big(V(x_0), c_{\#}S_{\min}(0)\big]$.

Then $\Phi(E_1) \leq \Phi(E_0) - c'_{\#}\Gamma(E_0)$.

From (13) and (14) follows at once

(15) Suppose $E_0, E_1 \in \big(V(x_0), c_{\#}S_{\min}(0)\big]$ and $|\Phi(E_0) - \Phi(E_1)| < c'_{\#}\Gamma(E_0)$.

Then $|E_1 - E_0| \leq c_{\#}S_{\min}(E_0)$.

For $|E - E_0| \leq c_\# S_{\min}(E_0)$ we have $\Phi'(E) \sim S_{\min}^{-1}(E_0)\Gamma(E_0)$ by (11) and (5), (6), (7). This and (15) yield the following.

$$\text{(16)} \quad \begin{array}{l} \text{Suppose } E_0, E_1 \in \big(V(x_0), c_\# S_{\min}(0)\big] \text{ and } |\Phi(E_0) - \Phi(E_1)| \leq c'_\# \Gamma(E_0)\ . \\ \text{Then } |E_0 - E_1| \sim S_{\min}(E_0)\Gamma^{-1}(E_0) \cdot |\Phi(E_0) - \Phi(E_1)|\ . \end{array}$$

Now we can start to control the eigenvalues of $-\frac{d^2}{dx^2} + V(x)$. Let k_{hi} be the greatest integer $\leq \Phi(0) + \overline{C}_\# \Lambda^{4\varepsilon - 2}$ for a suitable, large $\overline{C}_\#$.

Lemma 4. *For $0 \leq k \leq k_{hi} - 1$, there is an eigenvalue $E \leq 0$ with $|\Phi(E) - k| \leq C_\# \Lambda^{4\varepsilon-2}$.*

Proof.

As E varies from $V(x_0)$ to 0, $\Phi(E)$ varies from $-\frac{1}{2} + O(\Lambda^{-1}) < k$ to $\Phi(0) \geq k_{hi} - \overline{C}_\# \Lambda^{4\varepsilon-2} > k_{hi} - 1 \geq k$. Hence there is some $E_0 \in \big(V(x_0), 0\big)$ for which $\Phi(E_0) = k$.

We distinguish three cases:

(A) $E_0 - V(x_0) \geq c_\# S(x_0)$

(B) $c_\# \lambda^{-2\varepsilon} S(x_0) \leq E_0 - V(x_0) \leq c_\# S(x_0)$

(C) $E_0 - V(x_0) \leq c_\# \lambda^{-2\varepsilon}(x_0) S(x_0)$.

First suppose we are in Case A. Lemma 1 shows that the WKB Eigenvalue Theorem applies. Note that our present phase function $\Phi(E)$ is slightly different from the phase function called $\Phi(E)$ in the WKB Eigenvalue Theorem, which we now call $\Phi_{\text{WKB}}(E)$. In fact, $\Phi_{\text{WKB}}(E)$ is defined for $E \in [E_0 - c_\# S_{\min}(E_0), E_0 + c_\# S_{\min}(E_0)] \cap (-\infty, 0]$, and satisfies there $|\Phi_{\text{WKB}}(E) - \pi\Phi(E)| \leq C_\# \Lambda^{-2}$. This is a part of the conclusion of the WKB Eigenvalue Theorem. Let $E_+ = \min\big(0, E_0 + c_\# S_{\min}(E_0)\big)$, $E_- = E_0 - c_\# S_{\min}(E_0)$. These are the endpoints of the interval on which Φ_{WKB} is defined.

We shall check that k lies in the image of $\frac{1}{\pi}\Phi_{\mathrm{WKB}}$, by examining $\Phi_{\mathrm{WKB}}(E_+)$ and $\Phi_{\mathrm{WKB}}(E_-)$. Let's start with $\Phi_{\mathrm{WKB}}(E_+)$. Either $E_+ = 0$ or $E_+ = E_0 + c_\# S_{\min}(E_0)$. If $E_+ = 0$, then

(16bis)
$$\frac{1}{\pi}\Phi_{\mathrm{WKB}}(E_+) \geq \Phi(0) - C_\#\Lambda^{-2} \geq [k_{hi} - \overline{C}_\#\Lambda^{4\varepsilon-2}] - C_\#\Lambda^{-2} > k_{hi} - 1 \geq k \ ,$$
$$\text{so } \frac{1}{\pi}\Phi_{\mathrm{WKB}}(E_+) > k \ .$$

If $E_+ = E_0 + c_\# S_{\min}(E_0)$, then (5), (6), (7) give $\frac{1}{\pi}\Phi_{\mathrm{WKB}}(E_+) \geq \Phi(E_+) - C_\#\Lambda^{-2} \geq \Phi(E_0) + c'_\#\Gamma(E_0) - C_\#\Lambda^{-2} > \Phi(E_0) = k$, so again $\frac{1}{\pi}\Phi_{\mathrm{WKB}}(E_+) > k$. Hence in all cases $\frac{1}{\pi}\Phi_{\mathrm{WKB}}(E_+) > k$. On the other hand, $\frac{1}{\pi}\Phi_{\mathrm{WKB}}(E_-) \leq \Phi(E_-) + C_\#\Lambda^{-2} \leq \Phi(E_0) - c'_\#\Gamma(E_0) + C_\#\Lambda^{-2} < \Phi(E_0) = k$, so $\frac{1}{\pi}\Phi_{\mathrm{WKB}}(E_-) < k$. Therefore, $\pi k \in \Phi_{\mathrm{WKB}}\big([E_0 - c_\# S_{\min}(E_0), E_0 + c_\# S_{\min}(E_0)] \cap (-\infty, 0]\big)$ as claimed. The WKB Eigenvalue Theorem then asserts that

$$|\Phi_{\mathrm{WKB}}(\tilde{E}) - \pi k| \leq C_\#\Lambda^{-N''} \ ,$$

where $\tilde{E}$ is either an eigenvalue or 0. In particular

(17) $$|\Phi(\tilde{E}) - k| \leq C_\#\Lambda^{-2} \ .$$

Since

(18) $$\Phi(0) \geq k_{hi} - \overline{C}_\#\Lambda^{4\varepsilon-2} > (k_{hi} - 1) + \frac{1}{2} \geq k + \frac{1}{2} \ ,$$

(17) cannot hold with $\tilde{E} = 0$. Hence $\tilde{E}$ is an eigenvalue. Also (17) and (18) yield $\Phi(\tilde{E}) < \Phi(0)$, so $\tilde{E} < 0$. We have proven (17) for a negative eigenvalue $\tilde{E}$. Thus the conclusion of Lemma 4 holds in Case (A).

Next suppose we are in Case (B). Define an integer m by

(19) $$2^{-2(m+1)} c_\# S(x_0) < E_0 - V(x) \leq 2^{-2m} c_\# S(x_0) \ .$$

Since we are in Case (B), we have

$$c_\# \lambda^{-2\varepsilon}(x_0) \le 2^{-2m} \le 1 \ . \tag{20}$$

Lemma 2 above lets us apply the WKB Eigenvalue Theorem to the potential $V(x) - \tilde{E}_m$, with weight functions $2^{-2m}S(x_0)$, $2^{-m}B(x_0)$ in place of $S(x)$, $B(x)$; with $\{|x-x_0| < C_\# \cdot 2^{-m}B(x_0)\}$ in place of I; with $E_0 - \tilde{E}_m$ in place of E_0; with $E_\infty = 0$; and with $300K$ in place of K. Here, $\tilde{E}_m - E_0 \sim 2^{-2m}c_\# S(x_0)$. The WKB Eigenvalue Theorem tells us the following: There is a smooth phase function $\Phi_{\text{WKB}}(E)$ defined on $[E_0 - c'_\# 2^{-2m}S(x_0), E_0 + c'_\# 2^{-2m}S(x_0)]$ and satisfying there

$$|\Phi_{\text{WKB}}(E) - \pi\Phi(E)| \le C_\# \left(2^{-2m}\lambda(x_0)\right)^{-2} \ . \tag{21}$$

If πk belongs to $\Phi_{\text{WKB}}\big([E_0 - c''_\# 2^{-2m}S(x_0), E_0 + c''_\# 2^{-2m}S(x_0)] \cap (-\infty, \tilde{E}_m]\big)$ then we can find $\tilde{E} =$ either an eigenvalue of $-\frac{d^2}{dx^2} + V(x)$ or $\tilde{E}_m$, satisfying

$$|\Phi_{\text{WKB}}(\tilde{E}) - \pi k| \le C_\# \left(2^{-2m}\lambda(x_0)\right)^{-N''} \ . \tag{22}$$

Since $\tilde{E}_m - E_0 \sim 2^{-2m}c_\# S(x_0)$, we have

$$\begin{aligned} &[E_0 - c''_\# S(x_0)2^{-2m}, E_0 + c''_\# S(x_0)2^{-2m}] \cap (-\infty, \tilde{E}_m] \\ &\qquad = [E_-, E_+] \subset \big(V(x_0), c_\# S_{\min}(0)] \end{aligned} \tag{23}$$

with

$$(E_0 - E_-), (E_+ - E_0) \sim 2^{-2m}S(x_0) \ . \tag{24}$$

From (19), (20) we get $\Gamma(E_0) \sim \lambda(x_0)$, $S_{\min}(E_0) \sim S(x_0)$, so $\Phi'(E) \sim \Gamma(E_0)S^{-1}_{\min}(E_0)$ $\sim \lambda(x_0)S^{-1}(x_0)$ for $|E - E_0| < c_\# S(x_0)$, hence for $E \in [E_-, E_+]$. Therefore, (24) implies

$$\Phi(E_0) - \Phi(E_-) \sim 2^{-2m}\lambda(x_0) \ , \quad \Phi(E_+) - \Phi(E_0) \sim 2^{-2m}\lambda(x_0) \ . \tag{25}$$

Note that $2^{-2m}\lambda(x_0) \geq c_{\#}\lambda^{1-2\varepsilon}(x_0) >> 1$ by (20), and recall that $\Phi(E_0) = k$. Then (21) and (25) imply $\Phi_{\rm WKB}(E_+) > \pi k > \Phi_{\rm WKB}(E_-)$, so that $\pi k \in \Phi_{\rm WKB}\big([E_-, E_+]\big)$. Thus we can find $\tilde{E} = \tilde{E}_m$ or an eigenvalue, satisfying (22). From (21) and (22), we get

$$(26) \qquad |\Phi(\tilde{E}) - k| \leq C_{\#}\big(2^{-2m}\lambda(x_0)\big)^{-2} \; .$$

In particular, since $\Phi(E_0) = k$, we see from (25), (26) that $\Phi(\tilde{E}) < \Phi(E_+)$, hence $\tilde{E} < E_+ \leq \tilde{E}_m$. This shows that $\tilde{E} \neq \tilde{E}_m$, so $\tilde{E}$ is an eigenvalue of $-\frac{d^2}{dx^2} + V(x)$. Also since $\tilde{E} \leq \tilde{E}_m < 0$, we get $\tilde{E} < 0$. Hence (26) holds for a negative eigenvalue $\tilde{E}$. From (26) and (20), we get $|\Phi(\tilde{E}) - k| \leq C_{\#}\big(\lambda^{1-2\varepsilon}(x_0)\big)^{-2} \leq C'_{\#}\Lambda^{4\varepsilon-2}$. Thus, the conclusion of Lemma 4 holds in Case (B).

Finally, suppose we are in Case (C). Lemma 3 above lets us apply the WKB Theorem on Low Eigenvalues to the potential $V(x) - \tilde{E}$, with $S(x_0)$, $B(x_0)$ in place of S, B; with $E_\infty = 0$; and with $100K$ in place of K. Here, $\tilde{E} - E_0 \sim \lambda^{-2\varepsilon}(x_0)S(x_0)$, and $\tilde{E} < 0$. The WKB Theorem on Low Eigenvalues provides the following information:

Suppose $\frac{1}{\pi}\phi\big(\min\{\tilde{E}, V(x_0) + c_{\#}\lambda^{-2\varepsilon}(x_0)S(x_0)\}\big) - \frac{1}{2} \geq k+1$. Then there is an eigenvalue $E_k \leq \tilde{E}$ with

$$(27) \qquad |\Phi(E_k) - k| \leq C_{\#}\big(\lambda(x_0)\big)^{4\varepsilon-2} \; .$$

To check the condition on k, we note that $V(x_0) + c_{\#}\lambda^{-2\varepsilon}S(x_0) \geq E_0 + c'_{\#}\lambda^{-2\varepsilon}(x_0)S(x_0)$, provided we take the constant $c_{\#}$ in the definition of Case (C) small enough. Since also $\tilde{E} \geq E_0 + c''_{\#}\lambda^{-2\varepsilon}(x_0)S(x_0)$, we have

$$(28) \qquad E_+ = \min\big\{\tilde{E}, V(x_0) + c_{\#}\lambda^{-2\varepsilon}(x_0)S(x_0)\big\} \geq E_0 + c'''_{\#}\lambda^{-2\varepsilon}(x_0)S(x_0) \; .$$

We have $V(x_0) < E_0 \leq E_+ \leq V(x_0) + c_{\#}\lambda^{-2\varepsilon}(x_0)S(x_0)$, so $\phi'(E) \sim \lambda(x_0)S^{-1}(x_0)$

for E between E_0 and E_+. Hence, (28) implies

$$
\begin{aligned}
\Big[\frac{1}{\pi}\phi(E_+) - \frac{1}{2}\Big] &\geq \Big[\frac{1}{\pi}\phi(E_0) - \frac{1}{2}\Big] + c_{\#}\lambda^{1-2\varepsilon}(x_0) \\
&= \Phi(E_0) - \frac{1}{48\pi}\psi(E_0) + c_{\#}\lambda^{1-2\varepsilon}(x_0) \geq \Phi(E_0) + c'_{\#}\lambda^{1-2\varepsilon}(x_0) \\
&\quad (\text{since } |\psi(E_0)| \leq C_{\#}\Lambda^{-1} << 1 << \lambda^{1-2\varepsilon}(x_0)) = k + c'_{\#}\lambda^{1-2\varepsilon}(x_0)\ .
\end{aligned}
$$

This verifies the condition on k, which implies (27) for an eigenvalue $E_k \leq \tilde{E} < 0$. From (27) we get in particular $|\Phi(E_k) - k| \leq C_{\#}\Lambda^{4\varepsilon-2}$ for an eigenvalue $E_k < 0$.

Thus the conclusion of Lemma 4 holds also in Case (C). The proof of Lemma 4 is complete. ■

Corollary. *If* $\Phi(0) \geq k_{hi} + \overline{C}_{\#}\Lambda^{4\varepsilon-2}$, *then there is an eigenvalue* $E \leq 0$ *with* $|\Phi(E) - k_{hi}| \leq \overline{C}_{\#}\Lambda^{4\varepsilon-2}$.

Proof.

As E varies from $V(x_0)$ to 0, $\Phi(E)$ varies from $-\frac{1}{2}+O(\Lambda^{-1}) < k_{hi}$ to $\Phi(0) > k_{hi}$. Hence there is some $E_0 \in \big(V(x_0), 0\big)$ for which $\Phi(E_0) = k_{hi}$.

Now we can repeat the proof of Lemma 4, using k_{hi} in place of k, and making the following modifications. Instead of (16 bis), we note that $\frac{1}{\pi}\Phi_{\text{WKB}}(E_+) \geq \Phi(0) - C_{\#}\Lambda^{-2} \geq \big[k_{hi} + \overline{C}_{\#}\Lambda^{4\varepsilon-2}\big] - C_{\#}\Lambda^{-2} > k_{hi}$. Instead of (18), we note that $\Phi(0) \geq k_{hi} + \overline{C}_{\#}\Lambda^{4\varepsilon-2}$.

With these minor changes the proof of Lemma 4 goes through. ■

Lemma 5. *If* $E_0 \leq 0$ *is an eigenvalue of* $-\frac{d^2}{dx^2}+V(x)$, *then* $|\Phi(E_0)-k| \leq \overline{C}_{\#}\Lambda^{4\varepsilon-2}$ *for an integer* k $(0 \leq k \leq k_{hi})$.

Proof.

We distinguish three cases:

(A) $E_0 - V(x_0) \geq c_{\#}S(x_0)$

(B) $c_{\#}\lambda^{-2\varepsilon}(x_0)S(x_0) \leq E_0 - V(x_0) \leq c_{\#}S(x_0)$

(C) $E_0 - V(x_0) \leq c_{\#}\lambda^{-2\varepsilon}(x_0)S(x_0)$.

First suppose we are in Case (A). Lemma 1 and the WKB Eigenvalue Theorem show that $|\Phi(E)-(\text{integer})| \leq C_{\#}\Lambda^{-2}$ for any eigenvalue $E \in [E_0 - c'_{\#}S_{\min}(E_0), E_0 + c'_{\#}S_{\min}(E_0)] \cap (-\infty, 0]$. In particular, $|\Phi(E_0) - (\text{integer})| \leq C_{\#}\Lambda^{-2}$.

Next, suppose we are in Case (B). Define an integer m by $2^{-2(m+1)}c_\# S(x_0) < E_0 - V(x) \le 2^{-2m}c_\# S(x_0)$. Since we are in Case (B), we have $c_\# \lambda^{-2\varepsilon}(x_0) \le 2^{-2m} \le 1$. Lemma 2 and the WKB Eigenvalue Theorem show that $|\Phi(E) - (\text{integer})| \le C_\# \big(2^{-2m}\lambda(x_0)\big)^{-2}$ for any eigenvalue E in the interval $[E_0 - c'_\#(2^{-2m}S(x_0)), E_0 + c'_\#(2^{-2m}S(x_0))] \cap (-\infty, \tilde{E}_m]$. (Here $\tilde{E}_m > E_0$ is as in Lemma 2.) In particular, for E_0 itself we have $|\Phi(E_0) - (\text{integer})| \le C_\# \big(2^{-2m}\lambda(x_0)\big)^{-2} \le C_\# \big(\lambda^{1-2\varepsilon}(x_0)\big)^{-2} \le C_\# \Lambda^{4\varepsilon-2}$.

Next, suppose we are in Case (C). We apply Lemma 3 and the WKB Theorem for Low Eigenvalues. This shows that $|\Phi(E) - k| \le C_\# \big(\lambda(x_0)\big)^{4\varepsilon-2}$ for any eigenvalue E satisfying

$$E \le \min\big\{\tilde{E}, V(x_0) + c_\# \lambda^{-2\varepsilon}(x_0) S(x_0)\big\} \ . \tag{29}$$

(Here $\tilde{E} > E_0$ is as in Lemma 3.) In particular, $E = E_0$ satisfies (29), provided we pick $c_\#$ small enough in the statement of Case (C). Hence, $|\Phi(E_0) - (\text{integer})| \le C_\# \big(\lambda(x_0)\big)^{4\varepsilon-2} \le C_\# \Lambda^{4\varepsilon-2}$. We have proven in all cases that

$$|\Phi(E_0) - k| \le C_\# \Lambda^{4\varepsilon-2} \quad \text{for an integer } k \ . \tag{30}$$

As E varies from $V(x_0)$ to 0, $\Phi(E_0)$ varies from $-\frac{1}{2} + O\big(\lambda^{-1}(x_0)\big)$ to $\Phi(0)$. Hence (30) implies

$$-\frac{1}{2} - C_\# \lambda^{-1}(x_0) - C_\# \Lambda^{4\varepsilon-2} \le k \le \Phi(0) + C_\# \Lambda^{4\varepsilon-2} \ . \tag{31}$$

Since k is an integer, (31) implies

$$0 \le k \le \Phi(0) + \overline{C}_\# \Lambda^{4\varepsilon-2} \ , \tag{32}$$

provided we take $\overline{C}_\#$ larger then the $C_\#$ in (31) .

Then from (32) and the definition of k_{hi}, we see that $0 \le k \le k_{hi}$. Thus, $|\Phi(E_0) - k| \le \overline{C}_\# \Lambda^{4\varepsilon-2}$ and $0 \le k \le k_{hi}$, provided we also take $\overline{C}_\#$ larger than the $C_\#$ in

(30). The proof of Lemma 5 is complete. ■

Lemma 6. *Suppose* $\overline{E}_0$, $\overline{E}_1 \leq 0$ *are eigenvalues of* $-\frac{d^2}{dx^2} + V(x)$. *If* $|\Phi(\overline{E}_0) - \Phi(\overline{E}_1)| \leq C_\# \Lambda^{4\varepsilon - 2}$ *then* $\overline{E}_0 = \overline{E}_1$.

Proof.

Suppose not. We may suppose $\overline{E}_1 < \overline{E}_0 \leq 0$. Then (16) implies

$$\overline{E}_0 - C_\# S_{\min}(\overline{E}_0)\Gamma^{-1}(\overline{E}_0)\Lambda^{4\varepsilon-2} \leq \overline{E}_1 < \overline{E}_0 \leq 0 \ . \tag{33}$$

We distinguish three cases:

(A) $\overline{E}_0 - V(x_0) \geq c_\# S(x_0)$

(B) $c_\# \lambda^{-2\varepsilon} S(x_0) \leq \overline{E}_0 - V(x_0) \leq c_\# S(x_0)$

(C) $\overline{E}_0 - V(x) \leq c_\# \lambda^{-2\varepsilon}(x_0) S(x_0)$.

Suppose first we are in Case (A). Take $E_0 = \overline{E}_0$, and apply Lemma 1 and the WKB Eigenvalue Theorem. Thus, the eigenvalues in $\mathcal{E} = \left[\overline{E}_0 - c'_\# S_{\min}(\overline{E}_0), \overline{E}_0 + c'_\# S_{\min}(\overline{E}_0)\right] \cap (-\infty, 0]$ are among the E_k described in the WKB Eigenvalue Theorem, and distinct E_k have $|\Phi(E_k) - \Phi(E_{k'})| \geq 1 - C_\# \Lambda^{-2}$. Since (33) shows that $\overline{E}_0$, $\overline{E}_1 \in \mathcal{E}$, it follows that $|\Phi(\overline{E}_0) - \Phi(\overline{E}_1)| \geq 1 - C_\# \Lambda^{-2}$, contradicting the hypothesis of the Lemma.

Next suppose we are in Case (B). Define an integer m by $2^{-2(m+1)} c_\# S(x_0) < \overline{E}_0 - V(x_0) \leq 2^{-2m} c_\# S(x_0)$. Since we are in Case (B), we have

$$c_\# \lambda^{-2\varepsilon}(x_0) < 2^{-2m} \leq 1 \ . \tag{34}$$

Take $E_0 = \overline{E}_0$, and apply Lemma 2 and the WKB Eigenvalue theorem. Thus, the eigenvalues in $\mathcal{E} \equiv \left[\overline{E}_0 - c'_\# 2^{-2m} S(x_0), \overline{E}_0 + c'_\# 2^{-2m} S(x_0)\right] \cap (-\infty, \tilde{E}_m]$ are among the E_k described in the WKB Theorem, so that distinct E_k have $|\Phi(E_k) - \Phi(E_{k'})| \geq 1 - C_\# \cdot \left(2^{-2m} \lambda(x_0)\right)^{-2}$. (Here $\tilde{E}_m > \overline{E}_0$ is as in Lemma 2). In particular, (33) and

(3) show that $\overline{E}_0 - C_\# S(x_0)\lambda^{-1}(x_0)\Lambda^{4\varepsilon-2} \leq \overline{E}_1 < \overline{E}_0 < \tilde{E}_m \leq 0$, which implies $\overline{E}_0, \overline{E}_1 \in \mathcal{E}$ by virtue of (34). Therefore, $|\Phi(\overline{E}_0) - \Phi(\overline{E}_1)| \geq 1 - C_\# \left(2^{-2m}\lambda(x_0)\right)^{-2} \geq 1 - C'_\# \lambda^{4\varepsilon-2}(x_0) > \frac{1}{2}$, contradicting the hypothesis of the Lemma.

Finally, suppose we are in Case (C). We apply Lemma 3 and the WKB Theorem on Low Eigenvalues. Thus, all the eigenvalues in $\mathcal{E} = \left[V(x_0), V(x_0) + c_\# \lambda^{-2\varepsilon}(x_0)S(x_0)\right] \cap (-\infty, \tilde{E}]$ are among the E_k described in the WKB Theorem on Low Eigenvalues, and distinct E_k satisfy $|\Phi(E_k) - \Phi(E_{k'})| \geq 1 - C_\# \lambda^{4\varepsilon-2}(x_0)$. Here $\tilde{E} \leq 0$ is as in Lemma 3, so that $\tilde{E} - V(x_0) \sim \lambda^{-2\varepsilon}(x_0)S(x_0)$. Since we are in Case (C), we have $\overline{E}_0 < \tilde{E}$ and $\overline{E}_0 < V(x_0) + c_\# \lambda^{-2\varepsilon}(x_0)S(x_0)$, provided we take the $c_\#$ in the statement of Case (C) small enough. Thus $\overline{E}_0 \in \mathcal{E}$. Since $V(x_0) < \overline{E}_1 < \overline{E}_0$ we have also $\overline{E}_1 \in \mathcal{E}$. Hence $|\Phi(\overline{E}_0) - \Phi(\overline{E}_1)| \geq 1 - C_\# \lambda^{4\varepsilon-2}(x_0) > \frac{1}{2}$, contradicting the hypothesis of the Lemma.

In all three cases (A), (B), (C), we have reached a contradiction, starting with the assumption $\overline{E}_1 < \overline{E}_0$. The proof of Lemma 6 is complete. ■

Lemma 7. *The non-positive eigenvalues of* $-\frac{d^2}{dx^2} + V(x)$ *are* $E_0, E_1, \ldots, E_{k_{\max}}$, *with*

(35) $$|\Phi(E_k) - k| \leq \overline{C}_\# \Lambda^{4\varepsilon-2} \ ,$$

(36) $$k_{\max} = \text{ greatest integer } \leq \Phi(0) + w_{hi} \ ,$$

and

(37) $$|w_{hi}| \leq \overline{C}_\# \Lambda^{4\varepsilon-2} \ .$$

Proof.

This follows from Lemma 4 and its corollary, and from Lemmas 5 and 6. We take $k_{\max} = k_{hi}$ or $k_{hi} - 1$, depending on whether there is a non-positive eigenvalue

E satisfying $|\Phi(E) - k_{hi}| \leq \overline{C}_{\#}\Lambda^{4\varepsilon-2}$. For $0 \leq k \leq k_{\max}$, we can find exactly one non-positive eigenvalue E_k satisfying (35). (That follows from Lemmas 4 and 6.) Moreover, every non-positive eigenvalue is one of the E_k, by Lemma 5. If $k_{\max} = k_{hi}$, then we take $w_{hi} = \overline{C}_{\#}\Lambda^{4\varepsilon-2}$, so that (36) and (37) hold by definition of k_{hi} and w_{hi}. If instead $k_{\max} = k_{hi} - 1$, then

$$\Phi(0) < k_{hi} + \overline{C}_{\#}\Lambda^{4\varepsilon-2} \tag{38}$$

by the Corollary to Lemma 4. Hence we may set $w_{hi} = -\overline{C}_{\#}\Lambda^{4\varepsilon-2}$, so that (37) holds by definition and (36) amounts to saying that

$$k_{hi} - 1 \leq \Phi(0) - \overline{C}_{\#}\Lambda^{4\varepsilon-2} < k_{hi} \ . \tag{39}$$

Since $k_{hi} \leq \Phi(0) + \overline{C}_{\#}\Lambda^{4\varepsilon-2}$ by definition of k_{hi}, the first inequality in (39) is obvious. The second inequality in (39) is the known fact (38). The proof of Lemma 7 is complete. ■

Corollary. *The eigenvalues of* $-\frac{d^2}{dx^2} + V(x)$ *that belong to* $[E_*, 0]$ *are precisely the* E_k *for* $\Phi(E_*) \leq k \leq \Phi(0) + w_{hi}$.

Proof.

We must check that $E_k \in [E_*, 0]$ if and only if $k \geq \Phi(E_*)$. Since $\Phi(\cdot)$ is strictly monotone, $E_k \geq E_*$ if and only if $\Phi(E_k) \geq \Phi(E_*)$. We know that $\Phi(E_k)$ is near to k, and that $\Phi(E_*) = \frac{1}{\pi}\phi(E_*) + O(\Lambda^{-1}) - \frac{1}{2}$ is near (integer $-\frac{1}{2}$), hence not too near to any integer. (Here we use our assumption that $\frac{1}{\pi}\phi(E_*)$ is near to an integer.) It follows that $\Phi(E_k) \geq \Phi(E_*)$ if and only if $k \geq \Phi(E_*)$. Thus, $E_k \geq E_*$ if and only if $k \geq \Phi(E_*)$. ■

We prepare to use the above Corollary and the lemma on Riemann sums to compute the sum of the eigenvalues $E_k \in [E_*, 0]$. Define $(t_{\min}, t_{\max}] =$ Image of Φ on

$\big(V(x_0), c_\# S_{\min}(0)\big]$. For $t \in (t_{\min}, t_{\max}]$, define $E(t)$ to be the solution E of the equation $\Phi(E) = t$; and define $\sigma(t) = S_{\min}\big(E(t)\big)$, $\tau(t) = \Gamma\big(E(t)\big)$. Estimates (5), (6), (7) show that

$$(40) \qquad \left|\left(\frac{d}{dt}\right)^m E(t)\right| \leq C_\#^m \sigma(t)\tau^{-m}(t) \quad \text{for } t \in (t_{\min}, t_{\max}] \ ,$$

when $m \geq 1$. We check that (40) holds also for $m = 0$, which amounts to saying that

$$(41) \qquad |E| \leq C_\# S_{\min}(E) \quad \text{for } E \in \big(V(x_0), +c_\# S_{\min}(0)\big] \ .$$

When $E \in \big[0, +c_\# S_{\min}(0)\big]$, then (41) holds since $S_{\min}(E) = S_{\min}(0)$. When $E \in (V(x_0), 0]$, then for $x \in I$, $V(x) < E$ we have $-C_\# S(x) \leq V(x) < E \leq 0$, so $|E| \leq C_\# S(x)$. Hence, $|E| \leq C_\# \inf\big\{S(x) \mid x \in I, V(x) < E\big\} = C_\# S_{\min}(E)$, completing the proof of (41). Thus, (40) holds for all $m \geq 0$. Note also that

$$(42) \qquad \sigma(t_1) \sim \sigma(t_0) \text{ and } \tau(t_1) \sim \tau(t_0) \quad \text{for } t_0, t_1 \in (t_{\min}, t_{\max}] \ , \quad |t_0 - t_1| < c_\# \tau(t_0) \ .$$

This follows at once from (11) and (15). From (4) we get

$$(43) \qquad \tau(t) \geq c_\# \Lambda \quad \text{for} \quad t \in (t_{\min}, t_{\max}] \ .$$

From (5), (6), (7) follows also

$$(43\text{ bis}) \qquad \frac{dE(t)}{dt} \geq c_\# \sigma(t)\tau^{-1}(t) \quad \text{for} \quad t \in (t_{\min}, t_{\max}] \ .$$

Because of (8), (9), (10) with $E_0 = 0$, we have $t_{\max} = \Phi\big(c_\# S_{\min}(0)\big) \geq \Phi(0) + c'_\# \Gamma(0) \geq \Phi(0) + c''_\# \Lambda$ (by (4)). Therefore, setting

$$(44) \qquad a = \Phi(E_*) \ , \quad b = \Phi(0) + w_{hi}$$

with w_{hi} as in Lemma 7, $b_0 = \Phi(0)$, and recalling that $|w_{hi}| << 1 << c''_\# \Lambda$, we obtain

$$(45) \qquad t_{\min} < a \leq b < t_{\max} \ , \quad \text{and} \quad a \leq b_0 < t_{\max} \ .$$

(To see that $a \le b, b_0$ we use the assumption $\phi(E_*) \le \phi(0) - 1$, which we made at the start of this section.)

By definition, $S_{\min}(E)$ is constant for $E \ge 0$ and monotone decreasing for $E \le 0$. Hence,

$$\text{(45 bis)} \qquad S_{\min}(E) \le S_{\min}(E_*) \quad \text{for} \quad E \ge E_* \ ,$$

so that

$$\text{(46)} \qquad \sigma(t) \le S_{\min}(E_*) \quad \text{for} \quad t \in [a,b] \ .$$

Let $\{E_k\}$ be the non-positive eigenvalues of $-\frac{d^2}{dx^2} + V(x)$, as in Lemma 7. Then (35) and (40) imply

$$\text{(47)} \quad |E_k - E(k)| = |E(\Phi(E_k)) - E(k)| \le |\Phi(E_k) - k| \cdot \sup\left\{ \left|\frac{dE(t)}{dt}\right| : \ |t-k| \le |\Phi(E_k) - k| \right\} \le \overline{C}_\# \Lambda^{4\varepsilon - 2} \cdot \sup\left\{ C_\# \sigma(t) \tau^{-1}(t) : \ |t-k| \le \overline{C}_\# \Lambda^{4\varepsilon-2} \right\} \ .$$

Now (42), (43) imply that

$$\text{(48)} \qquad \sigma(t)\tau^{-1}(t) \text{ has constant order of magnitude for } |t-k| \le 1 \ .$$

Putting this into (47) yields

$$\text{(49)} \qquad |E_k - E(k)| \le C_\# \Lambda^{4\varepsilon - 2} \sigma(k) \tau^{-1}(k) \ .$$

Putting (48) into (43 bis), we get also

$$\text{(50)} \quad \int_{[a,b_0] \cap [k-1,k+1]} \left(\frac{dE(t)}{dt}\right) dt \ge c_\# \sigma(k) \tau^{-1}(k) \cdot \ \text{length}\big([a,b_0] \cap [k-1,k+1]\big)$$

for $k \in \mathbb{Z} \cap [a,b]$. Our assumption $\phi(E_*) < \phi(0) - 1$ implies that $[a, b_0]$ has length $\ge c_\#$, so length $\big([a,b_0] \cap [k-1,k+1]\big) \ge c'_\#$ provided distance $(k, [a,b_0]) \le \overline{C}_\# \Lambda^{4\varepsilon-2}$.

For $k \in \mathbb{Z} \cap [a,b]$ we have $\operatorname{dist}(k,[a,b_0]) \le \overline{C}_\# \Lambda^{4\varepsilon-2}$, since $|b-b_0| = |w_{hi}| \le \overline{C}_\# \Lambda^{4\varepsilon-2}$. Therefore, (50) implies

$$\int_{[a,b_0]\cap[k-1,k+1]} \Big(\frac{dE(t)}{dt}\Big)\, dt \ge c_\# \sigma(k)\tau^{-1}(k) \quad \text{for } k \in \mathbb{Z} \cap [a,b] \; .$$

Combining this with (49), we find that

$$|E_k - E(k)| \le C_\# \Lambda^{4\varepsilon-2} \int_{[a,b_0]\cap[k-1,k+1]} \Big(\frac{dE(t)}{dt}\Big)\, dt \quad \text{for } k \in \mathbb{Z} \cap [a,b] \; .$$

Summing on k, we get

$$\begin{aligned} \sum_{k\in\mathbb{Z}\cap[a,b]} |E_k - E(k)| &\le C_\# \Lambda^{4\varepsilon-2} \int_a^{b_0} \Big(\frac{dE(t)}{dt}\Big)\, dt \\ &= C_\# \Lambda^{4\varepsilon-2}\big(E(b_0) - E(a)\big) = C_\# \Lambda^{4\varepsilon-2}(0 - E_*) \end{aligned}$$

(by definition of a, b_0, $E(\cdot)$). Thus

$$(51) \qquad \sum_{k\in\mathbb{Z}\cap[a,b]} |E_k - E(k)| \le C_\# |E_*| \Lambda^{4\varepsilon-2} \; .$$

We apply the Lemma on Riemann sums to compute $\sum_{k\in\mathbb{Z}\cap[a,b]} E(k)$ modulo a small error. Estimates (40), (42), (43), (45) show that the hypotheses of the Lemma on Riemann sums are satisfied, and therefore:

$$(52) \qquad \begin{aligned} \sum_{k\in\mathbb{Z}\cap[a,b]} E(k) &= \int_a^b E(t)dt - E(b)\chi_-(b) - E(a)\chi_+(a) \\ &+ \frac{1}{2}\, \frac{dE(t)}{dt}\Big|_{t=b} \tilde{\chi}(b) - \frac{1}{2}\, \frac{dE(t)}{dt}\Big|_{t=a} \tilde{\chi}(a) + \text{Error}_0 \; , \end{aligned}$$

with

$$(53) \qquad |\text{Error}_0| \le C_\# \sigma(a)\tau^{-2}(a) + C_\# \sigma(b)\tau^{-2}(b) + C_\# \int_a^b \sigma(t)\tau^{-N}(t)\, dt \; .$$

Let us estimate $|\text{Error}_0|$. We have

$$\int_a^b \sigma(t)\tau^{-1}(t)dt \le C_\# \int_a^b \frac{dE(t)}{dt}\, dt \quad \text{(by (43 bis))} \;= \big(E(b) - E(a)\big)C_\# \; .$$

Since $E(a) = E_*$ and $E(b) \in \big(V(x_0), +c_\# S_{\min}(0)\big]$, it follows that

$$\int_a^b \sigma(t)\tau^{-1}(t)dt \le C_\# S_{\min}(0) + C_\# |E_*| \le C_\# S_{\min}(E_*) \ ,$$

by (41) and (45 bis). Therefore,

$$\int_a^b \sigma(t)\tau^{-N}(t)dt \le C_\# \Lambda^{1-N} S_{\min}(E_*) \ , \quad \text{by (43)} \ . \tag{54}$$

Also,

$$\sigma(a)\tau^{-2}(a) \ , \sigma(b)\tau^{-2}(b) \le C_\# S_{\min}(E_*) \cdot \Lambda^{-2} \ , \text{ by (43) and (46)} \ . \tag{55}$$

Combining (53), (54), (55), we get

$$|\text{Error}_0| \le C_\# S_{\min}(E_*) \cdot \Lambda^{-2} \ . \tag{56}$$

Also, $C_\# \Lambda^{4\varepsilon-2}|E_*| \le C'_\# \Lambda^{4\varepsilon-2} S_{\min}(E_*)$, so (51), (52), (56) imply

(57)

$$\sum_{k\in\mathbb{Z}\cap[a,b]} E_k = \int_a^b E(t)dt - E(b)\chi_-(b) - E(a)\chi_+(a) + \frac{1}{2}\,\frac{d}{dt}E(t)\Big|_{t=b}\,\tilde{\chi}(b)$$
$$-\frac{1}{2}\,\frac{d}{dt}E(t)\Big|_{t=a}\,\tilde{\chi}(a) + \text{ Error}_1$$

with

$$|\text{Error}_1| \le C_\# \Lambda^{4\varepsilon-2} S_{\min}(E_*) \ . \tag{58}$$

Let us study in turn each of the terms on the right in (57). Regarding the integral in (57), we note that

$$\Big|\int_a^b E(t)dt - \int_a^{b_0} E(t)dt\Big| \le \sup\big\{|E(t)| \colon t \text{ between } b \text{ and } b_0\big\} \cdot |b - b_0|$$
$$\le \sup\big\{|E(t)| \colon \ |t-b_0| \le \overline{C}_\# \Lambda^{4\varepsilon-2}\big\} \cdot \overline{C}_\# \Lambda^{4\varepsilon-2} \le C_\# \sigma(b) \Lambda^{4\varepsilon-2}$$
$$\le C_\# \Lambda^{4\varepsilon-2} S(E_*) \ , \text{ by (45) and (46)} \ .$$

Thus,

$$(59)\qquad \int_a^b E(t)dt = \int_a^{b_0} E(t)dt + \text{ Error}_2, \text{ with } |\text{Error}_2| \le C_\# \Lambda^{4\varepsilon-2} S_{\min}(E_*) \ .$$

We wait until later to analyze $\int_a^{b_0} E(t)dt$.

Next, note that $|E(b)| = |E(b) - E(b_0)| \le \sup\{ |\frac{dE(t)}{dt}|: t \text{ between } b\ , b_0\} \cdot |b - b_0| \le C_\# \sigma(b)\tau^{-1}(b)\Lambda^{4\varepsilon-2}$ (by (40)) $\le C_\# S_{\min}(E_*) \cdot \Lambda^{4\varepsilon-3}$, by (43), (46). Throwing away information, we conclude that

$$(60)\qquad |E(b)\chi_-(b)| \le C_\# \Lambda^{4\varepsilon-2} S_{\min}(E_*) \ .$$

Next we study $-E(a)\chi_+(a)$. By definition of $E(t)$ and of a, we have $E(a) = E_*$, and $a = \Phi(E_*) = \frac{1}{\pi}\phi(E_*) + \frac{1}{48\pi}\psi(E_*) - \frac{1}{2}$. Our assumption on E_* was that $\frac{1}{\pi}\phi(E_*) = m + \xi$ with m an integer, and $|\xi| \le \Lambda^{-2}$. Since $|\psi(E_*)| \le C_\# \Lambda^{-1}$, we see that $a = m - \frac{1}{2} + \frac{1}{48\pi}\psi(E_*) + \xi$ lies near to $m - \frac{1}{2}$. Hence $k =$ (smallest integer $\ge a) = m$, and so by definition we have $\chi_+(a) = k - a - \frac{1}{2} = m - \big(m - \frac{1}{2} + \frac{1}{48\pi}\psi(E_*) + \xi\big) - \frac{1}{2} = -\frac{1}{48\pi}\psi(E_*) - \xi$. Therefore, $-E(a)\chi_+(a) = -E_* \cdot \big(-\frac{1}{48\pi}\psi(E_*) - \xi\big)$. Since $|E_* \cdot \xi| \le C_\# \Lambda^{-2} S_{\min}(E_*)$ by (41), this may be rewritten as

(61)
$$-E(a)\chi_+(a) = +\frac{E_*}{48\pi}\psi(E_*) + \text{ Error}_3 \ , \text{with } |\text{Error}_3| \le C_\# \Lambda^{-2} S_{\min}(E_*) \ .$$

Next we study the term $+\frac{1}{2}E'(b)\tilde{\chi}(b)$. Recall that $\tilde{\chi}(\cdot)$ is Lipschitz continuous, and that $b = \frac{1}{\pi}\phi(0) + \frac{1}{48\pi}\psi(0) - \frac{1}{2} + w_{hi}$, with $|w_{hi}| \le C_\# \Lambda^{4\varepsilon-2}$ and $|\psi(0)| \le C_\# \Lambda^{-1}$. So

$$(62)\qquad \big|\tilde{\chi}(b) - \tilde{\chi}\big(\frac{1}{\pi}\phi(0) - \frac{1}{2}\big)\big| \le C_\# \big| \frac{1}{48\pi}\psi(0) + w_{hi} \big| \le C_\# \Lambda^{-1} \ .$$

Also, (40), (42), (43) and $|b - b_0| \le C_\# \Lambda^{4\varepsilon-2}$ imply $|E'(b) - E'(b_0)| \le C_\# |b - b_0| \sigma(b)\tau^{-2}(b) \le C_\# \Lambda^{4\varepsilon-2}\sigma(b)\tau^{-2}(b)$ and $|E'(b)| \le C_\# \sigma(b)\tau^{-1}(b)$. ¿From (43) and (46) we have therefore

$$(63)\qquad |E'(b)| \le C_\# \Lambda^{-1} S_{\min}(E_*) \ , \text{ and}$$

$$(64) \qquad |E'(b) - E'(b_0)| \leq C_\# \Lambda^{4\varepsilon - 4} S_{\min}(E_*) \ .$$

Applying (62), (63), (64), we get

$$(65) \quad \Big| \frac{1}{2}\ E'(b)\tilde{\chi}(b) - \frac{1}{2}\ E'(b_0)\tilde{\chi}\big(\frac{1}{\pi}\phi(0) - \frac{1}{2}\big)\Big| \leq$$
$$\frac{1}{2}\ \big|E'(b)\big|\ \big|\tilde{\chi}(b) - \tilde{\chi}\big(\frac{1}{\pi}\phi(0) - \frac{1}{2}\big)\big| + \frac{1}{2}\big|E'(b) - E'(b_0)\big|\ \big|\tilde{\chi}\big(\frac{1}{\pi}\phi(0) - \frac{1}{2}\big)\big|$$
$$\leq C_\# \Lambda^{-1} S_{\min}(E_*) \cdot \Lambda^{-1} + C_\# \Lambda^{4\varepsilon-4} S_{\min}(E_*) \leq C'_\# \Lambda^{-2} S_{\min}(E_*) \ .$$

By definition of $E(t)$, we have also $E'(t) = \Big(\frac{d}{dE}\Phi(E)\ \Big|_{E=E(t)}\Big)^{-1}$. Since $E(b_0) = 0$, this means

$$(66) \quad E'(b_0) = \Big(\frac{d}{dE}\Phi(E)\ \Big|_{E=0}\Big)^{-1} = \Big(\frac{1}{\pi}\phi'(0) + \frac{1}{48\pi}\psi'(0)\Big)^{-1}$$
$$= \pi(\phi'(0))^{-1}\Big(1 + \frac{1}{48}\frac{\psi'(0)}{\phi'(0)}\Big)^{-1} \ .$$

From (6), (7), we get $\big|\frac{\psi'(0)}{\phi'(0)}\big| \leq C_\# \Lambda^{-2}$, so $\Big|\big(1 + \frac{1}{48}\ \frac{\psi'(0)}{\phi'(0)}\big)^{-1} - 1\Big| \leq C_\# \Lambda^{-2}$. Hence, (66) implies

$$(67) \qquad |E'(b_0) - \pi(\phi'(0))^{-1}| \leq C_\# \Lambda^{-2} |E'(b_0)| \leq C_\# \Lambda^{-2} \sigma(b) \tau^{-1}(b) \ ,$$

by (40), (42), (43) and the fact that $|b - b_0| \leq C_\# \Lambda^{4\varepsilon - 2}$. Applying (43) and (46) to (67), we get $|E'(b_0) - \pi(\phi'(0))^{-1}| \leq C_\# \Lambda^{-3} S_{\min}(E_*)$. Combining this with (65), we conclude that

$$(68) \quad + \frac{1}{2}\ \frac{d}{dt}E(t)\ \Big|_{t=b}\ \cdot \tilde{\chi}(b) = +\frac{\pi}{2}\big(\phi'(0)\big)^{-1}\tilde{\chi}\big(\frac{1}{\pi}\phi(0) - \frac{1}{2}\big) + \ \mathrm{Error}_4\ ,$$
$$\text{with } |\mathrm{Error}_4| \leq C_\# \Lambda^{-2} S_{\min}(E_*) \ .$$

Next, we examine the term $-\frac{1}{2}E'(a)\tilde{\chi}(a)$. Since $E(a) = E_*$, we have (as in (66)) that

$$E'(a) = \Big(\frac{d\Phi}{dE}(E)\ \Big|_{E=E_*}\Big)^{-1} = \Big(\frac{1}{\pi}\phi'(E_*) + \frac{1}{48\pi}\psi'(E_*)\Big)^{-1}$$
$$(69) \qquad = \pi\big(\phi'(E_*)\big)^{-1} \cdot \Big(1 + \frac{\psi'(E_*)}{48\phi'(E_*)}\Big)^{-1} \ .$$

Again, (6), (7) imply $\left|\frac{\psi'(E_*)}{\phi'(E_*)}\right| \leq C_\# \Lambda^{-2}$, so that (69) yields

$$|E'(a) - \pi(\phi'(E_*))^{-1}| \leq C_\# \Lambda^{-2} |E'(a)| \leq C'_\# \Lambda^{-3} S_{\min}(E_*) \ , \tag{70}$$

since

$$|E'(a)| \leq C_\# \sigma(a) \tau^{-1}(a) \leq C_\# \Lambda^{-1} S_{\min}(E_*) \quad \text{by (40), (43), (46)} \ . \tag{71}$$

We turn to $\tilde{\chi}(a) = \tilde{\chi}\big(\frac{1}{\pi}\phi(E_*) + \frac{1}{48\pi}\psi(E_*) - \frac{1}{2}\big)$, using again our assumption

$$\frac{1}{\pi}\phi(E_*) = m + \xi \ , \ \text{with } m \text{ an integer and } |\xi| \leq \Lambda^{-2} \ .$$

We have therefore (by definition of $\tilde{\chi}$)

$$\tilde{\chi}(a) = \tilde{\chi}\big(m - \frac{1}{2} + \frac{1}{48\pi}\psi(E_*) + \xi\big) = \inf_{k\in\mathbb{Z}} \big|\big(m - \frac{1}{2} + \frac{1}{48\pi}\psi(E_*) + \xi\big) - (k + \frac{1}{2})\big|^2 - \frac{1}{12}$$
$$= \big(\frac{1}{48\pi}\psi(E_*) + \xi\big)^2 - \frac{1}{12}\ , \text{ with } |\psi(E_*)| \le C_\# \Lambda^{-1}\ .$$

Hence

$$|\tilde{\chi}(a) + \frac{1}{12}| \le C_\# \Lambda^{-2}\ . \tag{72}$$

Applying (70), (71), (72), we get

$$\Big| - \frac{1}{2}E'(a)\tilde{\chi}(a) - \big(-\frac{1}{2}\big)\big(\pi(\phi'(E_*))^{-1}\big) \cdot \big(-\frac{1}{12}\big)\Big|$$
$$\le \frac{1}{2}\big|E'(a) - \pi(\phi'(E_*))^{-1}\big| \cdot \big|\tilde{\chi}(a)\big| + \frac{1}{2}\ |\pi(\phi'(E_*))^{-1}| \cdot |\tilde{\chi}(a) + \frac{1}{12}|$$
$$\le C_\# \Lambda^{-3} S_{\min}(E_*) + C_\# \Lambda^{-1} S_{\min}(E_*) \cdot \Lambda^{-2} \le C_\# \Lambda^{-3} S_{\min}(E_*)\ .$$

That is,

$$-\frac{1}{2}\ \frac{d}{dt}E(t)\ \Big|_{t=a} \cdot \tilde{\chi}(a) = +\frac{\pi}{24}\big(\phi'(E_*)\big)^{-1} + \text{ Error}_5 \tag{73}$$
$$\text{with } |\text{Error}_5| \le C_\# \Lambda^{-3} S_{\min}(E_*)\ .$$

We can now substitute (59), (60), (61), (68), (73) into (57), to obtain the following result.

$$\sum_{k\in[a,b]} E_k = \int_a^{b_0} E(t)dt + \frac{E_*}{48\pi}\psi(E_*) + \frac{\pi}{2}\big(\phi'(0)\big)^{-1}\tilde{\chi}\big(\frac{1}{\pi}\phi(0) - \frac{1}{2}\big) + \frac{\pi}{24}\big(\phi'(E_*)\big)^{-1}$$
$$+ \text{ Error}_6\ , \quad \text{with} \tag{74}$$

$$|\text{Error}_6| \le C_\# \Lambda^{4\varepsilon-2} S_{\min}(E_*)\ . \tag{75}$$

Next we calculate the integral in (74). Changing variable from t to $E = E(t)$ in that integral, we obtain

$$\int_a^{b_0} E(t)dt = \int_{E_*}^0 E\frac{d\Phi}{dE}dE = \int_{E_*}^0 E \cdot \big(\frac{1}{\pi}\phi'(E) + \frac{1}{48\pi}\psi'(E)\big)dE \tag{76}$$
$$= \big(\frac{1}{\pi}\phi(E) + \frac{1}{48\pi}\psi(E)\big) \cdot E\big]_{E_*}^0 - \int_{E_*}^0 \big(\frac{1}{\pi}\phi(E) + \frac{1}{48\pi}\psi(E)\big)dE$$
$$= -\frac{E_*}{\pi}\phi(E_*) - \frac{E_*}{48\pi}\psi(E_*) - \frac{1}{\pi}\int_{E_*}^0 \phi(E)dE - \frac{1}{48\pi}\int_{E_*}^0 \psi(E)dE\ .$$

Thus, we have to integrate the phase functions ϕ and ψ. To integrate $\phi(E)$, we simply write

$$\int_{E_*}^{0} \phi(E)dE = \int\limits_{\substack{E_*<E<0 \\ V(x)<E}} \int \big(E - V(x)\big)^{1/2} dx\ dE$$

$$= \int\limits_{\substack{E<0 \\ V(x)<E}} \int \big(E - V(x)\big)^{1/2} dx dE - \int\limits_{\substack{E<E_* \\ V(x)<E}} \big(E - V(x)\big)^{1/2} dx dE$$

$$= \int \frac{2}{3}\big(-V(x)\big)_+^{3/2} dx - \int \frac{2}{3}\big(E_* - V(x)\big)_+^{3/2} dx\ ,$$

as follows from doing the dE-integration first. So

$$(77)\quad -\frac{1}{\pi}\int_{E_*}^{0} \phi(E)dE = -\frac{2}{3\pi}\int_{I_{\mathrm{BVP}}} \big(-V(x)\big)_+^{3/2} dx + \frac{2}{3\pi}\int_{I_{\mathrm{BVP}}} \big(E_* - V(x)\big)_+^{3/2} dx\ .$$

To integrate $\psi(E)$, we use the following observation.

Lemma 8. *The limit defining $\psi(E)$, namely*

$$(78)\qquad \psi(E) = \lim_{\delta\to 0+}\left[\int_{E-V(x)>\delta} V''(x)\cdot\big(E - V(x)\big)^{-3/2} dx - q(E)\delta^{-1/2}\right]\ ,$$

exists uniformly for $E \in [E_*, 0]$. Also, $q(E)$ is continuous on $[E_*, 0]$.

Proof.

It is enough to establish uniform convergence and continuity of $q(E)$ for $E \in [E_*, 0]$ in a small neighborhood of a given $E_0 \in [E_*, 0]$. Because E_* is assumed to be strictly greater than the minimum of the potential, we can make a partition of unity $V''(x) = G_{\mathrm{left}}(x)+G_{\mathrm{center}}(x)+G_{\mathrm{rt}}(x)$ in a neighborhood of $[x_{\mathrm{left}}(E_0), x_{\mathrm{rt}}(E_0)]$, with the following properties:

$$G_{\mathrm{center}}(x) \text{ is } C^\infty \text{ and is supported in } \{E_0 - V(x) > \delta_0\}$$

$$(79)\qquad \text{for some small, positive } \delta_0\ .$$

$G_{\text{left}}(x)$ is C^∞ and is supported in a small neighborhood of

(80) $x = x_{\text{left}}(E_0)$.

(81) $G_{\text{rt}}(x)$ is C^∞ and is supported in a small neighborhood of $x = x_{\text{rt}}(E_0)$.

We write $\psi(E)$ in (78) as a sum of $\psi_{\text{left}}(E)$, $\psi_{\text{center}}(E)$, $\psi_{\text{rt}}(E)$ by replacing $V''(x)$ respectively by $G_{\text{left}}(x)$, $G_{\text{center}}(x)$, $G_{\text{rt}}(x)$. In place of $q(E)$ we have functions $q_{\text{left}}(E)$, $q_{\text{center}}(E)$, $q_{\text{rt}}(E)$. Evidently, for E near E_0, the limit

$$\psi_{\text{center}}(E) = \lim_{\delta\to 0+}\Big[\int_{E-V(x)>\delta} G_{\text{center}}(x)\big(E - V(x)\big)^{-3/2}dx - q_{\text{center}}(E)\delta^{-1/2}\Big]$$

exists uniformly, and $q_{\text{center}}(E) \equiv 0$. The discussions for $\psi_{\text{left}}(E)$ and $\psi_{\text{rt}}(E)$ are analogous, so we just study

(82) $$\psi_{\text{left}}(E) = \lim_{\delta\to 0+}\Big[\int_{E-V(x)>\delta} G_{\text{left}}(x)\cdot\big(E - V(x)\big)^{-3/2}dx - q_{\text{left}}(E)\delta^{-1/2}\Big] .$$

For (x, E) near $(x_{\text{left}}(E_0), E_0)$, we make the smooth change of variable $(x, E) \to (\xi, E)$, $\xi = E - V(x)$. (Since $-V'(x) > 0$ at $x = x_{\text{left}}(E_0)$, this is a smooth change of variables. Here we make crucial use of our assumption $E_* > V(x_0) = \min V$.)

The change of variable is well-defined for $x \in \operatorname{supp} G_{\text{left}}$ and E near to E_0, and produces $|\xi| \le A$ for such (x, E). Here A is some positive number independent of x, E. Therefore, using ξ in place of x as the independent variable in (82), we get

(83) $$\psi_{\text{left}}(E) = \lim_{\delta\to 0+}\Big[\int_\delta^A \theta(\xi, E)\cdot\xi^{-3/2}d\xi - q_{\text{left}}(E)\delta^{-1/2}\Big]$$

for a smooth function $\theta(\xi, E)$. Writing

$$\int_\delta^A \theta(\xi, E)\cdot\xi^{-3/2}d\xi = \int_\delta^A \Big[\frac{\theta(\xi, E) - \theta(0, E)}{\xi}\Big]\xi^{-1/2}d\xi + \theta(0, E)\int_\delta^A \xi^{-3/2}d\xi ,$$

we see that the limit in (83) exists uniformly, and that $q_{\text{left}}(E) = 2\theta(0, E)$ is continuous. Since the quantity in brackets in (83) is equal to that in (82) the proof is complete. ■

Lemma 8 gives

$$\int_{E_*}^{0} \psi(E)dE = \lim_{\delta\to 0+} \Big[\int\limits_{\substack{E_*<E<0 \\ E-V(x)>\delta}} \int V''(x)\cdot \big(E-V(x)\big)^{-3/2} dx dE - \int_{E_*}^{0} q(E)dE \cdot \delta^{-1/2} \Big] . \tag{84}$$

In particular, the limit in (84) exists. We have

$$\begin{aligned}
\int\limits_{\substack{E_*<E<0 \\ E-V(x)>\delta}} \int V''(x)\cdot\big(E-V(x)\big)^{-3/2} dx dE &= \int\limits_{\substack{E<0 \\ E-V(x)>\delta}} \int V''(x)\cdot\big(E-V(x)\big)^{-3/2} dx dE \\
&\quad - \int\limits_{\substack{E<E_* \\ E-V(x)>\delta}} \int V''(x)\cdot\big(E-V(x)\big)^{-3/2} dx dE \\
&= \int_{V(x)<-\delta} V''(x) \Big[\int_{V(x)+\delta}^{0} \big(E-V(x)\big)^{-3/2} dE \Big] dx \\
&\quad - \int_{V(x)<E_*-\delta} V''(x) \Big[\int_{V(x)+\delta}^{E_*} \big(E-V(x)\big)^{-3/2} dE \Big] dx \\
&= -2 \int_{V(x)<-\delta} V''(x)\cdot\big(-V(x)\big)^{-1/2} dx + 2\delta^{-1/2} \int_{V(x)<-\delta} V''(x) dx \\
&\quad + 2 \int_{V(x)<E_*-\delta} V''(x)\cdot\big(E_*-V(x)\big)^{-1/2} dx - 2\delta^{-1/2} \int_{V(x)<E_*-\delta} V''(x) dx .
\end{aligned}$$

Putting this into (84), we find that

$$\begin{aligned}
&\int_{E_*}^{0} \psi(E)dE \\
&= \lim_{\delta\to 0+} \Big[-2 \int_{V(x)<-\delta} V''(x)\cdot\big(-V(x)\big)^{-1/2} dx + 2 \int_{V(x)<E_*-\delta} V''(x)\cdot\big(E_*-V(x)\big)^{-1/2} dx \\
&\qquad + \delta^{-1/2} \Big\{ 2 \int_{E_*-\delta<V(x)<-\delta} V''(x) dx - \int_{E_*}^{0} q(E) dE \Big\} \Big] .
\end{aligned} \tag{85}$$

In particular, the limit in (85) exists. As $\delta \to 0+$, we have

$$\begin{aligned}
&\int_{V(x)<-\delta} V''(x)\cdot\big(-V(x)\big)^{-1/2} dx \to \int_{I_{\mathrm{BVP}}} V''(x)\cdot\big(-V(x)\big)_+^{-1/2} dx \\
&\int_{V(x)<E_*-\delta} V''(x)\cdot\big(E_*-V(x)\big)^{-1/2} dx \to \int_{I_{\mathrm{BVP}}} V''(x)\cdot\big(E_*-V(x)\big)_+^{-1/2} dx \\
&\int_{E_*-\delta<V(x)<-\delta} V''(x) dx = \int_{E_*<V(x)<0} V''(x) dx + O(\delta) .
\end{aligned}$$

Hence, the limit in (85) can exist only when

$$\{2\int_{E_*<V(x)<0} V''(x)dx - \int_{E_*}^{0} q(E)dE\} = 0\ ; \tag{86}$$

and when (86) holds, then (85) simplifies to

$$\int_{E_*}^{0} \psi(E)dE = -2\int_{I_{\mathrm{BVP}}} V''(x)\big(-V(x)\big)_+^{-1/2}dx + 2\int_{I_{\mathrm{BVP}}} V''(x)\cdot\big(E_*-V(x)\big)_+^{-1/2}dx\ . \tag{87}$$

Putting (77) and (87) into (76), we find that

$$\begin{aligned}\int_a^{b_0} E(t)dt &= -\frac{E_*}{\pi}\phi(E_*) - \frac{E_*}{48\pi}\psi(E_*)\\ &\quad - \frac{2}{3\pi}\int_{I_{\mathrm{BVP}}}\big(-V(x)\big)_+^{3/2}dx + \frac{2}{3\pi}\int_{I_{\mathrm{BVP}}}\big(E_*-V(x)\big)_+^{3/2}dx\\ &\quad + \frac{1}{24\pi}\int_{I_{\mathrm{BVP}}} V''(x)\cdot\big(-V(x)\big)_+^{-1/2}dx - \frac{1}{24\pi}\int_{I_{\mathrm{BVP}}} V''(x)\cdot\big(E_*-V(x)\big)_+^{-1/2}dx\ .\end{aligned}$$

Substituting this into (74) and noting the cancellation of the terms $\frac{E_*\psi(E_*)}{48\pi}$, we get

$$\begin{aligned}\sum_{k\in[a,b]} E_k &= -\frac{2}{3\pi}\int_{I_{\mathrm{BVP}}}\big(-V(x)\big)_+^{3/2}dx + \frac{2}{3\pi}\int_{I_{\mathrm{BVP}}}\big(E_*-V(x)\big)_+^{3/2}dx\\ &\quad + \frac{1}{24\pi}\int_{I_{\mathrm{BVP}}} V''(x)\cdot\big(-V(x)\big)_+^{-1/2}dx - \frac{1}{24\pi}\int_{I_{\mathrm{BVP}}} V''(x)\cdot\big(E_*-V(x)\big)_+^{-1/2}dx\\ &\quad - \frac{E_*}{\pi}\phi(E_*) + \frac{\pi}{2}\big(\phi'(0)\big)^{-1}\tilde{\chi}\big(\frac{1}{\pi}\phi(0)-\frac{1}{2}\big) + \frac{\pi}{24}\big(\phi'(E_*)\big)^{-1}\\ &\quad + \ \mathrm{Error}_6\ ,\ \text{with Error}_6 \text{ estimated by (75)}\ .\end{aligned} \tag{88}$$

The Corollary to Lemma 7 and the definition (44) of a, b show that the left-hand side of (88) is the sum of the eigenvalues of $-\frac{d^2}{dx^2}+V(x)$ belonging to $[E_*,0]$.

We record our results (88), (75) in the following lemma.

Lemma on Truncated Eigenvalue Sums. *Suppose we are given* ε, K, N, $\hat{c}$, $x_0\in I\subset I_{\mathrm{BVP}}\subset\mathbb{R}^1$, $S(x)$, $B(x)$, $V(x)$ *satisfying hypotheses (Z0)…(Z9) of the WKB Density Theorem. Let* $E_*\in\big(V(x_0),0\big]$, *and assume that* $\phi(E_*)<\phi(0)-1$, *and that* $\frac{1}{\pi}\phi(E_*)$ *differs from an integer by at most* $\frac{1}{\pi}\Lambda^{-2}$. *Let* $H=-\frac{d^2}{dx^2}+V(x)$

on $I_{\rm BVP}$*, with Dirichlet or Neumann boundary conditions. Let* X *denote the sum of the eigenvalues of* H *belonging to the interval* $[E_*,0]$*. Then*

$$\begin{aligned} X = &-\frac{2}{3\pi}\int_{I_{\rm BVP}} \big(-V(x)\big)_+^{3/2} dx + \frac{1}{24\pi}\int_{I_{\rm BVP}} V''(x)\cdot\big(-V(x)\big)_+^{-1/2} dx \\ &+ \frac{2}{3\pi}\int_{I_{\rm BVP}} \big(E_* - V(x)\big)_+^{3/2} dx - \frac{1}{24\pi}\int_{I_{\rm BVP}} V''(x)\cdot\big(E_* - V(x)\big)_+^{-1/2} dx \\ &+ \frac{\pi}{24}\big(\phi'(E_*)\big)^{-1} - \frac{E_*}{\pi}\phi(E_*) + \frac{\pi}{2}\big(\phi'(0)\big)^{-1}\tilde{\chi}\Big(\frac{1}{\pi}\phi(0) - \frac{1}{2}\Big) \\ &+ \ \mathrm{Error}\ , \ \ \textit{with} \end{aligned}$$

$$|\mathrm{Error}| \le C_{\#}\Lambda^{4\varepsilon-2}\cdot \inf_{\substack{x\in I \\ V(x)<E_*}} S(x)\ .$$

The constant $C_{\#}$ *depends only on* ε, K, N, c, C, c_1, C_α, $\hat{c}$ *in hypotheses (Z0)... (Z9).*

Our setting is as in the WKB Density Theorem, and we assume the hypotheses (Z0)...(Z9) of that result. Thus, $H = -\frac{d^2}{dx^2} + V(x)$ on $I_{\rm BVP}$, with Dirichlet or Neumann boundary conditions. Let $\text{sneg}(H)$ denote the sum of the negative eigenvalues of H.

First WKB Eigenvalue Sum Theorem.

$$\begin{aligned}\text{sneg}(H) = &-\frac{2}{3\pi}\int_{I_{\rm BVP}} \big(-V(x)\big)_+^{3/2}\,dx + \frac{1}{24\pi}\int_{I_{\rm BVP}} V''(x)\cdot\big(-V(x)\big)_+^{-1/2}\,dx \\ &+ \frac{\pi}{2}\big(\phi'(0)\big)^{-1}\tilde{\chi}\Big(\frac{1}{\pi}\phi(0) - \frac{1}{2}\Big) + \text{ Error }, \text{with} \\ &|\text{Error}| \le \Lambda^{5\varepsilon-2}|V(x_0)|\ .\end{aligned}$$

Recall that $V(x)$ attains its minimum at $x = x_0$, and that $\phi(0) = \int_{I_{\rm BVP}} \big(-V(x)\big)_+^{1/2}\,dx$, $\phi'(0) = \frac{1}{2}\int_{I_{\rm BVP}} \big(-V(x)\big)_+^{-1/2}\,dx$, while $\tilde{\chi}(t) = \min_{k\in\mathbb{Z}} |t-k-\frac{1}{2}|^2 - \frac{1}{12}$.

Proof.

Take $E_* = V(x_0) + \tau^2$ with τ very small, apply the Lemma on Truncated Eigenvalue Sums, and let $\tau \to 0$. The hypotheses imposed on E_* in that lemma are $E_* \in \big(V(x_0), 0\big]$, $|\frac{1}{\pi}\phi(E_*) -$ integer$| < \frac{1}{\pi}\Lambda^{-2}$ and $\phi(E_*) < \phi(0) - 1$. These hypotheses hold trivially, for $\tau \ne 0$ small enough. In fact, $\frac{1}{\pi}\phi(E_*) \to 0$ as $\tau \to 0$, so the hypotheses on E_* follow, once we check that $\phi(0) > 2$. However, contained in the hypotheses (Z0)...(Z9) is the fact that $V(x) < -c_\# S(x_0)$ for $|x - x_0| < c_\# B(x_0)$. (Here we use $c_\#$, $C_\#$ as in the previous section, to denote constants depending only on ε, K, N, c, C, c_1, C_α, $\hat{c}$ in the hypotheses (Z0)...(Z9).) Therefore

$$\begin{aligned}\phi(0) = \int_{I_{\rm BVP}} \big(-V(x)\big)_+^{1/2}\,dx \ge \int_{|x-x_0|<c_\# B(x_0)} c_\# S^{1/2}(x_0)\,dx = c'_\# S^{1/2}(x_0)B(x_0) = \\ c'_\#\lambda(x_0) \ge c''_\#\Lambda > 2 \text{ as needed .}\end{aligned}$$

So the Lemma on Truncated Eigenvalue Sums holds for

(1) $$E_* = V(x_0) + \tau^2 \ , \tau \quad \text{small but nonzero} \ .$$

Since all the eigenvalues of $-\frac{d^2}{dx^2} + V(x)$ are greater than $V(x_0) + \delta_0$ for some small, fixed δ_0, it follows that $X =$ sum of eigenvalues of H belonging to $[E_*, 0] =$ sneg(H) for $\tau \neq 0$ small enough. Hence the Lemma on truncated eigenvalue sums gives for small non-zero τ that

$$\begin{aligned} \text{sneg}(H) = & -\frac{2}{3\pi} \int_{I_{\text{BVP}}} \big(-V(x)\big)_+^{3/2} \, dx + \frac{1}{24\pi} \int_{I_{\text{BVP}}} V''(x) \cdot \big(-V(x)\big)_+^{-1/2} \, dx \\ & + \frac{2}{3\pi} \int_{I_{\text{BVP}}} \big(E_* - V(x)\big)_+^{3/2} \, dx - \frac{1}{24\pi} \int_{I_{\text{BVP}}} V''(x) \cdot \big(E_* - V(x)\big)_+^{-1/2} \, dx \\ & + \frac{\pi}{24} \big(\phi'(E_*)\big)^{-1} - \frac{E_*}{\pi} \phi(E_*) + \frac{\pi}{2} \big(\phi'(0)\big)^{-1} \tilde{\chi}\big(\frac{1}{\pi}\phi(0) - \frac{1}{2}\big) + \text{ Error}(\tau) \ , \end{aligned}$$

(2)

with

(3) $$|\text{Error}(\tau)| \leq C_\# \Lambda^{4\varepsilon - 2} \cdot \inf_{\substack{x \in I \\ V(x) < V(x_0) + \tau^2}} S(x) \ .$$

We study what happens to the various terms on the right-hand side of (2) when $\tau \to 0$. The first two integrals and the term involving $\tilde{\chi}$ are of course independent of τ. Also, $-\frac{E_*}{\pi}\phi(E_*)$ and $\int_{I_{\text{BVP}}} \big(E_* - V(x)\big)_+^{3/2} \, dx$ evidently tend to zero as $E_* \to V(x_0)+$ i.e. as $\tau \to 0$. The terms that require a little work are $\frac{\pi}{24}\big(\phi'(E_*)\big)^{-1}$ and $-\frac{1}{24\pi} \int_{I_{\text{BVP}}} V''(x) \cdot \big(E_* - V(x)\big)_+^{-1/2} \, dx$. We shall see that

(4) $$\lim_{\tau \to 0} \Big\{ \frac{\pi}{24} \big(\phi'(E_*)\big)^{-1} - \frac{1}{24\pi} \int_{I_{\text{BVP}}} V''(x) \cdot \big(E_* - V(x)\big)_+^{-1/2} \, dx \Big\} = 0 \ .$$

Assuming (4) for a moment, we can pass to the limit in (2), to derive the equations:

(5)
$$\begin{aligned} \text{sneg}(H) = & -\frac{2}{3\pi} \int_{I_{\text{BVP}}} \big(-V(x)\big)_+^{3/2} \, dx + \frac{1}{24\pi} \int_{I_{\text{BVP}}} V''(x) \cdot \big(-V(x)\big)_+^{-1/2} \, dx \\ & + \frac{\pi}{2} \big(\phi'(0)\big)^{-1} \tilde{\chi}\big(\frac{1}{\pi}\phi(0) - \frac{1}{2}\big) + \text{ Error} \ , \text{ with} \end{aligned}$$

(6) $$|\text{Error}| \le C_\# \Lambda^{4\varepsilon-2} \cdot \limsup_{E_* \to V(x_0)+} \Big\{ \inf_{\substack{x\in I \\ V(x)<E_*}} S(x) \Big\} \; .$$

The right-hand side of (6) is dominated by $C_\# \Lambda^{4\varepsilon-2} S(x_0)$ since $x_0 \in I$, $V(x_0) < E_*$. Since the inequalities $V(x_0) < -c_\# S(x_0) < 0$ are contained in the hypotheses (Z0)...(Z9), it follows that $S(x_0) \le C_\# |V(x_0)|$. Therefore, (6) implies

(7) $$|\text{Error}| \le C'_\# \Lambda^{4\varepsilon-2} |V(x_0)| \le \Lambda^{5\varepsilon-2} |V(x_0)| \; .$$

Equations (5), (7) are the conclusions of the First WKB Eigenvalue Sum Theorem. Thus, our Theorem is reduced to proving (4).

To prove (4), it is convenient to make a change of variable, which we prepare to define. Since $V'(x_0) = 0$ Taylor's theorem gives

(8) $$V(x) = V(x_0) + h(x) \cdot (x - x_0)^2$$

for a smooth function $h(x)$ defined in a neighborhood of x_0. Differentiating (8) twice, we find that $V''(x_0) = 2h(x_0)$. Hence, $h(x_0) > 0$, so h has a smooth square root $f(x)$ defined in a neighborhood of x_0, with $f(x_0) = \big(\frac{1}{2} V''(x_0)\big)^{1/2}$. Setting $\xi(x) = f(x) \cdot (x - x_0)$, we obtain

(9) $$V(x) = V(x_0) + \big(\xi(x)\big)^2$$

from (8), and

(10) $$\xi(x_0) = 0 \; , \quad \xi'(x_0) = f(x_0) = \big(\frac{1}{2} V''(x_0)\big)^{1/2} > 0$$

from the definition of $\xi(x)$.

Here, $\xi(x)$ is smooth in a neighborhood of x_0. Our change of variable is $s = \xi(x)$, which is inverted by a function $x = y(s)$, satisfying

(11) $$y(0) = x_0 \; , \quad y'(0) = \big(\frac{1}{2} V''(x_0)\big)^{-1/2} > 0 \; .$$

The function $y(s)$ is smooth in a neighborhood of the origin. Changing variable from x to $s = \xi(x)$ gives $E_* - V(x) = \tau^2 - s^2$ by (1), (9), so that

$$(12) \qquad \int_{I_{\rm BVP}} V''(x)\big(E_* - V(x)\big)_+^{-1/2}\, dx = \int_{-\infty}^{\infty} V''\big(y(s)\big)\cdot(\tau^2 - s^2)_+^{-1/2} y'(s)\, ds$$

and

$$(13) \qquad \phi'(E_*) = \frac{1}{2}\int_{I_{\rm BVP}} \big(E_* - V(x)\big)_+^{-1/2}\, dx = \frac{1}{2}\int_{-\infty}^{\infty} (\tau^2 - s^2)_+^{-1/2} y'(s)\, ds\ .$$

For a general function $\theta(s)$ smooth in a neighborhood of the origin, and for small, nonzero τ we note that

$$\int_{-\infty}^{\infty}(\tau^2 - s^2)_+^{-1/2}\theta(s)ds = \theta(0)\cdot\Big[\int_{-\infty}^{\infty}(\tau^2 - s^2)_+^{-1/2}ds\Big] + \Big[\int_{-\infty}^{\infty}(\tau^2 - s^2)_+^{-1/2}\{\theta(s) - \theta(0)\}\, ds\Big]$$

The first integral on the right is identically equal to π. The second integral on the right is dominated by $C\int_{-\infty}^{\infty}|s|(\tau^2 - s^2)_+^{-1/2}ds \le C|\tau|\int_{-\infty}^{\infty}(\tau^2 - s^2)_+^{-1/2}ds = C\pi|\tau|$. It follows that $\lim_{\tau\to 0}\int_{-\infty}^{\infty}\theta(s)\cdot(\tau^2 - s^2)_+^{-1/2}ds = \pi\theta(0)$ whenever $\theta(s)$ is smooth near the origin. Taking $\theta(s) = V''\big(y(s)\big)\cdot y'(s)$ and $\theta(s) = y'(s)$, we see from (12), (13) that $\lim_{\tau\to 0}\int_{I_{\rm BVP}} V''(x)\cdot\big(E_* - V(x)\big)_+^{-1/2}dx = \pi V''\big(y(0)\big)y'(0)$ and $\lim_{\tau\to 0}\phi'(E_*) = \frac{\pi}{2}y'(0)$. Substituting (11), we get

(14)

$$\lim_{\tau\to 0}\int_{I_{\rm BVP}} V''(x)\cdot\big(E_* - V(x)\big)_+^{-1/2}\, dx = \pi V''(x_0)\cdot\big(\tfrac{1}{2}V''(x_0)\big)^{-1/2} = \pi\sqrt{2}\big(V''(x_0)\big)^{+1/2}$$

and

$$(15) \qquad \lim_{\tau\to 0}\ \phi'(E_*) = \frac{\pi}{2}\ \big(\tfrac{1}{2}V''(x_0)\big)^{-1/2} = \frac{\pi}{\sqrt{2}}\big(V''(x_0)\big)^{-1/2}\ .$$

Hence, (4) is reduced to checking that

$$\frac{\pi}{24}\Big[\frac{\pi}{\sqrt{2}}\big(V''(x_0)\big)^{-1/2}\Big]^{-1} - \frac{1}{24\pi}\Big[\pi\sqrt{2}\big(V''(x_0)\big)^{+1/2}\Big] = 0\ ,$$

which is correct.

The proof of the First WKB Eigenvalue Theorem is complete. ■

In this section and the next, we prove a variant of the first WKB Theorem on Eigenvalue Sums, which applies to potentials that look like

$$V_c(x) = \frac{\ell(\ell+1)}{x^2} + E_0 - \frac{Z}{x} \tag{1}$$

near $x = 0$. Here, we show that if $V(x)$ is given exactly by (1) near the origin, then the low eigenvalues of $-\frac{d^2}{dx^2} + V(x)$ on $(0,\infty)$ are very close to the familiar eigenvalues of $-\frac{d^2}{dx^2} + V_c(x)$. In the next section, we combine this information with the Lemma on Truncated Eigenvalue Sums, to compare the eigenvalue sum for $-\frac{d^2}{dx^2} + V(x)$ with that for $-\frac{d^2}{dx^2} + V_c(x)$. We begin with a slight variant of the Agmon Lemma from [FS2].

Lemma 1. *Suppose* $-\frac{d^2}{dx^2}u + Vu = Eu$ *on* $(0,\infty)$*, with* $\|u\|_{L^2(0,\infty)} = 1$*. Suppose also that* V *is smooth on* $(0,\infty)$*, and that*

$$V(x) - E \geq \frac{1}{2}|E_*| \quad \textit{for } x > x_* \ .$$

Then

$$\|u\|^2_{L^2(2x_*,\infty)} \leq C_N\big(x_*^2|E_*|\big)^{-N}\big\{1 + x_*^2 \sup_{\frac{1}{8}x_*<x<x_*} |V(x) - E|\big\}$$

and

$$\|u'\|^2_{L^2(2x_*,\infty)} \leq C_N x_*^{-2}(x_*^2|E_*|)^{-N}\big\{1 + x_*^2 \sup_{\frac{1}{8}x_*<x<x_*} |V(x) - E|\big\}$$

with C_N *depending only on* N*.*

Proof.

We may assume u is real. For $y > 0$ we have $Av_{[y/2,y]}|u|^2 = \frac{2}{y}\|u\|^2_{L^2(y/2,y)} \leq \frac{2}{y}$. Putting $y/4$ for y gives also $Av_{[y/8,y/4]}|u|^2 \leq \frac{8}{y}$. Hence we can find $x_1 \in (\frac{y}{8}, \frac{y}{4})$ and

$x_2 \in (\frac{y}{2}, y)$ with $|u(x_1)|^2 \leq \frac{8}{y}$, $|u(x_2)|^2 \leq \frac{2}{y}$. The mean-value theorem produces an $x \in [x_1, x_2] \subset (\frac{y}{8}, y)$ with $2u'(x)u(x) = \frac{u^2(x_2) - u^2(x_1)}{x_2 - x_1}$, hence

$$|u'(x)u(x)| \leq \frac{1}{2} \frac{\frac{8}{y} + \frac{2}{y}}{y/4} = \frac{20}{y^2} .$$

Putting $y = x_*$, we obtain an $x_0 \in (\frac{x_*}{8}, x_*)$ with

$$\text{(2)} \qquad |u'(x_0)u(x_0)| \leq 20 x_*^{-2} .$$

Instead letting $y \to \infty$, we obtain a sequence $x_\nu \to \infty$, with

$$\text{(3)} \qquad u'(x_\nu)u(x_\nu) \to 0 .$$

Now suppose $\phi \in C^\infty(0, \infty)$. Integration by parts yields

$$\begin{aligned}
\text{(4)} \quad 0 = \int_{x_0}^{x_\nu} e^\phi u\Big\{-\frac{d^2}{dx^2}u + (V - E)u\Big\}dx &= -e^\phi uu'\Big]_{x_0}^{x_\nu} \\
&+ \int_{x_0}^{x_\nu} e^\phi\{(u')^2 + (V - E)u^2 + \phi' uu'\}dx \\
&\geq -e^\phi uu'\Big]_{x_0}^{x_\nu} + \int_{x_0}^{x_\nu} e^\phi\{\frac{1}{2}(u')^2 + (V - E - 100(\phi')^2)u^2\}dx
\end{aligned}$$

since $\phi' uu' \geq -\frac{1}{2}(u')^2 - 100(\phi')^2 u^2$.

We want to take $\phi(x) = 0$ for $x < x_*$, $\phi(x) = 10^{-3}|E_*|^{1/2}(x - x_*)$ for $x_* \leq x \leq y$, $\phi(x) = 10^{-3}|E_*|^{1/2}(y - x_*)$ for $y \leq x$. Here y is a given point with $2x_* < y < x_\nu$.

Although ϕ is not C^∞, we can find $\phi_\varepsilon \in C^\infty$ with $\phi_\varepsilon(x) \to \phi(x)$ and $\phi'_\varepsilon(x) \to \phi'(x)$ for $x \neq x_*, y$; and with $|\phi_\varepsilon|$, $|\phi'_\varepsilon|$ bounded uniformly in ε. Applying (4) to ϕ_ε and passing to the limit, we obtain (4) for the desired ϕ.

For $x < x_*$ we have $\phi = 0$, $\phi' = 0$.

For $x_* < x$ we have $V - E - 100(\phi')^2 \geq V - E - 10^{-4}|E_*| > \frac{1}{4}|E_*| > 0$. Hence (4) implies

$$\begin{aligned}
|u(x_0)u'(x_0)| + |e^{\phi(x_\nu)}u(x_\nu)u'(x_\nu)| + \int_{x_0}^{x_*} |V(x) - E| \cdot |u(x)|^2 dx \\
\geq \int_{x_*}^{x_\nu} e^\phi \Big\{\frac{1}{2}(u')^2 + \frac{1}{4}|E_*|(u(x))^2\Big\}dx .
\end{aligned}$$

Note that $\phi(x_\nu) = \phi(y)$ since $x_\nu \geq y$, and that

$$\int_{x_0}^{x_*} |V(x) - E|\ |u(x)|^2 dx \leq \sup_{x \in (x_0, x_*)} |V(x) - E| \cdot \|u\|^2$$
$$\leq \sup_{\frac{1}{8}x_* < x < x_*} |V(x) - E|\ .$$

Hence

$$|u(x_0)u'(x_0)| + |e^{\phi(y)} u(x_\nu)u'(x_\nu)| + \sup_{\frac{1}{8}x_* < x < x_*} |V(x) - E|$$
$$\geq \int_{x_*}^{x_\nu} e^{\phi} \{\frac{1}{2}(u')^2 + \frac{1}{4}|E_*|u^2\} dx\ .$$

The expression in curly brackets is non-negative everywhere, and $\phi(x) \geq 10^{-3}|E_*|^{1/2} x_*$ for $x \in [2x_*, y] \subset [x_*, x_\nu]$. Hence,

(5) $$|u(x_0)u'(x_0)| + |e^{\phi(y)} u(x_\nu)u'(x_\nu)| + \sup_{\frac{1}{8}x_* < x < x_*} |V(x) - E|$$
$$\geq \int_{2x_*}^{y} \{\frac{1}{2}(u')^2 + \frac{1}{4}|E_*|(u^2)\} dx \cdot \exp(10^{-3}|E_*|^{1/2} x_*)\ .$$

In (5) we fix $y > 2x_*$ and let $\nu \to \infty$. We obtain from (2), (3), (5) that

$$\exp(-10^{-3}|E_*|^{1/2} x_*) \cdot \{20x_*^{-2} + \sup_{\frac{1}{8}x_* < x < x_*} |V(x) - E|\}$$
$$\geq \int_{2x_*}^{y} \{\frac{1}{2}(u')^2 + \frac{1}{4}|E_*|(u)^2\} dx\ .$$

This holds for any $y > 2x_*$, so we can let $y \to \infty$ to get:

$$\int_{2x_*}^{\infty} (u')^2 dx \leq 40x_*^{-2}\{1 + x_*^2 \sup_{\frac{1}{8}x_* < x < x_*} |V(x) - E|\} \cdot \exp(-10^{-3}|E_*|^{1/2} x_*)$$

$$\int_{2x_*}^{\infty} (u)^2 dx \leq 80(|E_*|x_*^2)^{-1}\{1 + x_*^2 \sup_{\frac{1}{8}x_* < x < x_*} |V(x) - E|\} \cdot \exp(-10^{-3}|E_*|^{1/2} x_*)\ .$$

These estimates are much stronger than the conclusion of the Lemma. ∎

Lemma 2. *Suppose $V_1(x)$ and $V_2(x)$ are smooth potentials on $(0, \infty)$, with $V_1(x) = V_2(x)$ for $x < 10x_*$. Assume also $V_1(x)$, $V_2(x) \geq \frac{1}{2}E_*$ for $x > x_*$, with a given*

$E_* < 0$. Define $H_1 = -\frac{d^2}{dx^2} + V_1(x)$ and $H_2 = -\frac{d^2}{dx^2} + V_2(x)$ on $(0, \infty)$ with Dirichlet boundary conditions. Let $E_1 < E_*$ be an eigenvalue of H_1. Assume also that $\sup_{\frac{1}{8}x_* < x < x_*} |V_1(x) - E_1| \le (|E_*|x_*^2)^K |E_*|$, and that $|E_*|x_*^2$ is greater than a certain large, positive number depending only on K, N. Then there is a point E_2 in the spectrum of H_2, with

$$|E_1 - E_2| \le |E_*| \cdot (|E_*|x_*^2)^{-N} .$$

Proof.

Let $u(x)$ be the (normalized) eigenfunction corresponding to E_1. For $x > x_*$ we have $V_1(x) - E_1 \geq \frac{1}{2}E_* - E_* = \frac{1}{2}|E_*|$, so Lemma 1 applies. In view of our estimate on $\sup_{\frac{1}{8}x_*<x<x_*} |V_1(x) - E_1|$, we get from Lemma 1 that

$$\|u\|^2_{L^2(2x_*,\infty)} \leq C_{N_1}(x_*^2|E_*|)^{K+1-N_1} \ , \text{ and} \tag{6}$$

$$\|u'\|^2_{L^2(2x_*,\infty)} \leq C_{N_1}x_*^{-2}(x_*^2|E_*|)^{K+1-N_1} \tag{7}$$

for N_1 to be picked, depending on K and N .

Take $\chi(x)$ satisfying $\chi(x) = 1$ for $x \leq 2x_*$, $\chi(x) = 0$ for $x \geq 10x_*$, $0 \leq \chi \leq 1$ everywhere, $\left|\left(\frac{d}{dx}\right)^m \chi(x)\right| \leq C_m x_*^{-m}$. Then set $w = \chi u$.
Note that $\|w\| \geq \|u\| - \|(1-\chi)u\| = 1 - \|(1-\chi)u\| \geq 1 - \|u\|_{L^2(2x_*,\infty)} > \frac{1}{2}$ by (6), if we take N_1 large enough. Also

$$\begin{aligned}(H_2 - E_1)w &= \big(-\frac{d^2}{dx^2} + V_2 - E_1\big)(\chi u)\\ &= \chi\big(-\frac{d^2}{dx^2} + V_2 - E_1\big)u - 2\chi' u' - \chi'' u\\ &= -2\chi' u' - \chi'' u \ ,\end{aligned} \tag{8}$$

since $V_2 = V_1$ in supp χ, and $(H_1 - E_1)u = 0$.
Since χ', χ'' are supported in $\{x > 2x_*\}$ and $|\chi'(x)| \leq Cx_*^{-1}$, $|\chi''(x)| \leq Cx_*^{-2}$, it follows from (6), (7), (8) that

$$\begin{aligned}\|(H_2 - E_1)w\|^2 &\leq Cx_*^{-2}\|u'\|^2_{L^2(2x_*,\infty)} + Cx_*^{-4}\|u\|^2_{L^2(2x_*,\infty)}\\ &\leq C_{N_1}x_*^{-4}(x_*^2|E_*|)^{K+1-N_1} = C_{N_1}|E_*|^2(x_*^2|E_*|)^{K-1-N_1} \ .\end{aligned}$$

Taking $N_1 > K - 1 + 2N$ and assuming $x_*^2|E_*|$ large enough, we conclude that

$$\|(H_2 - E_1)w\| \leq \frac{1}{2}|E_*|(x_*^2|E_*|)^{-N} \ . \tag{9}$$

Since $\|w\| > \frac{1}{2}$, it follows from spectral theory and (9) that $|E_1 - E_2| \leq |E_*|(x_*^2|E_*|)^{-N}$ for some E_2 in the spectrum of H_2. This is the conclusion of the lemma. ■

Corollary. *Under the assumptions of Lemma 2, let u be the normalized eigenfunction of H_1 corresponding to E_1. Then there is a function w on $(0,\infty)$, satisfying $\|u - w\| \leq (|E_*|x_*^2)^{-N}$ and $\|(H_2 - E_1)w\| \leq |E_*| \cdot (x_*^2|E_*|)^{-N}$.*

Proof.

The second estimate is (9) and the first is immediate from (6), provided we take N_1 large enough and use the function w constructed in the proof of Lemma 2. ■

Lemma on Low Eigenvalues in Coulomb Singularities. *Suppose we are given E_0, E_*, Z, ℓ, x_* satisfying the following*

(a) $cZ^{4/3} < E_0 < CZ^{4/3}$

(b) $-Z^2 < E_* < -Z^{4/3}$

(c) $\min_{n \in \mathbb{Z}} \left| E_* - \left(E_0 - \frac{Z^2}{4n^2}\right)\right| > Z^{-500}$

(d) ℓ is a non-negative integer, and $\ell < CZ^{1/3}$

(e) $|E_*|x_*^2 > Z^{\frac{1}{100}}$ and $x_* > Z^{-1}$

(f) Z is greater than a large positive number determined by c and C in (a) and (d).

Let $V(x)$ be a smooth potential in $(0, \infty)$, equal to $V_c(x) \equiv \frac{\ell(\ell+1)}{x^2} + E_0 - \frac{Z}{x}$ for $x \leq 10x_*$. Assume also that $V(x)$, $V_c(x) > \frac{1}{2}E_*$ for $x > x_*$. Let $H = -\frac{d^2}{dx^2} + V(x)$ on $(0, \infty)$, with Dirichlet boundary conditions. Then the spectrum of H in $(-\infty, E_*)$ consists of eigenvalues $E_0 - \frac{Z^2}{4n^2} +$ $Error_n$ for all integers n in the range $\ell < n < \frac{Z}{2(E_0 - E_*)^{1/2}}$, and with $|Error_n| \leq Z^{-500}$.

Proof.

We start with a few preliminary remarks. For $x > x_*$ we have $V(x) \geq \frac{1}{2}E_* > E_*$, and therefore the spectrum of H in $(-\infty, E_*)$ consists at most of finitely many eigenvalues. Also, note that $V(x) \geq \frac{1}{2}E_* \geq -\frac{1}{2}Z^2$ for $x > x_*$, while $V(x) = V_c(x) \geq -\frac{Z}{x}$ for $x \leq x_*$. Hence $V(x) \geq -\frac{Z}{x} - \frac{1}{2}Z^2$ for all $x \in (0, \infty)$, which implies that the lowest eigenvalue of H is greater than or equal to that of $-\frac{d^2}{dx^2} - \frac{Z}{x} - \frac{1}{2}Z^2$, which is $-\frac{3}{4}Z^2$.

If $E < 0$ is an eigenvalue of either H or $H_c = -\frac{d^2}{dx^2} + V_c(x)$, then $-\frac{3}{4}Z^2 \le E < 0$, and

$$\sup_{\frac{1}{8}x_* < x < x_*} |V(x) - E| = \sup_{\frac{1}{8}x_* < x < x_*} |V_c(x) - E|$$
$$\le \sup_{\frac{1}{8}x_* < x < x_*} \left(\frac{\ell(\ell+1)}{x^2} + E_0 + \frac{Z}{x}\right) + |E|$$
$$\le CZ^{2/3}x_*^{-2} + CZ^{4/3} + CZx_*^{-1} + \frac{3}{4}Z^2 \le CZ^{\frac{8}{3}} ,$$
$$\text{whereas } (|E_*|x_*^2)^K |E_*| \ge Z^{\frac{K}{100}} \cdot Z^{\frac{4}{3}} >> CZ^{8/3} \text{ for large } K .$$

Hence

$$(10) \qquad \sup_{\frac{1}{8}x_* < x < x_*} |V(x) - E| = \sup_{\frac{1}{8}x_* < x < x_*} |V_c(x) - E| \le (|E_*|x_*^2)^K |E_*| ,$$

as in the hypotheses of Lemma 2.

We will prove three statements:

(11)

Every eigenvalue $E < E_*$ of H may be written in the form $E = E_0 - \dfrac{Z^2}{4n^2} +$ Error ,

where $|\text{Error}| < Z^{-500}$ and n is an integer satisfying $\ell < n < \dfrac{Z}{2(E_0 - E_*)^{1/2}}$.

(12) If $E, \tilde{E} < E_*$ are eigenvalues of H , and if for an integer n satisfying $\ell < n < \dfrac{Z}{2(E_0 - E_*)^{1/2}}$ we have $E = E_0 - \dfrac{Z^2}{4n^2} +$ Error and $\tilde{E} = E_0 - \dfrac{Z^2}{4n^2} + \widetilde{\text{Error}}$, with $|\text{Error}|, |\widetilde{\text{Error}}| < Z^{-500}$, then $E = \tilde{E}$.

(13)

Given an integer n with $\ell < n < \dfrac{Z}{2(E_0 - E_*)^{1/2}}$, we can find an eigenvalue $E < E_*$ of H , which can be written as $E = E_0 - \dfrac{Z^2}{4n^2} +$ Error with $|\text{Error}| < Z^{-500}$.

Together these statements make up the conclusion of the lemma. First we prove (11). We take $H_1 = H$, $E_1 = E$, $H_2 = H_c = -\frac{d^2}{dx^2} + V_c$ and apply Lemma 2. (Note that the hypotheses of Lemma 2 hold, by virtue of (10).) Thus there is an E_2 in the spectrum of H_c, with $|E-E_2| \le |E_*|(x_*^2|E_*|)^{-N} \le Z^2 \cdot Z^{-\frac{N}{100}} < Z^{-500}$ for suitable large N. In particular, $E_2 < E + Z^{-500} < E_* + Z^{-500} < 0 < E_0$. The spectrum of H_c below energy E_0 consists of the eigenvalues $E_0 - \frac{Z^2}{4n^2}$ for integers $n > \ell$, so we have $E_2 = E_0 - \frac{Z^2}{4n^2}$, $n > \ell$. Hence $|E - (E_0 - \frac{Z^2}{4n^2})| < Z^{-500}$. Since $E < E_*$ we have $E_0 - \frac{Z^2}{4n^2} < E_* + Z^{-500}$, so that by (c) we have also $E_0 - \frac{Z^2}{4n^2} < E_*$, i.e. $n < \frac{Z}{2(E_0-E_*)^{1/2}}$. Thus, $\ell < n < \frac{Z}{2(E_0-E_*)^{1/2}}$ and $|E-(E_0-\frac{Z^2}{4n^2})| < Z^{-500}$. This completes the proof of (11).

Next we prove (12). Let u, $\tilde{u}$ be the normalized eigenfunctions associated to E, $\tilde{E}$. Again we take $H_1 = H$, $E_1 = E$, $H_2 = H_c$ and note that the hypotheses of Lemma 2 hold, by virtue of (10). Hence the Corollary to Lemma 2 produces a function w on $(0,\infty)$ with

(14) $$\|u-w\| < Z^{-\frac{N}{100}} \;,\; \|(H_c - E)w\| < Z^2 \cdot Z^{-\frac{N}{100}} \;.$$

Similarly, using $\tilde{E}$ in place of E in the proof of (14), we get a function $\tilde{w}$ on $(0,\infty)$, satisfying

(15) $$\|\tilde{u}-\tilde{w}\| < Z^{-\frac{N}{100}} \;,\; \|(H_c - E)\tilde{w}\| < Z^2 \cdot Z^{-\frac{N}{100}} \;.$$

Taking N suitable large and recalling that $\|u\| = \|\tilde{u}\| = 1$, $|E-(E_0-\frac{Z^2}{4n^2})|$, $|\tilde{E}-(E_0-\frac{Z^2}{4n^2})| < Z^{-500}$, we conclude that

(16) $$\|(H_c - [E_0 - \frac{Z^2}{4n^2}])w\| \;,\; \|(H_c - [E_0 - \frac{Z^2}{4n^2}])\tilde{w}\| < 2 \cdot Z^{-500} \;.$$

Let ψ_n be the (normalized) eigenfunction of H_c corresponding to the eigenvalue $E_0 - \frac{Z^2}{4n^2}$. Then we can write

(17) $$w = a\psi_n + \varphi \text{ with } a \in \mathbb{C},\; \varphi \perp \psi_n \;; \qquad \text{and}$$

(18) $$\tilde{w} = \tilde{a}\psi + \tilde{\varphi} \quad \text{with} \quad \tilde{a} \in \mathbb{C} \;, \tilde{\varphi} \perp \psi_n \;.$$

Note that the eigenvalue $E_0 - \frac{Z^2}{4n^2}$ is separated from the rest of the spectrum of H_c by at least

$$\frac{cZ^2}{n^3} > cZ^2 \cdot \left[\frac{Z}{2(E_0 - E_*)^{1/2}}\right]^{-3} = c'Z^{-1}(E_0 - E_*)^{3/2}$$
$$\geq c''Z^{-1}(Z^{4/3})^{3/2} = c''Z > 1 \ .$$

It follows by spectral theory that

$$\|(H_c - [E_0 - \frac{Z^2}{4n^2}])w\| = \|(H_c - [E_0 - \frac{Z^2}{4n^2}])\varphi\| \geq \|\varphi\| \ .$$

Hence (16) implies $\|\varphi\| \leq 2Z^{-500}$, and similarly $\|\tilde{\varphi}\| \leq 2Z^{-500}$. Thus, (17) and (18) show that

$$\text{(19)} \qquad \|w - a\psi_n\| \ , \ \|\tilde{w} - \tilde{a}\psi_n\| \leq 2Z^{-500} \ .$$

On the other hand, $\|\psi_n\| = 1$ and $\|w\|$, $\|\tilde{w}\| \sim 1$ by (14) and (15), since $\|u\| = \|\tilde{u}\| = 1$. Therefore, (19) implies that $|a|$, $|\tilde{a}| \sim 1$. Putting this back into (19), we conclude that $\|w - a'\tilde{w}\| < CZ^{-500}$ for a complex number a' with $|a'| \sim 1$. Hence by (14) and (15) we have also $\|u - a'\tilde{u}\| < C'Z^{-500} << 1$. If E, $\tilde{E}$ were distinct, then $u \perp \tilde{u}$, contradicting $\|u - a'\tilde{u}\| << 1$. Hence $E = \tilde{E}$, which completes the proof of (12).

It remains to prove (13). We invoke Lemma 2 with $H_1 = H_c$, $E_1 = E_0 - \frac{Z^2}{4n^2} < E_*$ (by our restriction on n), $H_2 = H$. Again the hypotheses of Lemma 2 hold, by virtue of (10).

The conclusion of Lemma 2 gives us an $E \in$ spectrum(H) with $|E - (E_0 - \frac{Z^2}{4n^2})| < |E_*|(|E_*|x_*^2)^{-N} < Z^2 \cdot Z^{-\frac{N}{100}} < Z^{-500}$ for suitable large N. In particular, $E < E_0 - \frac{Z^2}{4n^2} + Z^{-500} < E_*$ by virtue of (c). Since the spectrum of H below E_* contains only eigenvalues, we see that $E < E_*$ is an eigenvalue of H satisfying $|E - (E_0 - \frac{Z^2}{4n^2})| < Z^{-500}$. This completes the proof of (13) and with it, that of the Lemma. ■

Suppose we are given a smooth potential $V(x)$ defined on $(0,\infty)$, an interval $I \subset (0,\infty)$ containing $\{V(x) < 0\}$, and two positive functions $S(x)$, $B(x)$ defined on I. We will say that $V(x)$ has an *exact Coulomb singularity* with parameters (ℓ, E_0, Z, x_*) if the following conditions are satisfied:

(CS1) $V(x) = V_c(x) \equiv \frac{\ell(\ell+1)}{x^2} + E_0 - \frac{Z}{x}$ for $0 < x < 10x_*$

(CS2) $S(x) = \frac{Z}{x}$ and $B(x) = x$ for $x \in I$, $x < 10x_*$

(CS3) ℓ is an integer, and $0 \leq \ell \leq \frac{1}{16}(Zx_*)^{1/2}$

(CS4) $cZ^{4/3} < E_0 < CZ^{4/3}$

(CS5) $Z^{-\frac{98}{100}} < x_* < Z^{-\frac{1}{3}-\frac{1}{100}}$

(CS6) Z is greater than a large, positive constant determined by c and C in (CS4) above.

Strictly speaking, these conditions pertain to $V(x)$, $S(x)$, $B(x)$ rather than just $V(x)$, but this should cause no confusion.

In this section, we study potentials $V(x)$ satisfying the hypotheses (Z0)...(Z9) of the WKB Density Theorem, and also having an exact Coulomb singularity. We set $H = -\frac{d^2}{dx^2} + V(x)$ on $(0,\infty)$ with Dirichlet boundary conditions. Our goal is to make an accurate computation of the sum $\text{sneg}(H)$ of the negative eigenvalues of H. The First WKB Eigenvalue Sum Theorem gives a formula for $\text{sneg}(H)$ with an error dominated by $\Lambda^{4\varepsilon-2}|\min_{x\in(0,\infty)} V(x)|$. However, for $\ell << Z^{1/3}$, a Coulomb singularity has $\min_{x\in(0,\infty)} V(x)$ very large, so that the error grows too big. We will overcome this difficulty by comparing H with $H_c = -\frac{d^2}{dx^2} + V_c(x)$ and using the known eigenvalues of H_c. Thus, instead of computing $\text{sneg}(H)$ directly, we will compute $\text{sneg}(H) - \text{sneg}(H_c)$ and get an error small compared to that in $\text{sneg}(H)$. This trick has been used repeatedly in the literature, and is obvious to anyone interested in Coulomb singularities.

We will pick a suitable cutoff energy $E_* < 0$, and define:

(1) $$X = \text{ sum of the eigenvalues of } H \text{ belonging to } [E_*, 0]$$

(2) $$X_c = \text{ sum of the eigenvalues of } H_c \text{ belonging to } [E_*, 0]$$

(3) $$Y = \text{ sum of the eigenvalues of } H \text{ belonging to } (-\infty, E_*)$$

(4) $$Y_c = \text{ sum of the eigenvalues of } H_c \text{ belonging to } (-\infty, E_*) \ .$$

So

(5) $$\text{sneg}(H) - \text{sneg}(H_c) = X - X_c + (Y - Y_c) \ .$$

Using the results of the previous section, we will show that $(Y - Y_c)$ is negligibly small. We will then compute X and X_c using the Lemma on Truncated Eigenvalue Sums. Putting these results into (5), we will get an accurate approximation to $\text{sneg}(H) - \text{sneg}(H_c)$.

To apply the Lemma on Truncated Eigenvalue Sums, we must pick E_* so that the phase $\phi(E_*)$ is close to a multiple of π. We compute $\phi(E)$ in the following elementary lemma, whose proof we include for the reader's convenience.

Lemma 1. *Suppose E, ℓ, Z are positive numbers, with $Z^2 > 4\ell(\ell+1)E$ so that $\frac{\ell(\ell+1)}{x^2} + E - \frac{Z}{x}$ is negative somewhere in $(0,\infty)$. Then*

$$\int_0^\infty \Big(-\big(\frac{\ell(\ell+1)}{x^2} + E - \frac{Z}{x}\big)\Big)_+^{1/2} dx = \pi\Big(\frac{Z}{2E^{1/2}} - \sqrt{\ell(\ell+1)}\Big) \ .$$

Proof.

For $0 < a < b$, we compute

$$I = \int_a^b \sqrt{-(t-a)(t-b)} \ \frac{dt}{t} \ .$$

Multiplying together suitable branches of $(z-a)^{1/2}$ and $(z-b)^{1/2}$, we get a single-valued branch of $F(z) = \sqrt{(z-a)(z-b)}$ defined for $z \in \mathbb{C} \backslash [a,b]$, with the following properties:

$F(z) \sim z$ for $|z|$ large (rather than $F(z) \sim -z$);

$F(t+i0) = i\sqrt{-(t-a)(t-b)}$ and $F(t-i0) = -i\sqrt{-(t-a)(t-b)}$ for $t \in [a,b]$;

$F(0) = -\sqrt{ab}$.

Then define contours γ, $\tilde{\gamma}$, Γ in $\mathbb{C}$ as follows:

$\tilde{\gamma}$ is a small circle about 0

Γ is a huge circle about 0

γ is a thin rectangle around the interval $[a,b]$.

We suppose all these contours have counterclockwise orientation. Cauchy's theorem gives

$$\oint_\Gamma F(z)\frac{dz}{z} - \oint_\gamma F(z)\frac{dz}{z} - \oint_{\tilde{\gamma}} F(z)\frac{dz}{z} = 0\ . \tag{6}$$

If Γ has radius R, then

$$\begin{aligned}\oint_\Gamma F(z)\frac{dz}{z} &= \oint_\Gamma z\sqrt{(1-\frac{a}{z})(1-\frac{b}{z})}\cdot\frac{dz}{z} \\ &= \oint_\Gamma\Big\{1-(\frac{a+b}{2})z^{-1}+O(|z|^{-2})\Big\}\ dz = -(\frac{a+b}{2})\cdot 2\pi i + O(R^{-1})\ .\end{aligned} \tag{7}$$

We will let $R \to \infty$ in (6). Also

$$\oint_{\tilde{\gamma}} F(z)\frac{dz}{z} = 2\pi i F(0) = -2\pi i\sqrt{ab}\ . \tag{8}$$

To evaluate $\oint_\gamma F(z)\frac{dz}{z}$, we write $\gamma = \big\{(t+i\varepsilon) \mid t \in [a-\varepsilon, b+\varepsilon]\big\} \cup \big\{(t-i\varepsilon) \mid t \in [a-\varepsilon, b+\varepsilon]\big\} \cup \big\{a-\varepsilon+it \mid t\in[-\varepsilon,+\varepsilon]\big\} \cup \big\{b+\varepsilon+it \mid t \in [-\varepsilon,\varepsilon]\big\}$ and let $\varepsilon \to 0+$. In view of our formulas for $F(t+i0)$, $F(t-i0)$, we get

$$\oint_\gamma F(z)\frac{dz}{z} = -2i\int_a^b \sqrt{-(t-a)(t-b)}\ \frac{dt}{t} = -2i\,I\ . \tag{9}$$

Putting (7), (8), (9) into (6) we get $I = \pi\cdot\big(\frac{a+b}{2}-\sqrt{ab}\big)$. Equivalently, if $\{t^2-Qt+P < 0\}$ is a non-empty subinterval of $(0,\infty)$, then

$$\int_0^\infty \Big(-(t^2-Qt+P)\Big)_+^{1/2}\ \frac{dt}{t} = \pi(\frac{Q}{2}-\sqrt{P})\ .$$

This shows that

$$(10)\qquad \int_0^\infty \Big(-\big(\frac{P}{t^2}-\frac{Q}{t}+A\big)\Big)_+^{1/2} dt = \int_0^\infty \Big(-\big(At^2-Qt+P\big)\Big)_+^{1/2} \frac{dt}{t}$$
$$= \pi\Big(\frac{Q}{2\sqrt{A}}-\sqrt{P}\Big) \text{ when } Q, A, P, Q^2-4AP>0 \ .$$

The conclusion of the Lemma is immediate from (10). ■

In view of Lemma 1, we will have

$$(11)\qquad \phi_c(E_*) = \int_0^\infty \big(E_* - V_c(x)\big)_+^{1/2} dx = \pi\big(\frac{Z}{2\sqrt{E_0-E_*}} - \sqrt{\ell(\ell+1)}\,\big) = \pi m_*$$

for an integer m_*, provided we take

$$(12)\qquad E_* = E_0 - \frac{Z^2}{4[m_* + \sqrt{\ell(\ell+1)}]^2} \ ,$$

and provided

$$(13)\qquad Z^2 > 4\ell(\ell+1)(E_0-E_*) \ .$$

We will pick

$$(14)\qquad m_* = \text{ largest integer } < \frac{1}{16}(Zx_*)^{1/2} \ ,$$

and check that (13) holds, since

$$4\ell(\ell+1)(E_0-E_*) = 4\ell(\ell+1)\cdot\frac{Z^2}{4[m_*+\sqrt{\ell(\ell+1)}]^2}$$
$$< 4\ell(\ell+1)\cdot\frac{Z^2}{4[\sqrt{\ell(\ell+1)}]^2} = Z^2 \ .$$

Lemma 2. *Suppose $V(x)$ is smooth on $(0,\infty)$ and satisfies the hypotheses (Z0)...(Z9) and (CS1)...(CS6). Assume also that $\ell > Z^{1/K} + 10^9$. Define E_* by (12), (14). Then all the hypotheses of the Lemma on Low Eigenvalues in Coulomb Singularities are satisfied.*

Proof.

The hypotheses (a)...(f) of that Lemma are as follows. (a) says that $cZ^{4/3} < E_0 < CZ^{4/3}$, which is (CS4). (b) says that $-Z^2 < E_* < -Z^{4/3}$. We have from (12) that $E_* > -\frac{Z^2}{4[m_*+\sqrt{\ell(\ell+1)}]^2} > -Z^2$, which is the desired lower bound for E_*. To prove the desired upper bound for E_*, it's enough to check that

$$\text{(14 bis)} \qquad \frac{Z^2}{4[m_* + \sqrt{\ell(\ell+1)}]^2} > c\,Z^{\frac{4}{3}+\frac{1}{100}} ,$$

i.e. $[m_* + \sqrt{\ell(\ell+1)}]^2 < CZ^{\frac{2}{3}-\frac{1}{100}}$, i.e. $m_*, \ell < CZ^{\frac{1}{3}-\frac{1}{200}}$. In fact we have by (14) and (CS3) that $m_*, \ell \le \frac{1}{16}(Zx_*)^{1/2} < \frac{1}{16}(Z^{\frac{2}{3}-\frac{1}{100}})^{1/2} = \frac{1}{16}Z^{\frac{1}{3}-\frac{1}{200}}$ by (CS5). This completes the proof of (b).

(c) says that $\min_{n\in\mathbb{Z}}\left|E_* - \left(E_0 - \frac{Z^2}{4n^2}\right)\right| > Z^{-500}$, i.e. $\min_{n\in\mathbb{Z}}\left|\frac{Z^2}{4[m_*+\sqrt{\ell(\ell+1)}]^2} - \frac{Z^2}{4n^2}\right| > Z^{-500}$, i.e.

$$\text{(15)} \qquad \min_{n\in\mathbb{Z}}\left|[m_* + \sqrt{\ell(\ell+1)}]^{-2} - n^{-2}\right| > 4\;Z^{-502} .$$

We have $\sqrt{\ell(\ell+1)} = \ell\sqrt{1+\ell^{-1}} = \ell + \frac{1}{2} + O(\ell^{-1})$. Since $\ell > Z^{1/K} + 10^9$, it follows that $\left|\sqrt{\ell(\ell+1)} - (\ell+\frac{1}{2})\right| < \frac{1}{100}$. Thus, $m_* + \sqrt{\ell(\ell+1)} = (m_* + \ell + \frac{1}{2}) + \xi$ with $|\xi| < \frac{1}{100}$. So in (15), the minimum is attained by $n = m_* + \ell$ or by $n = m_* + \ell + 1$. In either case we have $\frac{1}{4} < |n - (m_* + \sqrt{\ell(\ell+1)})| < 1$, so $|[m_* + \sqrt{\ell(\ell+1)}]^{-2} - n^{-2}| \sim n^{-3} \sim (m_* + \ell)^{-3} > c(Zx_*)^{-3/2}$ since we saw in the proof of (b) that $m_*, \ell \le \frac{1}{16}(Zx_*)^{1/2}$. Thus,

$$\min_{n\in\mathbb{Z}}\left|[m_* + \sqrt{\ell(\ell+1)}]^{-2} - n^{-2}\right| > c(Zx_*)^{-3/2} > c\left(Z^{\frac{2}{3}-\frac{1}{100}}\right)^{-3/2} >> 4Z^{-502} ,$$

completing the proof of (c).

(d) says that ℓ is an integer and $0 \le \ell \le CZ^{1/3}$, which is immediate from (CS3) and (CS5).

(e) says that $x_* > Z^{-1}$ and $|E_*|x_*^2 > Z^{\frac{1}{100}}$. Now $x_* > Z^{-1}$ is contained in (CS5). From (CS4), (12), (14 bis), we have $|E_*| \sim \frac{Z^2}{4[m_*+\ell]^2}$. Hence, since $\ell \le \frac{1}{16}(Zx_*)^{1/2} \sim$

m_*, we get $|E_*|x_*^2 \sim \frac{Z^2 x_*^2}{[m_*+\ell]^2} \sim \frac{Z^2 x_*^2}{m_*^2} \sim \frac{Z^2 x_*^2}{(Zx_*)}$ (by (14)) $= Zx_*$, so $|E_*|x_*^2 > cZ^{\frac{2}{100}}$ by (CS5). This completes the proof of (e).

(f) asserts that Z is large, which is just (CS6). Thus we have proven (a)...(f).

The remaining hypotheses of the Lemma on Low Eigenvalues in Coulomb Singularities are $V \in C^\infty(0,\infty)$, $V(x) = V_c(x)$ for $x \le 10x_*$, and $V(x)$, $V_c(x) > \frac{1}{2}E_*$ for $x > x_*$. Since (CS1)...(CS6) include the second of these assertions, and we are assuming the first, it remains only to show that

$$(16) \qquad V(x), V_c(x) > \frac{1}{2}E_* \quad \text{for } x \ge x_* \ .$$

Now $V_c(x)$ is decreasing for $x < x_0 = \frac{2\ell(\ell+1)}{Z}$ and increasing for $x > x_0$. We have $x_0 < \frac{4\ell^2}{Z} \le \frac{4}{Z}\left[\frac{1}{16}(Zx_*)^{1/2}\right]^2 = \frac{1}{64}x_*$, so $V_c(x)$ is increasing in $[x_*,\infty)$. Hence to check $V_c(x) > \frac{1}{2}E_*$ for $x \ge x_*$ it's enough to check that $V_c(x_*) = \frac{\ell(\ell+1)}{x_*^2} + E_0 - \frac{Z}{x_*} > \frac{1}{2}E_*$. This follows from $\frac{1}{2}E_0 - \frac{Z}{x_*} > \frac{1}{2}E_*$, i.e. $\frac{Z}{x_*} < \frac{1}{2}(E_0 - E_*) = \frac{Z^2}{8[m_*+\sqrt{\ell(\ell+1)}]^2}$, i.e.

$$(17) \qquad \left[m_* + \sqrt{\ell(\ell+1)}\right]^2 < \left(\frac{Zx_*}{8}\right) .$$

Since $m_* < \frac{1}{16}(Zx_*)^{1/2}$ and $\sqrt{\ell(\ell+1)} < \ell + 1 \le \frac{2}{16}(Zx_*)^{1/2}$, we have $[m_* + \sqrt{\ell(\ell+1)}]^2 < \left(\frac{3}{16}\right)^2 Zx_* < \frac{Zx_*}{8}$, proving (17). So $V_c(x) > \frac{1}{2}E_*$ for $x \ge x_*$.

To handle $V(x)$, we argue as follows. There is a local minimum of $V_c(x)$ at $x = x_0 < x_*$; and $V_c(x_0) < 0$. Since $V(x) = V_c(x)$ for $x < x_*$, it follows that $V(x)$ has a local minimum at the same x_0, and $V(x_0) < 0$. Hypotheses (Z0)...(Z9) show that x_0 is the global minimum of $V(x)$, which is also called x_0 in (Z0)...(Z9). From (Z0)...(Z9), we have $V(x)$ increasing in $[x_0, x_{\mathrm{rt}}]$, and $V(x) > 0$ for $x > x_{\mathrm{rt}}$. We know that $V(x_*) = V_c(x_*) > \frac{1}{2}E_*$. Now suppose $x > x_*$. Either $x \le x_{\mathrm{rt}}$, in which case $V(x) \ge V(x_*) > \frac{1}{2}E_*$; or else $x > x_{\mathrm{rt}}$, in which case $V(x) > 0 > \frac{1}{2}E_*$. Thus $V(x) > \frac{1}{2}E_*$ for all $x \ge x_*$. The proof of Lemma 2 is complete. ■

By Lemma 2 and the Lemma on Low Eigenvalues in Coulomb Singularities, we have

$$|Y - Y_c| < Z^{-100} \ . \quad \text{(See (3), (4).)} \tag{18}$$

Next we prepare to compute X, X_c using the Lemma on Truncated Eigenvalue Sums.

Lemma 3. *The hypotheses (Z0)...(Z9) of the WKB Density Theorem are satisfied when we take* $V(x) = V_c(x)$; $S_c(x) = \frac{Z}{x}$, $B_c(x) = x$, $I_{\mathrm{BVP}} = (0,\infty)$, $I_c = \left[10^{-9}\frac{\ell^2}{Z}, 10^{+9}\frac{Z}{E_0}\right]$, *with* ℓ *and* E_0 *as in (CS1)...(CS6) and* $\ell > Z^{(10^{-9})}$ *and with* $K = 10^9$ *and* $\hat{c}$ *suitably picked. Moreover, with* $\lambda_c(x) = S_c^{1/2}(x)B_c(x)$ *and* $\Lambda_c = \left(\int_{V_c<0} \frac{dx}{\lambda_c(x)B_c(x)}\right)^{-1}$, *we have* $\Lambda_c \sim \ell$.

Proof.

We provide the elementary details for the reader's convenience. (Z0) says that $c < y/x < C$ and $c < (\frac{Z}{x})/(\frac{Z}{y}) < C$ for $|x-y| < c|x|$, which is obvious. (Z1) says that $\left|\left(\frac{d}{dx}\right)^\alpha\left\{\frac{\ell(\ell+1)}{x^2} + E_0 - \frac{Z}{x}\right\}\right| \le C_\alpha Z x^{-1-\alpha}$ in I_c. In fact,

$$\begin{aligned}\left|\left(\frac{d}{dx}\right)^\alpha\left\{\frac{\ell(\ell+1)}{x^2}\right\}\right| &= c_\alpha \ell(\ell+1)x^{-2-\alpha} \\ &= c_\alpha\left\{\frac{\ell(\ell+1)}{Z}x^{-1}\right\}Zx^{-1-\alpha} \le C_\alpha Z x^{-1-\alpha} \quad \text{in } I_c \ .\end{aligned} \tag{19}$$

Also, when $\alpha = 0$ we have $\left|\left(\frac{d}{dx}\right)^\alpha E_0\right| = E_0 \le 10^{+9}\frac{Z}{x}$ in I_c, while $\left(\frac{d}{dx}\right)^\alpha E_0 = 0$ for $\alpha \neq 0$. So

$$\left|\left(\frac{d}{dx}\right)^\alpha E_0\right| \le 10^9 Z x^{-1-\alpha} \quad \text{in } I_c \ . \tag{20}$$

Evidently,

$$\left|\left(\frac{d}{dx}\right)^\alpha\left\{\frac{Z}{x}\right\}\right| \le C_\alpha Z\, x^{-1-\alpha} \ . \tag{21}$$

Estimates (19), (20), (21) imply (Z1).

(Z2) and (Z5) follow when we show that $\{x \in (0,\infty) \mid V_c(x) < 0\} = (x_{\rm left}, x_{\rm rt}) \subset I_c$ with $x_{\rm left}, x_{\rm rt} \in \left[10^{-6}\frac{\ell^2}{Z}, 10^{+6}\frac{Z}{E_0}\right]$. In fact, $\{x^2 V_c(x) < 0\} = \{\ell(\ell+1) - Zx + E_0 x^2 < 0\}$ has the form $(x_{\rm left}, x_{\rm rt})$ with $-\infty < x_{\rm left} < x_{\rm rt} < +\infty$, since the discriminant $Z^2 - 4E_0\ell(\ell+1)$ is positive. (Recall $E_0 < CZ^{4/3}$, $\ell \le \frac{1}{16}(Zx_*)^{1/2} < CZ^{\frac{1}{3}-\frac{1}{200}}$, so

$$E_0\ell(\ell+1) < CZ^{2-\frac{1}{100}}.\ ) \tag{22}$$

We have $x_{\rm left} = \frac{Z - \sqrt{Z^2 - 4E_0\ell(\ell+1)}}{2E_0} > 0$, so $(x_{\rm left}, x_{\rm rt}) \subset (0,\infty)$. Note that

$$x_{\rm rt} = \frac{Z + \sqrt{Z^2 - 4E_0\ell(\ell+1)}}{2E_0} > \frac{Z}{2E_0} > cZ^{-1/3}\ . \tag{22 bis}$$

For $x \le 10^{-6}\frac{\ell^2}{Z}$ we have $\frac{\ell(\ell+1)}{x^2} + E_0 - \frac{Z}{x} > \frac{Z}{x^2}\{\frac{\ell(\ell+1)}{Z} - x\} > 0$, so $(x_{\rm left}, x_{\rm rt}) = \{\frac{\ell(\ell+1)}{x^2} + E_0 - \frac{Z}{x} < 0\} \subset (10^{-6}\frac{\ell^2}{Z}, \infty)$. For $x \ge 10^{+6}\frac{Z}{E_0}$ we have $\frac{\ell(\ell+1)}{x^2} + E_0 - \frac{Z}{x} \ge \frac{E_0}{x}\{x - \frac{Z}{E_0}\} > 0$, so $(x_{\rm left}, x_{\rm rt}) \subset (10^{-6}\frac{\ell^2}{Z}, 10^{+6}\frac{Z}{E_0})$ as claimed. We have proven (Z2) and (Z5).

(Z3) says that $V_c(x_0) < -c\frac{Z}{x_0}$, $V'(x_0) = 0$, $V''(x) > c\frac{Z}{x^3}$ for $|x - x_0| < c_1 x_0$. To find x_0, set $V_c'(x) = -\frac{2\ell(\ell+1)}{x^3} + \frac{Z}{x^2}$ equal to zero. Thus, $x_0 = \frac{2\ell(\ell+1)}{Z}$, so $V_c(x_0) = \frac{\ell(\ell+1)}{x_0^2} + E_0 - \frac{Z}{x_0} = \frac{Z}{x_0}\{\frac{\ell(\ell+1)}{Zx_0} + \frac{E_0 x_0}{Z} - 1\}$ i.e. $V_c(x_0) = \frac{Z}{x_0}\{\frac{1}{2} + \frac{E_0x_0}{Z} - 1\} \sim -\frac{1}{2}\frac{Z}{x_0}$ since $\frac{E_0 x_0}{Z} \sim \frac{E_0\ell(\ell+1)}{Z^2} < CZ^{-\frac{1}{100}}$ by (22).

Also for $|x - x_0| < \frac{1}{100}x_0$, we have

$$\begin{aligned} V_c''(x) = \frac{6\ell(\ell+1)}{x^4} - \frac{2Z}{x^3} = \frac{6Z}{x^3}\left\{\frac{\ell(\ell+1)}{Zx} - \frac{1}{3}\right\} &> \frac{6Z}{x^3}\left\{\frac{100}{101}\,\frac{\ell(\ell+1)}{Zx_0} - \frac{1}{3}\right\} \\ &= \frac{6Z}{x^3}\left\{\frac{100}{101}\cdot\frac{1}{2} - \frac{1}{3}\right\} > \frac{cZ}{x^3}\ . \end{aligned}$$

This proves (Z3) with $c_1 = \frac{1}{100}$.

(Z4) says that

$$\begin{aligned} V_c'(x) &< -\frac{cZ}{x^2} \quad \text{for } x_{\rm left} < x < \frac{99}{100}x_0\ \text{, and} \\ V_c'(x) &> +\frac{cZ}{x^2} \quad \text{for } \frac{101}{100}x_0 < x < x_{\rm rt}\ . \end{aligned}$$

For $x < \frac{99}{100}x_0$ we have

$$V_c'(x) = \frac{Z}{x^2}\{\frac{-2\ell(\ell+1)}{Zx} + 1\} < \frac{Z}{x^2}\{-\frac{100}{99}\cdot\frac{2\ell(\ell+1)}{Zx_0} + 1\} = -\frac{1}{99}\frac{Z}{x^2} .$$

For $x > \frac{101}{100}x_0$ we have

$$V_c'(x) = \frac{Z}{x^2}\{\frac{-2\ell(\ell+1)}{Zx} + 1\} > \frac{Z}{x^2}\{-\frac{100}{101}\cdot\frac{2\ell(\ell+1)}{Zx_0} + 1\} = +\frac{1}{101}\frac{Z}{x^2} ,$$

which proves (Z4).

Next we compute $\lambda_c(x)$ and $\Lambda_c = \left(\int_{x_{\rm left}}^{x_{\rm rt}} \frac{dx}{\lambda_c(x)B_c(x)}\right)^{-1}$. We have $\lambda_c(x) = \left(\frac{Z}{x}\right)^{1/2}$. $x = Z^{1/2}x^{1/2}$, so

$$\Lambda_c^{-1} = \int_{x_{\rm left}}^{x_{\rm rt}} \frac{dx}{Z^{1/2}x^{3/2}} = \text{(const)}\ Z^{-\frac{1}{2}} \cdot (x_{\rm left}^{-1/2} - x_{\rm rt}^{-1/2}) .$$

Now $x_{\rm left} = \inf\{x \mid V_c(x) < 0\} \le x_0$ since $V_c(x_0) < 0$ by (Z3). Thus $x_{\rm left} < \frac{2\ell(\ell+1)}{Z}$, while $x_{\rm rt} > cZ^{-1/3}$. Since $\ell \le \frac{1}{16}(Zx_*)^{1/2} < CZ^{\frac{1}{3}-\frac{1}{200}}$, we have $x_{\rm left} << x_{\rm rt}$ and therefore $\Lambda_c^{-1} \sim Z^{-1/2}x_{\rm left}^{-1/2}$. We know that $\frac{10^{-6}\ell^2}{Z} < x_{\rm left} < \frac{2\ell(\ell+1)}{Z}$, so $x_{\rm left} \sim \frac{\ell^2}{Z}$, and thus

$$\Lambda_c \sim Z^{1/2}x_{\rm left}^{1/2} \sim \ell . \tag{23}$$

We have already checked (Z5).

(Z6) says that $V_c(x) \ge \frac{1000}{|x-x_{\rm left}|^2}$ for $0 < x < x_{\rm left} - \Lambda_c^{(10^9)}x_{\rm left}$, and that $V_c(x) \ge \frac{1000}{|x-x_{\rm rt}|^2}$ for $x_{\rm rt} + \Lambda_c^{(10^9)}x_{\rm rt} < x < \infty$.

The first assertion holds vacuously since $x_{\rm left} - \Lambda_c^{(10^9)}x_{\rm left} < 0$. The second assertion follows if we can prove that

$$V_c(x) > \frac{5000}{x^2} \quad \text{for } \Lambda_c^{(10^9)}x_{\rm rt} < x < \infty . \tag{23 bis}$$

In fact, $V_c(x) = \frac{\ell(\ell+1)}{x^2} + E_0 - \frac{Z}{x} \ge \frac{\ell(\ell+1)}{x^2}$ for $x > \frac{Z}{E_0}$, hence for $x > CZ^{-1/3}$, hence for $x > C'x_{\rm rt}$, by virtue of (22 bis). This implies (23 bis) since we are taking here $\ell \ge Z^{(10^{-9})}$. So we have proven (Z6).

(Z7) says that $(\max I_c)/(\min I_c) < \Lambda_c^{10^9}$, i.e. $10^{18}\frac{Z^2}{\ell^2 E_0} < C\ \ell^{(10^9)}$, by (23) and the definition of I_c. In fact, $\frac{10^{18}Z^2}{\ell^2 E_0} < \frac{Z^2}{E_0} \sim Z^{2/3}$ while $\ell^{(10^9)} > Z$. This proves (Z7).

(Z8) says that the constant $\hat{c}$, which hasn't yet been specified, is small enough. Just pick $\hat{c}$ small, and (Z8) is satisfied.

(Z9) says that Λ_c is large, i.e. that ℓ is large. Here we take $\ell > Z^{(10^{-9})}$ with Z very large. Hence ℓ is large. So (Z9) is proven.

Thus (Z0)...(Z9) are all satisfied. ■

Now we know that both $V(x)$ and $V_c(x)$ satisfy (Z0)...(Z9). To apply the Lemma on Truncated Eigenvalue Sums, we still must check the assumptions made there on E_*, namely $\frac{1}{\pi}\phi(E_*)$, $\frac{1}{\pi}\phi_c(E_*)$ are integers, $\phi(E_*) < \phi(0) - 1$, $\phi_c(E_*) < \phi_c(0) - 1$. Here $\phi(E) = \int_0^\infty \big(E - V(x)\big)_+^{1/2} dx$, $\phi_c(E) = \int_0^\infty \big(E - V_c(x)\big)_+^{1/2} dx$. Equation (11) shows that $\frac{1}{\pi}\phi_c(E_*)$ is an integer. Moreover, for $E \le \frac{1}{2}E_*$ we have $\phi_c(E) = \phi(E)$. In fact, Lemma 2 shows that $V(x)$, $V_c(x) > \frac{1}{2}E_*$ for $x \ge x_*$, hence the region of integration in the definitions of $\phi(E)$, $\phi_c(E)$ may be changed to $(0, x_*)$. In that region, $V(x) = V_c(x)$, so $\phi(E) = \phi_c(E)$ for $E \le \frac{1}{2}E_*$. In particular, $\frac{1}{\pi}\phi(E)$ is an integer. To check that $\phi(E_*) < \phi(0) - 1$ and $\phi_c(E_*) < \phi_c(0) - 1$, we prove the stronger statements $\phi(E_*) < \phi(\frac{1}{2}E_*) - 1$ and $\phi_c(E_*) < \phi_c(\frac{1}{2}E_*) - 1$. These two statements are the same, so we may just look at ϕ_c, which is given by $\frac{1}{\pi}\phi_c(E) = \frac{Z}{2(E_0 - E)^{1/2}} - \sqrt{\ell(\ell+1)}$, by Lemma 1. Thus we must show that

$$(24) \qquad \frac{\pi Z}{2}(E_0 - E_*)^{-1/2} < \frac{\pi Z}{2}(E_0 - \frac{1}{2}E_*)^{-1/2} - 1$$

in order to verify the hypotheses of the Lemma on Truncated Eigenvalue Sums. From (14 bis) we have $-E_* > cZ^{\frac{4}{3}+\frac{1}{100}}$, hence $-E_* >> E_0$. Therefore, $(E_0 - E_*) > \frac{3}{2}(E_0 - \frac{1}{2}E_*)$, so $(E_0 - \frac{1}{2}E_*)^{-1/2} - (E_0 - E_*)^{-1/2} > (E_0 - \frac{1}{2}E_*)^{-1/2} \cdot \big(1 - (\frac{3}{2})^{-1/2}\big) > c|E_*|^{-1/2}$. Hence, (24) holds provided $|E_*| < Z^{2-\frac{1}{100}}$. Since $m_* \sim (Zx_*)^{1/2}$, we have

$$(25) \qquad -E_* = -E_0 + \frac{Z^2}{4[m_* + \sqrt{\ell(\ell+1)}]^2} < \frac{Z^2}{4[m_* + \sqrt{\ell(\ell+1)}]^2} < \frac{Z^2}{4m_*^2} \sim \frac{Z}{x_*}\ .$$

In particular, $|E_*| = -E_* < Z^{2-\frac{1}{100}}$ since $x_* > Z^{-1+\frac{1}{100}}$.

This completes the proof of (24) and allows us to use the Lemma on Truncated Eigenvalue Sums. We get:

$$
\begin{aligned}
X = &-\frac{2}{3\pi}\int_0^\infty \big(-V(x)\big)_+^{3/2}dx + \frac{1}{24\pi}\int_0^\infty V''(x)\cdot\big(-V(x)\big)_+^{-1/2}dx \\
&+\frac{\pi}{2}\big(\phi'(0)\big)^{-1}\tilde{\chi}\big(\frac{1}{\pi}\phi(0)-\frac{1}{2}\big) + \big\{\frac{2}{3\pi}\int_0^\infty \big(E_* - V(x)\big)_+^{3/2}dx \\
&-\frac{1}{24\pi}\int_0^\infty V''(x)\big(E_* - V(x)\big)_+^{-1/2}\,dx + \frac{\pi}{24}\big(\phi'(E_*)\big)^{-1} - \frac{E_*}{\pi}\phi(E_*)\big\} + \text{ Error}
\end{aligned}
\tag{26}
$$

with

$$
|\text{Error}| \le C_\# \Lambda^{4\varepsilon-2} \inf_{\substack{x\in I\\ V(x)<E_*}} S(x)\ . \tag{27}
$$

$$
\begin{aligned}
X_c = &-\frac{2}{3\pi}\int_0^\infty \big(-V_c(x)\big)_+^{3/2}dx + \frac{1}{24\pi}\int_0^\infty V_c''(x)\cdot\big(-V_c(x)\big)_+^{-1/2}dx \\
&+\frac{\pi}{2}\big(\phi_c'(0)\big)^{-1}\tilde{\chi}\big(\frac{1}{\pi}\phi_c(0)-\frac{1}{2}\big) \\
&+\big\{\frac{2}{3\pi}\int_0^\infty \big(E_* - V_c(x)\big)_+^{3/2}dx - \frac{1}{24\pi}\int_0^\infty V_c''(x)\cdot\big(E_* - V_c(x)\big)_+^{-1/2}dx \\
&+\frac{\pi}{24}\big(\phi_c'(E_*)\big)^{-1} - \frac{E_*}{\pi}\phi_c(E_*)\big\} + \text{ Error}_c
\end{aligned}
\tag{28}
$$

with

$$
|\text{Error}_c| \le C_\# \Lambda_c^{4\varepsilon-2} \inf_{\substack{x\in I_c\\ V_c(x)<E_*}} S_c(x)\ . \tag{29}
$$

Here, $C_\#$ is as in the Lemma on Truncated Eigenvalue Sums.

We know that $\phi(E) = \phi_c(E)$ for $E < \frac{1}{2}E_*$, so $\phi(E_*) = \phi_c(E_*)$, $\phi'(E_*) = \phi_c'(E_*)$. Also, $V(x)$, $V_c(x) > E_*$ for $x \ge x_*$, and $V(x) = V_c(x)$ for $x < x_*$. Hence $\big(E_* - V(x)\big)_+^{3/2} = \big(E_* - V_c(x)\big)_+^{3/2}$ and $V''(x)\cdot\big(E_* - V(x)\big)_+^{-1/2} = V_c''(x)\cdot\big(E_* - V(x)\big)_+^{-1/2}$.

This shows that the expressions in curly brackets in (26) and (28) are equal. So (26)...(29) imply:

$$
\begin{aligned}
X - X_c = &-\frac{2}{3\pi}\int_0^\infty \big(-V(x)\big)_+^{3/2} dx + \frac{1}{24\pi}\int_0^\infty V''(x)\cdot\big(-V(x)\big)_+^{-1/2} dx \\
&+ \frac{\pi}{2}\big(\phi'(0)\big)^{-1}\tilde{\chi}\big(\frac{1}{\pi}\phi(0) - \frac{1}{2}\big) \\
&+ \frac{2}{3\pi}\int_0^\infty \big(-V_c(x)\big)_+^{3/2} dx - \frac{1}{24\pi}\int_0^\infty V_c''(x)\cdot\big(-V_c(x)\big)_+^{-1/2} dx \\
&- \frac{\pi}{2}\big(\phi_c'(0)\big)^{-1}\tilde{\chi}\big(\frac{1}{\pi}\phi_c(0) - \frac{1}{2}\big) + \text{ Error}_1 \ , \qquad (30)
\end{aligned}
$$

with

$$
|\text{Error}_1| \le C_\# \Lambda^{4\varepsilon-2} \inf_{\substack{x\in I \\ V(x)<E_*}} S(x) + C_\# \Lambda_c^{4\varepsilon-2} \inf_{\substack{x\in I_c \\ V_c(x)<E_*}} S_c(x) \ . \qquad (31)
$$

Again recall that $V(x), V_c(x) > \frac{1}{2}E_*$ for $x \ge x_*$. Hence the inf's in (31) occur over subsets of $(0, x_*)$, where $V(x) = V_c(x)$ and $S(x) = S_c(x)$. Also, from (Z0)...(Z9) applied to V and V_c, we know that $V(x) < E_*$ implies $x \in I$, and $V_c(x) < E_*$ implies $x \in I_c$. Therefore,

$$
\inf_{\substack{x\in I \\ V(x)<E_*}} S(x) = \inf_{\substack{x\in I_c \\ V_c(x)<E_*}} S_C(x) \ . \qquad (32)
$$

Let us compute the right side of (32). In fact, $\{V_c(x) < E_*\}$ is a non-empty interval (x_-, x_+) whose endpoints are obtained by solving $\frac{\ell(\ell+1)}{x^2} + (E_0 - E_*) - \frac{Z}{x} = 0$. Hence

$$
\begin{aligned}
\inf_{\substack{x\in I_c \\ V_c(x)<E_*}} S_c(x) &= \inf_{(x_-,x_+)} \frac{Z}{x} = \frac{Z}{x_+} = Z\cdot\Big[\frac{Z - \sqrt{Z^2 - 4\ell(\ell+1)(E_0-E_*)}}{2\ell(\ell+1)}\Big] \\
= \frac{Z^2}{2\ell(\ell+1)}\Big[1 - \big\{1 - \frac{4\ell(\ell+1)(E_0-E_*)}{Z^2}\big\}^{1/2}\Big] &\le \frac{Z^2}{2\ell(\ell+1)}\cdot\frac{C\ell(\ell+1)(E_0-E_*)}{Z^2} \\
&= C'\big(E_0 - E_*\big) \le C''\cdot(-E_*) \le C'''\frac{Z}{x_*} \ ,
\end{aligned}
$$

by (25). Putting this into (31), (32), we obtain

$$
|\text{Error}_1| \le C_\#\big(\Lambda^{4\varepsilon-2} + \ell^{4\varepsilon-2}\big)\cdot\frac{Z}{x_*} \ , \qquad (33)
$$

since $\Lambda_c \sim \ell$. We can simplify (33) a little, because

$$\Lambda^{-1} \geq c\,\ell^{-1} \,. \tag{34}$$

To see this, write

$$\begin{aligned} \Lambda^{-1} &= \int_{\{V(x)<0\}} \frac{dx}{\lambda(x)B(x)} \geq \int_{\{V(x)<0\}\cap(0,x_*)} \frac{dx}{\lambda(x)B(x)} \\ &= \int_{\{V_c(x)<0\}\cap(0,x_*)} \frac{dx}{\lambda_c(x)B_c(x)} \geq \int_{\{|x-x_0|<cx_0\}\cap(0,x_*)} \frac{dx}{Z^{1/2}x^{3/2}} \,. \end{aligned} \tag{35}$$

Here, $x_0 = \frac{2\ell(\ell+1)}{Z} \leq \frac{3\ell^2}{Z} \leq \frac{3\cdot\left[\frac{1}{16}(Zx_*)^{1/2}\right]^2}{Z} = \frac{3}{256}x_*$, so $\{|x-x_0|<cx_0\}\cap(0,x_*) = \{|x-x_0|<cx_0\}$, and (35) becomes $\Lambda^{-1} \geq c(Zx_0)^{-1/2} = c(2\ell(\ell+1))^{-1/2} \geq c'\ell^{-1}$, proving (34). Hence, (33) simplifies to

$$|\text{Error}_1| \leq C_{\#}\Lambda^{4\varepsilon-2}\frac{Z}{x_*} \,. \tag{36}$$

Our basic results are (5), (18), (30) and (36). We rewrite (30) using the fact that $\frac{1}{\pi}\phi_c(E) = \frac{Z}{2(E_0-E)^{1/2}} - \sqrt{\ell(\ell+1)}$. In particular, $\frac{\pi}{2}\left(\phi_c'(0)\right)^{-1} = \frac{1}{2}\left(\frac{1}{\pi}\phi_c'(0)\right)^{-1} = \frac{1}{2}\left(\frac{Z}{4(E_0)^{3/2}}\right)^{-1} = \frac{2E_0^{3/2}}{Z}$, so $\frac{\pi}{2}\left(\phi_c'(0)\right)^{-1}\tilde{\chi}\left(\frac{1}{\pi}\phi_c(0)-\frac{1}{2}\right) = \frac{2E_0^{3/2}}{Z}\tilde{\chi}\left(\frac{Z}{2E_0^{1/2}} - \sqrt{\ell(\ell+1)} - \frac{1}{2}\right)$. Also, note that $\Lambda^{4\varepsilon-2}\frac{Z}{x_*} \geq c\,\ell^{4\varepsilon-2}\frac{Z}{x_*} >> Z^{-100}$.

Thus, (5), (18), (30) and (36) imply the following result, which strengthens the First WKB Eigenvalue Sum Theorem when $\ell << (Zx_*)^{1/2}$.

Second WKB Eigenvalue Sum Theorem. *Suppose $V(x)$ is a smooth potential defined on $(0,\infty)$, $I \subset (0,\infty)$ is an interval containing $\{V(x)<0\}$, and $S(x)$, $B(x)$ are positive functions defined on I. Also, let ε, K, N, ℓ, E_0, Z, x_* be given.*

Assume the hypotheses (Z0)...(Z9) of the WKB Density Theorem, and assume that $V(x)$ has an exact Coulomb singularity with parameters (ℓ, E_0, Z, x_*). (That is, assume (CS1)...(CS6)). Finally, suppose $\ell > Z^{(10^{-9})}$.

Set $H = -\frac{d^2}{dx^2} + V(x)$ *on* $(0,\infty)$ *with Dirichlet boundary conditions. Then the sum of the negative eigenvalues of* H *is given by the equation*

$$
\begin{aligned}
&\operatorname{sneg}(H) - \operatorname{sneg}(H_c) = \\
&\quad -\frac{2}{3\pi}\int_0^\infty \big(-V(x)\big)_+^{3/2} dx + \frac{1}{24\pi}\int_0^\infty V''(x)\cdot\big(-V(x)\big)_+^{-1/2} dx \\
&\quad + \frac{\pi}{2}\big(\phi'(0)\big)^{-1}\tilde{\chi}\big(\frac{1}{\pi}\phi(0) - \frac{1}{2}\big) + \frac{2}{3\pi}\int_0^\infty \big(-V_c(x)\big)_+^{3/2} dx \\
&\quad - \frac{1}{24\pi}\int_0^\infty V_c''(x)\cdot\big(-V_c(x)\big)_+^{-1/2} dx - \frac{2E_0^{3/2}}{Z}\tilde{\chi}\big(\frac{Z}{2E_0^{1/2}} - \sqrt{\ell(\ell+1)} - \frac{1}{2}\big) \\
&\quad + \text{ Error },
\end{aligned}
$$

with $H_c = -\frac{d^2}{dx^2} + V_c(x)$ *on* $(0,\infty)$ *subject to Dirichlet boundary conditions,* $V_c(x) = \frac{\ell(\ell+1)}{x^2} + E_0 - \frac{Z}{x}$, *and* $|\text{Error}| \le \Lambda^{5\varepsilon-2}\frac{Z}{x_*}$. *Recall that* $\phi(E) = \int_0^\infty \big(E - V(x)\big)_+^{1/2} dx$, $\phi'(E) = \frac{1}{2}\int_0^\infty \big(E - V(x)\big)_+^{-1/2} dx$, $\tilde{\chi}(t) = \min_{k\in\mathbb{Z}} |t - k - \frac{1}{2}|^2 - \frac{1}{12}$.

In this section, we derive crude versions of the Second WKB Eigenvalue Sum Theorem that hold under weakened hypotheses. The idea is as follows. Suppose $H = -\frac{d^2}{dx^2} + V(x)$ on $(0,\infty)$, (Dirichlet b.c.) with $V(x)$ having an exact Coulomb singularity (i.e. smooth on $(0,\infty)$ and satisfying (CS1)...(CS6) for suitable (ℓ, E_0, Z, x_*)). We pick $E_* \in \left(-\frac{8Z}{x_*}, -\frac{8Z}{3x_*}\right)$ so that

$$\min_{n\in\mathbb{Z}} \left|E_* - \left(E_0 - \frac{Z^2}{4n^2}\right)\right| > Z^{-500} \ .$$

If $V(x) > \frac{1}{2}E_*$ for $x > x_*$, then the hypotheses of the Lemma on Low Eigenvalues in Coulomb Singularities apply. Hence, with $H_c = -\frac{d^2}{dx^2} + V_c(x)$, $V_c(x) = \frac{\ell(\ell+1)}{x^2} + E_0 - \frac{Z}{x}$, we have

$$\text{(1)} \qquad \left|\sum_k (E_* - E_k)_+ - \sum_k (E_* - E_k^c)_+\right| \le Z^{-100} \ ,$$

where $\{E_k\}$ are the negative eigenvalues of H and $\{E_k^c\}$ are those of H_c.

Now let

$$\mathcal{N}(E) = \text{ number of eigenvalues of } H \text{ which are } < E$$
$$\mathcal{N}_c(E) = \text{ number of eigenvalues of } H_c \text{ which are} < E \ ,$$

and suppose we can prove that

$$\text{(2)} \qquad \left|\mathcal{N}(E) - \frac{1}{\pi}\int_0^\infty \left(E - V(x)\right)_+^{1/2} dx\right| \le C \quad \text{for } E \in [E_*, 0] \ .$$

We have also

$$\text{(3)} \qquad \left|\mathcal{N}_c(E) - \frac{1}{\pi}\int_0^\infty \left(E - V_c(x)\right)_+^{1/2} dx\right| \le C \quad \text{for } E \in [E_*, 0] \ ,$$

by virtue of our explicit knowledge of the eigenvalues of H_c and of the integral $\frac{1}{\pi}\int_0^\infty \left(E - V_c(x)\right)_+^{1/2} dx = \frac{Z}{2(E_0-E)^{1/2}} - \sqrt{\ell(\ell+1)}$.

For each $E_k < 0$, we have

$$E_k + (E_* - E_k)_+ = -\int_{E_*}^{0} \chi_{E_k < E} dE \ .$$

Summing over k, we get

$$\text{sneg}(H) + \sum_k (E_* - E_k)_+ = -\int_{E_*}^{0} \mathcal{N}(E) dE \ . \tag{4}$$

Similarly,

$$\text{sneg}(H_c) + \sum_k (E_* - E_k^c)_+ = -\int_{E_*}^{0} \mathcal{N}_c(E) dE \ . \tag{5}$$

Subtracting (5) from (4) and using (1), (2), (3), we conclude that

$$\begin{aligned} \text{sneg}(H) - \text{sneg}(H_c) = &-\frac{1}{\pi}\int_{E_*}^{0}\int_0^{\infty} \big(E - V(x)\big)_+^{1/2} dx\, dE \\ &+\frac{1}{\pi}\int_{E_*}^{0}\int_0^{\infty} \big(E - V_c(x)\big)_+^{1/2} dx\, dE + \text{ Error} \ , \end{aligned} \tag{6}$$

$$|\text{Error}| < Z^{-100} + C|E_*| < C'\frac{Z}{x_*} \ . \tag{7}$$

Note that

$$\begin{aligned} \frac{1}{\pi}\int_{E_*}^{0}\int_0^{\infty} \big(E - V(x)\big)_+^{1/2} dx\, dE &= \frac{1}{\pi}\int_0^{\infty} \Big\{\int_{E_*}^{0} \big(E - V(x)\big)_+^{1/2} dE\Big\} dx \\ &= \frac{2}{3\pi}\int_0^{\infty} \Big\{\big(-V(x)\big)_+^{3/2} - \big(E_* - V(x)\big)_+^{3/2}\Big\} dx \ , \text{ and similarly for } V_c(x) \ . \end{aligned}$$

Since $V(x) = V_c(x)$ for $x \le x_*$, while $V(x)$, $V_c(x) > E_*$ for $x > x_*$, it follows that $\big(E_* - V(x)\big)_+^{3/2} = \big(E_* - V_c(x)\big)_+^{3/2}$ everywhere. Thus

$$\begin{aligned} &-\frac{1}{\pi}\int_{E_*}^{0}\int_0^{\infty} \big(E - V(x)\big)_+^{1/2} dx\, dE + \frac{1}{\pi}\int_{E_*}^{0}\int_0^{\infty} \big(E - V_c(x)\big)_+^{1/2} dx\, dE \\ &= -\frac{2}{3\pi}\int_0^{\infty} \Big\{\big(-V(x)\big)_+^{3/2} - \big(-V_c(x)\big)_+^{3/2}\Big\} dx \ . \end{aligned} \tag{8}$$

Note that the quantity in curly brackets vanishes near $x = 0$, although $\big(-V(x)\big)^{3/2}$ fails to be integrable when $V(x) \sim -\frac{Z}{x}$ near $x = 0$.

Putting (8) into (6), (7), we get

$$\text{(9)} \qquad \mathrm{sneg}(H) - \mathrm{sneg}(H_c) = -\frac{2}{3\pi}\int_0^\infty \{(-V(x))_+^{3/2} - (-V_c(x))_+^{3/2}\}dx + \text{ Error}$$

with

$$\text{(10)} \qquad |\text{Error}| < C\frac{Z}{x_*} \; .$$

So proving (9), (10) for potentials with exact Coulomb singularities is reduced to proving (2), which concerns the number of eigenvalues $< E$. To establish (2), we use our previous results on degenerate potentials.

Theorem 1. *Suppose $V(x)$, $S(x)$, $B(x)$ satisfy hypotheses $(Z\bar{0})\dots(Z\bar{8})$, and also suppose $V(x)$ has an exact Coulomb singularity (CS1)... (CS6) for a given (ℓ, E_0, Z, x_*). Assume $E_{\text{crit}} < -\frac{3Z}{x_*}$, with E_{crit} as in $(Z\bar{0})\dots(Z\bar{8})$. Assume also $V(x) > -\frac{4}{3}\frac{Z}{x_*}$ for $x > x_*$. Set $V_c(x) = \frac{\ell(\ell+1)}{x^2} + E_0 - \frac{Z}{x}$. Then for $H = -\frac{d^2}{dx^2} + V(x)$, $H_c = -\frac{d^2}{dx^2} + V_c(x)$ on $(0, \infty)$ with Dirichlet boundary conditions, we have*

$$\mathit{sneg}(H) - \mathit{sneg}(H_c) = -\frac{2}{3\pi}\int_0^\infty \{(-V(x))_+^{3/2} - (-V_c(x))_+^{3/2}\}dx + \text{ Error} \; ,$$

with $|\text{Error}| < C\frac{Z}{x_*}$. Here, C depends only on the constants in $(Z\bar{0})\dots(Z\bar{8})$ and in (CS1)... (CS6).

Proof.

Take $E_* \in \left(-\frac{3Z}{x_*}, -\frac{8}{3}\frac{Z}{x_*}\right)$, such that $\min_n |E_* - (E_0 - \frac{Z^2}{4n^2})| > Z^{-500}$. Then $E_{\text{crit}} \le E_* \le 0$, and $V(x) > \frac{1}{2}E_*$ for $x > x_*$. For $E \in [E_{\text{crit}}, 0]$, and hence for $E \in [E_*, 0]$, Theorem D2 from the section on WKB Theory with weak turning points implies (2). Hence (9), (10) follow, and Theorem 1 is proven. ■

For Hypotheses $(Z\bar{0})\dots(Z\bar{8})$, see "Review of Previous Results", part G.

Theorem 2. *Suppose $V(x)$, $S(x)$, $B(x)$ satisfy hypotheses $(Z\hat{0})\dots(Z\hat{9})$, and suppose also that $V(x)$ has an exact Coulomb singularity (CS1)... (CS6) for a given*

(ℓ, E_0, Z, x_*). *Assume* $V(x) > -\frac{2Z}{x_*}$ *for* $x > x_*$. *Assume* $V(2x_0) < -\frac{8Z}{x_*}$, *and assume*

$$(11) \qquad \int_0^\infty \{(-V(x))_+^{1/2} - (V(\tfrac{1}{2}x_1) - V(x))_+^{1/2}\} dx < \underline{C} \, ,$$

where x_0, x_1 *are as in* $(\hat{Z0}) \ldots (\hat{Z9})$. *Set* $V_c(x) = \frac{\ell(\ell+1)}{x^2} + E_0 - \frac{Z}{x}$. *Then for* $H = -\frac{d^2}{dx^2} + V(x)$, $H_c = -\frac{d^2}{dx^2} + V_c(x)$ *on* $(0, \infty)$ *with Dirichlet boundary conditions, we have*

$$sneg(H) - sneg(H_c) = -\frac{2}{3\pi} \int_0^\infty \{(-V(x))_+^{3/2} - (-V_c(x))_+^{3/2}\} dx + \text{ Error } ,$$

with $|\text{Error}| < C\frac{Z}{x_*}$. *Here* C *depends only on* $\underline{C}$ *in (11), and on the constants in* $(\hat{Z0}) \ldots (\hat{Z9})$ *and in (CS1)...(CS6).*

For hypotheses $(\hat{Z0}) \ldots (\hat{Z9})$ see "Review of Earlier Results," section E.

Proof.

Take $E_* \in (-\frac{8Z}{x_*}, -\frac{4Z}{x_*})$, such that $\min_n |E_* - (E_0 - \frac{Z^2}{4n^2})| > Z^{-500}$. Then $V(x) > \frac{1}{2}E_*$ for $x > x_*$, and $V(2x_0) < E_*$. It is enough to establish (2) for $E \in [E_*, 0]$. Hence, it's enough to establish (2) for $E \in [V(2x_0), 0]$. Since $\mathcal{N}(E)$ is monotone, (11) shows that it's enough to establish (2) for $E = 0$ and for $E \in [V(2x_0), V(\frac{1}{2}x_1)]$. These follow from Theorem D1 in the section on WKB Theory with Weak Turning Points, together with Lemmas E1 and E2 from the section "Review of Earlier Results," part E. Note that the semiclassical approximation to $\mathcal{N}(E)$ is given in Theorem D1 as $\frac{1}{\pi} \int_{I_{\text{center}}} (E - V(x))_+^{1/2} dx$. Since $\int_{(0,\infty)\setminus I_{\text{center}}} (E - V(x))_+^{1/2} dx < C$ here, the conclusion of Theorem D1 is equivalent to (2). ■

Theorem 3. *Suppose* $V(x)$, $S(x)$, $B(x)$ *satisfy hypotheses* $(Z0^\dagger) \ldots (Z7^\dagger)$. *Suppose also that* $V(x)$ *has an exact Coulomb singularity (CS1)...(CS6) for given*

(ℓ, E_0, Z, x_*). *We take the same* x_* *in (Z0*†*)... (Z7*†*) as in (CS1)...(CS6)]. We make the following additional assumptions:*

(12) $$V\left(\frac{x_*}{10}\right) < -\frac{3Z}{x_*}$$

(13) $$V(x) > -\frac{4}{3}\,\frac{Z}{x_*} \quad \text{for} \quad x > x_*$$

(14) $$V(x) > -C x_{\rm big}^2 x^{-4} \quad \text{for } x > x_{\rm big}\ .$$

(15) $$\int_0^\infty \left\{ \left(-V(x)\right)_+^{1/2} - \left(V(\frac{1}{4}x_1) - V(x)\right)_+^{1/2} \right\} dx < \underline{C}\ ,$$

with x_1 *as in (Z0*†*)...(Z7*†*).*

Set $V_c(x) = \frac{\ell(\ell+1)}{x^2} + E_0 - \frac{Z}{x}$. *Then for* $H = -\frac{d^2}{dx^2} + V(x)$, $H_c = -\frac{d^2}{dx^2} + V_c(x)$ *on* $(0,\infty)$ *with Dirichlet boundary conditions, we have*

$$\mathrm{sneg}(H) - \mathrm{sneg}(H_c) = -\frac{2}{3\pi}\int_0^\infty \left\{ \left(-V(x)\right)_+^{3/2} - \left(-V_c(x)\right)_+^{3/2} \right\} dx + \text{ Error }\ ,$$

with $|\text{Error}| < \frac{CZ}{x_*}$. *Here, C depends only on* $\underline{C}$ *in (15), and on the constants in (Z0*†*)...(Z7*†*), (CS1)...(CS6), and (14).*

For hypotheses (Z0†)...(Z7†), see "Review of Earlier Results," part F.

Proof.

Take $E_* \in \left(-\frac{3Z}{x_*}, -\frac{8}{3}\,\frac{Z}{x_*}\right)$, with $\min_n \left|E_* - \left(E_0 - \frac{Z^2}{4n^2}\right)\right| > Z^{-500}$. Then $V(x) > \frac{1}{2}E_*$ for $x > x_*$, by (13). As before, it is enough to prove (2), and by (15) we need only look at $E = 0$ and at $E \in [E_*, V(\frac{1}{4}x_1)]$. By (12), $[E_*, V(\frac{1}{2}x_1)] \subset [V(\frac{1}{10}x_*), V(\frac{1}{4}x_1)]$. For $E \in [V(\frac{1}{10}x_*), V(\frac{1}{4}x_1)]$, Lemma F2 in the section "Review of Earlier Results" applies. Therefore, Theorem D1 in the section on WKB Theory with Weak Turning Points gives

(16) $$\mathcal{N}(E) = O(1) + \frac{1}{\pi}\int_{x_0}^{\tilde{x} - \hat{C}_\# \lambda^{-2/3}(\tilde{x})\cdot\tilde{x}} \left(E - V(x)\right)_+^{1/2} dx\ ,$$

where $V(\tilde{x}) = E$, $\frac{1}{10}x_* \le \tilde{x} \le \frac{1}{4}x_1$. ¿From (Z4†) we have

$$\int_0^{x_0} \big(E - V(x)\big)_+^{1/2} dx \le \int_0^{x_0} \big(-V(x)\big)_+^{1/2} dx = O(1) \ . \tag{17}$$

Lemma F2 shows that $|E - V(x)| \le C(\lambda^{-2/3}(\tilde{x}) \cdot \tilde{x})^{-2}$ for $x \in [\tilde{x} - \hat{C}_\# \lambda^{-2/3}(\tilde{x}) \cdot \tilde{x}, \tilde{x}]$, because that's contained in condition (H$\hat{5}$) in the conclusion of Lemma F2. This and (16), (17) imply

$$\mathcal{N}(E) = O(1) + \frac{1}{\pi} \int_0^{\tilde{x}} \big(E - V(x)\big)_+^{1/2} dx \ . \tag{18}$$

Now (Z2†), (Z5†) show that $V(x)$ is increasing in $[\tilde{x}, x_{\text{big}}] \subset [x_0, x_1] \cup [x_1, x_{\text{big}}]$. Since $V(\tilde{x}) = E$, we conclude that $V(x) > E$ for $x \in [\tilde{x}, x_{\text{big}}]$. Hence, (18) is equivalent to

$$\mathcal{N}(E) = O(1) + \frac{1}{\pi} \int_0^{x_{\text{big}}} \big(E - V(x)\big)_+^{1/2} dx \ . \tag{19}$$

From (14) we get $\big(E - V(x)\big)_+ \le \big(-V(x)\big)_+ \le C x_{\text{big}}^2 x^{-4}$ for $x > x_{\text{big}}$, so $\int_{x_{\text{big}}}^\infty \big(E - V(x)\big)_+^{1/2} dx = O(1)$, and (19) implies $\mathcal{N}(E) = O(1) + \frac{1}{\pi} \int_0^\infty \big(E - V(x)\big)_+^{1/2} dx$, which is (2).

Thus we have proven (2) for $E \in [V(\frac{1}{10}x_*), V(\frac{1}{4}x_1)]$.

To complete the proof of Theorem 3, we must check (2) for $E = 0$. Lemma F1 and Theorem D1 from the section on WKB Theory with Weak Turning Points together yield

$$\mathcal{N}(0) = O(1) + \frac{1}{\pi} \int_{x_0}^{x_1} \big(-V(x)\big)_+^{1/2} dx \ . \tag{20}$$

From (Z4†) we get $\int_0^{x_0} \big(-V(x)\big)_+^{1/2} dx = O(1)$.
From (Z5†) we get $\int_{x_1}^{x_{\text{big}}} \big(-V(x)\big)_+^{1/2} dx = O(1)$.
From (14) we get $\int_{x_{\text{big}}}^\infty \big(-V(x)\big)_+^{1/2} dx = O(1)$. Therefore, (20) implies $\mathcal{N}(0) = O(1) + \frac{1}{\pi} \int_0^\infty \big(-V(x)\big)_+^{1/2} dx$, which is (2). The proof of Theorem 3 is complete. ∎

The Third WKB Eigenvalue Sum Theorem

Suppose we are in the setting of the WKB Theorem on Low Eigenvalues, with $E_\infty = 0$. We assume the hypotheses (H0*)...(H6*) of that Theorem, and we make the further assumption that

$$-\lambda^{-3\varepsilon} S < V(x_0) < 0 \ . \tag{1}$$

Our goal is to control the sum of the negative eigenvalues of $H = -\frac{d^2}{dx^2} + V(x)$ on I_{BVP}, with Dirichlet or Neumann boundary conditions.

In view of (1), the minimum of the potential is relatively small, so the hypotheses of the First WKB Eigenvalue Sum Theorem break down. However, the ideas used in its proof apply here, with fewer technical difficulties.

We begin by quoting Lemma B1 from the section on the WKB Theorem on Low Eigenvalues. For $E \in (E_{\min}, E_{\max}] \equiv \big(V(x_0), V(x_0)+\lambda^{-2\varepsilon}S\big]$, we have phases $\phi(E)$, $\psi(E)$ with

$$\Big|\big(\frac{d}{dE}\big)^{\beta}\phi(E)\Big| \le C_{\#}^{\beta}\lambda S^{-\beta} \tag{2}$$

$$\Big|\big(\frac{d}{dE}\big)^{\beta}\psi(E)\Big| \le C_{\#}^{\beta}\lambda^{-1} S^{-\beta} \tag{3}$$

$$\frac{d}{dE}\phi(E) \ge c_{\#}\lambda S^{-1} \ . \tag{4}$$

Throughout this section, $c_{\#}$, $C_{\#}$ etc. denote constants depending only on c, c', c'', C_α, ε, K, N in (H0*)...(H6*). Setting $\Phi(E) = \frac{1}{\pi}\phi(E) + \frac{1}{48\pi}\psi(E) - \frac{1}{2}$, we get

$$\Big|\big(\frac{d}{dE}\big)^{\beta}\Phi(E)\Big| \le C_{\#}^{\beta}\lambda S^{-\beta} \tag{5}$$

$$\big(\frac{d}{dE}\big)\Phi(E) \ge c_{\#}\lambda S^{-1} \tag{6}$$

for $E \in (E_{\min}, E_{\max}]$.

Let $(t_{\min}, t_{\max}] = \Phi\big((E_{\min}, E_{\max}]\big)$. Then for $t \in (t_{\min}, t_{\max}]$, the equation $\Phi(E) = t$ has a unique solution $E = E(t) \in (E_{\min}, E_{\max}]$, with

$$\text{(7)} \qquad \left|\left(\frac{d}{dt}\right)^m E(t)\right| \le C_{\#}^m S\lambda^{-m}$$

$$\text{(8)} \qquad \frac{d}{dt}E(t) \ge c_{\#}S\lambda^{-1} \ .$$

We pick $E_* \in \big(V(x_0), 0\big)$ with $\phi(E_*) < \lambda^{-2}$, and with E_* less than all the eigenvalues of H.

Let $a_0 = \Phi(E_*)$, and let $b_0 = \Phi(0)$. Since $E_{\max} = V(x_0) + \lambda^{-2\varepsilon}S > \frac{1}{2}\lambda^{-2\varepsilon}S$ by (1), it follows from (6) that

$$t_{\max} = \Phi(E_{\max}) \ge c_{\#}\lambda S^{-1} \cdot \frac{1}{2}\lambda^{-2\varepsilon}S + \Phi(0) = c'_{\#}\lambda^{1-2\varepsilon} + b_0 \ .$$

Thus,

$$\text{(9)} \qquad t_{\min} < a_0 < b_0 < t_{\max} - c_{\#}\lambda^{1-2\varepsilon} \ , \quad \text{and}$$

$$\text{(10)} \qquad |a_0 + \frac{1}{2}| \le C_{\#}\lambda^{-1} \ .$$

Next, we control the non-positive eigenvalues of H, by applying the WKB Theorem on Low Eigenvalues. The non-positive eigenvalues are $E_0, E_1, \dots, E_{k_{\max}}$, with

$$\text{(11)} \qquad |\Phi(E_k) - k| < C_{\#}\lambda^{4\varepsilon-2} \ ,$$

$$\text{(12)} \qquad \{0, \dots, k_{\max}\} = \mathbb{Z} \cap [a, b] \ ,$$

$$\text{(13)} \qquad a = a_0$$

$$\text{(14)} \qquad b = b_0 + w_{hi}$$

$$|w_{hi}| < C_\# \lambda^{4\varepsilon - 2} \tag{15}$$

$$w_{hi} = 0 \quad \text{if} \quad \min_{k \in \mathbb{Z}} |b_0 - k| > C_\# \lambda^{4\varepsilon-2} \; . \tag{16}$$

We leave to the reader the task of deducing (12)...(16) from the WKB Theorem on Low Eigenvalues. Note that $a \le b$. If $|a_0 - b_0| > C_\# \lambda^{4\varepsilon-2}$, then this follows from (9), (13), (14), (15). If $|a_0 - b_0| \le C_\# \lambda^{4\varepsilon-2}$, then (10) gives $|b_0 + \frac{1}{2}| \le C'_\# \lambda^{-1} << 1$, so $\min_{k\in\mathbb{Z}} |b_0 - k| > \frac{1}{2} - C_\# \lambda^{-1} > \frac{1}{4}$, and (16) gives $w_{hi} = 0$. Then (9), (13), (14) show that $a \le b$. Hence $a \le b$ in all cases.

We use (11)...(15) and the Lemma on Riemann Sums to control

$$\mathrm{sneg}(H) = \sum_{k=0}^{k_{\max}} E_k \; .$$

Since $t_{\min} < a_0 = a \le b \le b_0 + 1 \le t_{\max}$ by (9), we have $[a, b] \subset (t_{\min}, t_{\max}]$, so that each k from 0 to $k_{\max}$ belongs to the domain of $E(t)$ by (12). Hence for $0 \le k \le k_{\max}$, we have

$$\begin{aligned} |E_k - E(k)| &\le \max_{t \in (t_{\min}, t_{\max}]} \Big|\frac{d}{dt} E(t)\Big| \cdot \Big|\Phi(E_k) - \Phi(E(k))\Big| \\ &= \max_{t \in (t_{\min}, t_{\max}]} \Big|\frac{d}{dt} E(t)\Big| \cdot |\Phi(E_k) - k| \le C_\# S \lambda^{-1} \cdot C_\# \lambda^{4\varepsilon - 2} \end{aligned}$$

by (7), (11). Thus,

$$|E_k - E(k)| \le C_\# S \lambda^{4\varepsilon-3} \quad \text{for } 0 \le k \le k_{\max} \; . \tag{17}$$

We estimate the number of E_k. Evidently,

$$\begin{aligned} k_{\max} &\le C_\# + (b - a) \le C'_\# + (b_0 - a_0) \\ &\le C'_\# + C_\# \max_{E \in (E_{\min}, E_{\max}]} \Big|\frac{d\Phi}{dE}(E)\Big| \cdot |E_{\max} - E_{\min}| \\ &\le C'_\# + C_\# \lambda S^{-1} \cdot \lambda^{-2\varepsilon} S \le C''_\# \lambda^{1-2\varepsilon} \; . \end{aligned}$$

Thus,

$$0 \le b - a \le C_\# \lambda^{1-2\varepsilon} \ , \quad \text{and} \tag{18}$$

$$k_{\max} \le C_\# \lambda^{1-2\varepsilon} \ . \tag{19}$$

From (12), (17), (19), we see that $\left| \sum_{k=0}^{k_{\max}} E_k - \sum_{k \in \mathbb{Z} \cap [a,b]} E(k) \right| \le C_\# S \lambda^{2\varepsilon - 2}$, i.e.

$$|\mathrm{sneg}(H) - \sum_{k \in [a,b] \cap \mathbb{Z}} E(k)| \le C_\# S \lambda^{2\varepsilon-2} \ . \tag{20}$$

Since $a \le b$, estimates (7) and the Lemma on Riemann sums yield

$$\sum_{k \in [a,b] \cap \mathbb{Z}} E(k) = \int_a^b E(t) dt - E(b)\chi_-(b) - E(a)\chi_+(a) \tag{21}$$
$$+ \frac{1}{2} \frac{dE(t)}{dt} \Big|_{t=b} \tilde{\chi}(b) - \frac{1}{2} \frac{dE(t)}{dt} \Big|_{t=a} \tilde{\chi}(a) + \ \mathrm{Error}_0 \ .$$

with

$$|\mathrm{Error}_0| \le C_\# S \lambda^{-2} + C_\# \int_a^b S \lambda^{-N} dt \le C'_\# S \lambda^{-2} \ , \tag{22}$$

by (18). Combining (20)...(22), we get

$$\mathrm{sneg}(H) = \int_a^b E(t) dt - E(b)\chi_-(b) - E(a)\chi_+(a) \tag{23}$$
$$+ \frac{1}{2} E'(b)\tilde{\chi}(b) - \frac{1}{2} E'(a)\tilde{\chi}(a) + \ \mathrm{Error}_1$$

with

$$|\mathrm{Error}_1| \le C_\# S \lambda^{2\varepsilon-2} \ . \tag{24}$$

We study in turn each of the terms on the right in (23). Regarding the integral, we have

$$\left| \int_a^b E(t) dt - \int_{a_0}^{b_0} E(t) dt \right| \le \max_{t \in (t_{\min}, t_{\max}]} |E(t)| \cdot |b - b_0|$$
$$\le C_\# \lambda^{-2\varepsilon} S \cdot C_\# \lambda^{4\varepsilon - 2} = C_\# \lambda^{2\varepsilon-2} S \ , \quad \text{i.e.}$$

$$(25) \qquad \int_a^b E(t)dt = \int_{a_0}^{b_0} E(t)dt + \text{Error}_2 \ , \quad \text{with} \quad |\text{Error}_2| \le C_\# S\lambda^{2\varepsilon-2} \ .$$

We postpone discussion of the integral from a_0 to b_0.

Next, $|E(b)| = |E(b) - E(b_0)| \le \max_{t\in(t_{\min},t_{\max}]} \left|\frac{dE(t)}{dt}\right| \cdot |b - b_0| \le C_\# S\lambda^{-1} \cdot C_\# \lambda^{4\varepsilon-2} \le C_\# S\lambda^{4\varepsilon-3}$. Throwing away information, we conclude that

$$(26) \qquad |E(b)\chi_-(b)| \le C_\# S\lambda^{2\varepsilon-2} \ .$$

Next we study $-E(a)\chi_+(a)$. Recall that $E(a) = E_*$ and $a = \Phi(E_*) = \frac{1}{\pi}\phi(E_*) + \frac{1}{48\pi}\psi(E_*) - \frac{1}{2} = \xi + \frac{1}{48\pi}\psi(E_*) - \frac{1}{2}$ with $|\xi| \le \lambda^{-2}$, $|\psi(E_*)| \le C_\# \lambda^{-1}$. Hence, $k \equiv$ (smallest integer $\ge a) = 0$, so $\chi_+(a) = k - a - 1/2 = -\xi - \frac{1}{48\pi}\psi(E_*)$ with $|\xi| \le \lambda^{-2}$. We conclude that

$$(27) \qquad -E(a)\chi_+(a) = +\frac{E_*\psi(E_*)}{48\pi} + \text{ Error}_3 \ ,$$

with

$$(28) \qquad |\text{Error}_3| \le \lambda^{-2}|E_*| \le C_\# S\lambda^{-2} << S\lambda^{2\varepsilon-2} \ .$$

Next we study $E'(b)\tilde{\chi}(b)$. Recall that $\tilde{\chi}(\cdot)$ is Lipschitz continuous, and that $b = \frac{1}{\pi}\phi(0) + \frac{1}{48\pi}\psi(0) - \frac{1}{2} + w_{hi} = \frac{1}{\pi}\phi(0) - \frac{1}{2} + \xi$ with $|\xi| \le C_\# \lambda^{-1}$. Hence

$$\left|\tilde{\chi}(b) - \tilde{\chi}\Big(\frac{1}{\pi}\phi(0) - \frac{1}{2}\Big)\right| \le C_\# \lambda^{-1} \ .$$

Also,

$$|E'(b)| \le C_\# S\lambda^{-1} \quad \text{and}$$

$$\begin{aligned} |E'(b) - E'(b_0)| \le \max_{t\in(t_{\min},t_{\max}]} |E''(t)| \cdot |b - b_0| &\le C_\# S\lambda^{-2} \cdot C_\# \lambda^{4\varepsilon-2} \\ &\le C_\# S\lambda^{4\varepsilon-4} \ . \end{aligned}$$

These estimates show that

$$\begin{aligned} (29) \quad & \left|\frac{1}{2}E'(b)\tilde{\chi}(b) - \frac{1}{2}E'(b_0)\tilde{\chi}(\frac{1}{\pi}\phi(0) - \frac{1}{2})\right| \\ & \le \frac{1}{2}|E'(b)| \cdot \left|\tilde{\chi}(b) - \tilde{\chi}(\frac{1}{\pi}\phi(0) - \frac{1}{2})\right| + \frac{1}{2}|E'(b) - E'(b_0)| \cdot \left|\tilde{\chi}(\frac{1}{\pi}\phi(0) - \frac{1}{2})\right| \\ & \le C_\# S\lambda^{-1} \cdot C_\# \lambda^{-1} + C_\# S\lambda^{4\varepsilon-4} \le C_\# S\lambda^{-2} \ . \end{aligned}$$

Also, by definition of $\Phi(E)$ and $E(t)$, we have $E'(b_0) = \big(\Phi'(E)\ \big|_{E=E(b_0)}\big)^{-1} = \big(\frac{1}{\pi}\phi'(0) + \frac{1}{48\pi}\psi'(0)\big)^{-1}$ since $E(b_0) = 0$. Thus,

$$\frac{1}{2}E'(b_0) = \frac{\pi}{2}\big(\phi'(0)\big)^{-1}\Big(1 + \frac{1}{48}\ \frac{\psi'(0)}{\phi'(0)}\Big)^{-1} .$$

From (3), (4), we get $\big|\frac{\psi'(0)}{\phi'(0)}\big| \le C_{\#}\lambda^{-2}$, so

$$\Big|\frac{1}{2}E'(b_0) - \frac{\pi}{2}\big(\phi'(0)\big)^{-1}\Big| \le C_{\#}\lambda^{-2}|E'(b_0)| \le C_{\#}\lambda^{-2}\cdot S\lambda^{-1} = C_{\#}S\lambda^{-3} .$$

Combining this with (29), we find that

$$\begin{aligned}(30)\qquad +\frac{1}{2}E'(b)\tilde{\chi}(b) = +\frac{\pi}{2}\big(\phi'(0)\big)^{-1}\tilde{\chi}\big(\frac{1}{\pi}\phi(0) - \frac{1}{2}\big) + \ \mathrm{Error}_4\ ,\\ \text{with } |\mathrm{Error}_4| \le C_{\#}S\lambda^{-2}\ .\end{aligned}$$

Next, we examine the term $-\frac{1}{2}E'(a)\tilde{\chi}(a)$. Since $E(a) = E_*$, we have as above

$$\begin{aligned}E'(a) = \big(\Phi'(E_*)\big)^{-1} &= \big(\frac{1}{\pi}\phi'(E_*) + \frac{1}{48\pi}\psi'(E_*)\big)^{-1}\\ &= \pi\big(\phi'(E_*)\big)^{-1}\big(1 + \frac{\phi'(E_*)}{48\psi'(E_*)}\big)^{-1}\end{aligned}$$

with $\big|\frac{\phi'(E_*)}{\psi'(E_*)}\big| \le C_{\#}\lambda^{-2}$ by (3), (4). Therefore, $\big|E'(a) - \pi\big(\phi'(E_*)\big)^{-1}\big| \le C_{\#}\lambda^{-2}|E'(a)| \le C_{\#}S\lambda^{-3}$, since $|E'(a)| \le C_{\#}S\lambda^{-1}$.

We turn our attention to

$$\begin{aligned}\tilde{\chi}(a) = \tilde{\chi}\big(\frac{1}{\pi}\phi(E_*) + \frac{1}{48\pi}\psi(E_*) - \frac{1}{2}\big) = \tilde{\chi}(\xi - \frac{1}{2})\\ \text{with } |\xi| \le C_{\#}\lambda^{-1}\ .\end{aligned}$$

The definition of $\tilde{\chi}$ gives therefore $\tilde{\chi}(a) = \min_{k\in\mathbb{Z}}|(\xi - 1/2) - k - 1/2|^2 - \frac{1}{12} = \xi^2 - \frac{1}{12}$, i.e. $|\tilde{\chi}(a) + \frac{1}{12}| \le C_{\#}\lambda^{-2}$. Thus,

$$\begin{aligned}&\Big| -\frac{1}{2}E'(a)\tilde{\chi}(a) - \big(-\frac{1}{2}\big)\big(\pi(\phi'(E_*))^{-1}\big)\cdot\big(-\frac{1}{12}\big)\Big|\\ &\qquad \le \frac{1}{2}|E'(a)|\cdot\big|\tilde{\chi}(a) + \frac{1}{12}\big| + \frac{1}{2}\cdot\big(\frac{1}{12}\big)\cdot\big|E'(a) - \pi\big(\phi'(E_*)\big)^{-1}\big|\\ &\qquad\qquad \le C_{\#}S\lambda^{-1}\cdot C_{\#}\lambda^{-2} + C_{\#}S\lambda^{-3} \le C'_{\#}S\lambda^{-3}\ .\end{aligned}$$

So

$$-\frac{1}{2}E'(a)\tilde{\chi}(a) = +\frac{\pi}{24}\big(\phi'(E_*)\big)^{-1} + \text{ Error}_5 \ , \qquad \text{with } |\text{Error}_5| < C_\# S\lambda^{-3} \ . \tag{31}$$

Now we can substitute (25), (26), (27), (28), (30), (31), into (23), (24) to obtain the following result.

$$\text{sneg}(H) = \int_{a_0}^{b_0} E(t)dt + \frac{E_*}{48\pi}\psi(E_*) + \frac{\pi}{2}\big(\phi'(0)\big)^{-1}\tilde{\chi}\big(\frac{1}{\pi}\phi(0) - \frac{1}{2}\big) + \frac{\pi}{24}\big(\phi'(E_*)\big)^{-1} + \text{ Error}_6 \ , \quad \text{with } |\text{Error}_6| \le C_\# S\lambda^{2\varepsilon-2} \ . \tag{32}$$

Next, we evaluate the integral in (32). The computation of the corresponding integral in the section on Truncated Sums of Eigenvalues carries over here word for word, to tell us once more that

$$\begin{aligned}\int_{a_0}^{b_0} E(t)dt = &-\frac{E_*}{\pi}\phi(E_*) - \frac{E_*}{48\pi}\psi(E_*) \\ &- \frac{2}{3\pi}\int_{I_{\text{BVP}}}\big(-V(x)\big)_+^{3/2}dx + \frac{2}{3\pi}\int_{I_{\text{BVP}}}\big(E_* - V(x)\big)_+^{3/2}dx \\ &+ \frac{1}{24\pi}\int_{I_{\text{BVP}}}V''(x)\big(-V(x)\big)_+^{-1/2}dx - \frac{1}{24\pi}\int_{I_{\text{BVP}}}V''(x)\cdot\big(E_* - V(x)\big)_+^{-1/2}dx \ .\end{aligned}$$

Putting this into (32), we find that

$$\begin{aligned}\text{sneg}(H) = &-\frac{2}{3\pi}\int_{I_{\text{BVP}}}\big(-V(x)\big)_+^{3/2}dx + \frac{1}{24\pi}\int_{I_{\text{BVP}}}V''(x)\cdot\big(-V(x)\big)_+^{-1/2}dx \\ &+ \frac{2}{3\pi}\int_{I_{\text{BVP}}}\big(E_* - V(x)\big)_+^{3/2} - \frac{1}{24\pi}\int_{I_{\text{BVP}}}V''(x)\cdot\big(E_* - V(x)\big)_+^{-1/2}dx \\ &+ \frac{\pi}{24}\big(\phi'(E_*)\big)^{-1} - \frac{E_*}{\pi}\phi(E_*) + \frac{\pi}{2}\big(\phi'(0)\big)^{-1}\tilde{\chi}\big(\frac{1}{\pi}\phi(0) - \frac{1}{2}\big) \\ &+ \text{ Error}_6 \ ,\end{aligned} \tag{33}$$

with $|\text{Error}_6| \le C_\# S\lambda^{2\varepsilon-2}$.

Next, we let E_* tend to $V(x_0)$ and pass to the limit in (33). As in the proof of the First WKB Eigenvalue Sum Theorem, we have

$$\lim_{E_* \searrow V(x_0)}\int_{I_{\text{BVP}}}\big(E_* - V(x)\big)_+^{3/2}dx = 0 \ ,$$

$$\lim_{E_* \searrow V(x_0)} \frac{E_*}{\pi} \phi(E_*) = 0 \ ,$$

$$\lim_{E_* \searrow V(x_0)} \Big\{ \frac{\pi}{24} \big(\phi'(E_*)\big)^{-1} - \frac{1}{24\pi} \int_{I_{\rm BVP}} V''(x) \cdot \big(E_* - V(x)\big)_+^{-1/2} dx \Big\} = 0 \ .$$

Using these results to pass to the limit in (33), we obtain the following Theorem.

Third WKB Eigenvalue Sum Theorem. *Suppose we are in the setting of the WKB Theorem on Low Eigenvalues, with* $E_\infty = 0$*. Assume hypotheses (H0*)... (H6*) of that Theorem, and assume also* $-\lambda^{-3\varepsilon} S < V(x_0) < 0$*. Then the sum of the negative eigenvalues of* $H = -\frac{d^2}{dx^2} + V(x)$ *on* $I_{\rm BVP}$*, with Dirichlet or Neumann boundary conditions, is given by*

$$\begin{aligned} sneg(H) = & -\frac{2}{3\pi} \int_{I_{\rm BVP}} \big(-V(x)\big)_+^{3/2} dx + \frac{1}{24\pi} \int_{I_{\rm BVP}} V''(x) \cdot \big(-V(x)\big)_+^{-1/2} dx \\ & + \frac{\pi}{2} (\phi'(0))^{-1} \tilde{\chi}\Big(\frac{1}{\pi}\phi(0) - \frac{1}{2}\Big) + \ \text{Error} \ , \end{aligned}$$

with $|\text{Error}| \leq \lambda^{5\varepsilon - 2} S$*.*

Recall that $\phi(E) = \int_{I_{\rm BVP}} \big(E - V(x)\big)_+^{1/2} dx$*,* $\phi'(E) = \frac{1}{2} \int_{I_{\rm BVP}} \big(E - V(x)\big)_+^{-1/2} dx$*, and* $\tilde{\chi}(t) = \min_{k \in \mathbb{Z}} |t - k - \frac{1}{2}|^2 - \frac{1}{12}$*.*

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